# A COMPUTER PROOF OF TURÁN'S INEQUALITY

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Abstract. We show how Turán's inequality $P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0$ for Legendre polynomials and related inequalities can be proven by means of a computer procedure. The use of this procedure simplifies the daily work with inequalities. For instance, we have found the stronger inequality $|x|P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0$, $-1 \leq x \leq 1$, effortlessly with the aid of our method.

## 1. Introduction

Turán showed in a 1946 letter to Szegő that

$$\Delta_n(x) := P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0, \qquad x \in [-1,1],\ n \geq 1, \tag{1.1}$$

where $P_n(x)$ denotes the $n$-th Legendre polynomial. Szegő [9] gave four non-trivial proofs. Several authors have proven analogous statements for other families of orthogonal polynomials, and there is now a substantial body of literature [6] devoted to these and related results. The aim of the present note is to describe a computer algebra proof of Turán's inequality that requires as input only the three term recurrence of the Legendre polynomials and the first two polynomials. Our method [4] is applicable to many other inequalities, including the following refinement of Turán's result, which appears to be new.

**Theorem 1.1.** *Let $P_n(x)$ denote the $n$-th Legendre polynomial. Then*

$$|x|P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0, \qquad x \in [-1,1],\ n \geq 1, \tag{1.2}$$

*with equality holding if and only if either $x = 0$ and $n$ is even, or $|x| = 1$.*

## 2. The Proving Method

We exemplify our proving method on the classical Turán inequality (1.1). The idea underlying the method is complete induction on $n$. That is, we establish the induction step

$$\Delta_n(x) \geq 0 \implies \Delta_{n+1}(x) \geq 0, \qquad x \in [-1,1], n \geq 1, \tag{2.1}$$

and afterwards we verify that the original inequality holds for $n = 1$. For proving (2.1) automatically, we construct a so-called Tarski formula whose truth implies the validity of the induction step. Tarski formulas are quantified formulas built via logical connectives from polynomial equations and inequalities over the reals. Upon replacing $P_{n-1}(x)$, $P_n(x)$, $P_{n+1}(x)$, $P_{n+2}(x)$ in (2.1) by indeterminates $Y_{-1}$, $Y_0$, $Y_1$, $Y_2$, we obtain the formula

$$\Phi := \big(\forall\, Y_{-1}, Y_0, Y_1, Y_2 \in \mathbb{R} : Y_0^2 - Y_{-1}Y_1 \geq 0 \implies Y_1^2 - Y_0Y_2 \geq 0\big).$$

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Three things have to be remarked about this formula. (i) It can be decided algorithmically whether or not a given Tarski formula is true. The classical decision procedure of Tarski [11] as well as the more efficient method of Cylindrical Algebraic Decomposition (CAD) due to Collins [1] are available for this purpose. (ii) If $\Phi$ holds, then (2.1) is also true, for if the implication $Y_0^2 - Y_{-1}Y_1 \geq 0 \implies Y_1^2 - Y_0Y_2 \geq 0$ holds for all real numbers, then it holds in particular for any real number $P_{n+i}(x)$ ($n$ and $x$ arbitrary) in place of $Y_i$. (iii) Of course, $\Phi$ is false.

In order to make the proof go through, additional knowledge about the Legendre polynomials has to be encoded into the hypothesis part of formula $\Phi$. Remarkably enough, in case of Turán's inequality it is sufficient to throw in the inequality's domain of validity and the classic recurrence [10]

$$(n+2)P_{n+2}(x) = (2n+3)xP_{n+1}(x) - (n+1)P_n(x), \qquad n \geq 0,$$

of the Legendre polynomials. This requires additional indeterminates $N$ (representing $n$) and $X$ (representing $x$). The refined formula is

$$\begin{aligned}\forall\, N, X, Y_{-1}, Y_0, Y_1, Y_2 \in \mathbb{R} : \big(&N \geq 1 \wedge -1 \leq X \leq 1 \wedge \\ &(N+2)Y_2 = -(N+1)Y_0 + (3X + 2NX)Y_1 \wedge \\ &(N+1)Y_1 = -NY_{-1} + (X + 2NX)Y_0\big) \\ \implies \big(&Y_0^2 - Y_{-1}Y_1 \geq 0 \implies Y_1^2 - Y_0Y_2 \geq 0\big).\end{aligned}$$

Using CAD, this formula can be easily verified by the computer, and by the remarks above we may regard this as a computer proof for the fact that Turán's inequality holds for $n+1$ whenever it holds for $n$.

To complete the proof, we have to consider the induction base $n = 1$. Since $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = (3x^2 - 1)/2$, we just have to verify the obvious formula

$$\forall\, X \in \mathbb{R} : -1 \leq X \leq 1 \implies \tfrac{1}{2}(1 - X^2) \geq 0,$$

which we can again leave to the computer, if we want.

Strict positivity of $\Delta_n(x)$ for $-1 < x < 1$ can be shown analogously.

## 3. Remarks and Further Applications

We have to dispel any hopes that our method yields a decision procedure for inequalities involving orthogonal polynomials or other special functions. Needless to say, there are many special functions that do not fit into our recursive framework. Roughly speaking, our procedure requires functions of $n$ (and possibly other real parameters) such that the $n$-th value depends polynomially on a finite number, independent of $n$, of previous values. For instance, the Bernoulli polynomials $B_n(x)$ cannot be handled, since their recurrence "goes all the way back". Even if an inequality is in the input class, our method may be doomed to failure because the sufficient condition that we check might not be satisfied although the conjectured inequality is true. In some cases the user can remedy this by inputting extra equations or inequalities that the functions in question satisfy. A third reason for failure are excessive computing time and memory overflows; this is what happened when we tried to reprove Gasper's extension [3] of Turán's inequality to Jacobi polynomials. Using Mathematica's implementation of CAD, we ran out of memory (3 GB) after having spent forty hours of CPU time (1.5 GHz). This is in contrast to the computation time needed for Turán's original inequality, whose proof was completed in just a second. The reason for this discrepancy are the additional two parameters appearing in the Jacobi polynomials.

In view of the doubly exponential complexity of CAD, it is surprising that our method is able to verify quite a few inequalities from the literature in a reasonable amount of time. For instance, it is a matter of seconds to verify Turán's inequality also for the following quantities in place of $P_n(x)$:

- Hermite polynomials $H_n(x)$ (for $x \in \mathbb{R}$),
- Laguerre polynomials $L_n^\alpha(x)$ (for $x > 0, \alpha > 0$),
- normalized Laguerre polynomials $L_n^\alpha(x)/L_n^\alpha(0)$ (for $x \geq 0, \alpha > -1$),
- differentiated Legendre polynomials $P_n'(x)$ (For $-1 \leq x \leq 1$; the inequality actually holds for all $x \in \mathbb{R}$, but our method fails outside $[-1, 1]$).

None of these results are new. We can also prove the inequality

$$\Delta_n(x) \geq \frac{n-1}{n+1}\Delta_{n-1}(x), \qquad x \in [-1, 1], n \geq 2,$$

which is due to Constantinescu [2].

Our method lends itself to playing around with conjectured inequalities; this is how Theorem 1.1 was obtained. Note that the absolute value function can be easily accommodated by Tarski formulas. The cases where we claim equality in (1.2) follow from the well-known facts $P_n(1) = 1$, $P_n(-1) = (-1)^n$, and

$$P_n(0) = \begin{cases} 0, & n \text{ odd}, \\ \frac{(-1)^{n/2}}{2^n}\binom{n}{n/2}, & n \text{ even}. \end{cases}$$

These can also be proven automatically by the method described above. However, there are a lot of established methods available for which proving identities like these is offendingly trivial [8, 7, 5].

We believe that our method could become a helpful tool for researchers working with inequalities. It might not be capable of proving difficult inequalities that are of interest in their own right (Turán's inequality seems to be exceptional in this respect), but it might be helpful for proving elementary inequalities that appear as subproblems in the proof of more involved theorems.

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