Symbolische Summation in Differenzen Körpern

Carsten.Schneider@risc.uni-linz.ac.at

1 A Bonus Problem in "Concrete Mathematics"

Chapter 6. Special Numbers, Bonus problem 69:

Find a closed form for

$$\sum_{k=1}^{n} k^2 \mathrm{H}_{n+k},$$

where $\mathrm{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

Knuth's answer to the problem is

$$\frac{1}{3}n\left(n+\frac{1}{2}\right)(n+1)\left(2\mathrm{H}_{2n}-\mathrm{H}_n\right)-\frac{1}{36}n\left(10n^2+9n-1\right)$$

with the remark

> *"It would be nice to automate the derivation of formulas such as this."*

In[1]:= **<< Sigma`**

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Sigma -A summation package by Carsten Schneider
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In[2]:= **Problem69 = SigmaSum[k^2 SigmaHNumber[n + k], {k, 1, n}]**

Out[2]= $\sum_{k=1}^{n} \left(k^2\ H_{k+n}\right)$

In[3]:= **Sigma[Problem69]//Simplify**

Out[3]= $-\frac{1}{36}\ n\ (1+n)\ (-1+10\ n+6\ (1+2\ n)\ H_n-12\ (1+2\ n)\ H_{2n})$

- First implementation of Karr's algorithm in a major computer algebra system

2 Multisums

An Identity from Physics (Essam, Guttmann) - Case 5 -

We eliminate the sum quantifiers in

In[4]:= **mySum** =

$$\sum_{k1=0}^{n}\left(\sum_{k2=0}^{k1}\left(\sum_{k3=0}^{k2}\left(\sum_{k4=0}^{k3}\left(\sum_{k5=0}^{k4}\Big((k1-k2)(k1-k3)(k2-k3)(k1-k4)(k2-k4)(k3-k4)(k1-k5)(k2-k5)(k3-k5)(k4-k5)\binom{n}{k1}\binom{n}{k2}\binom{n}{k3}\binom{n}{k4}\binom{n}{k5}\Big)\right)\right)\right)\right)$$

by using the two sums

In[5]:= **tower** = $\{\sum_{k=0}^{a}((\binom{n}{k})^2),\sum_{k=0}^{a}(\binom{n}{k})\}$;

We get:

In[6]:= **result** = **Sigma**[**mySum**, **Tower** → **tower**]

$$\text{Out[6]}=\frac{3\,(-3+n)\,(-2+n)^2(-1+n)^3\,n^5\left(\sum_{\iota_1=0}^{n}(\binom{n}{\iota_1})\right)\left(\sum_{\iota_1=0}^{n}((\binom{n}{\iota_1})^2)\right)^2}{256\,(-5+2\,n)\,\left(3-8\,n+4\,n^2\right)^2}$$

By the substitution

In[7]:= $\mathbf{subst} = \{\sum_{\iota_1=0}^{\mathbf{n}}(\binom{\mathbf{n}}{\iota_1}^{\cdot}) \to (\mathbf{2})^{\mathbf{n}\cdot},$
$\sum_{\iota_1=0}^{\mathbf{n}}((\binom{\mathbf{n}}{\iota_1}^{\cdot})^{\mathbf{2}}) \to \binom{\mathbf{2\ n}}{\mathbf{n}}^{\cdot}\};$

we obtain the final result:

In[8]:= **result/.subst**

$$\text{Out[8]=}\ \frac{3\ (-3+\mathtt{n})\ (-2+\mathtt{n})^2\ (-1+\mathtt{n})^3\ \mathtt{n}^5\ (\binom{2\ \mathtt{n}}{\mathtt{n}}^{\cdot})^2\ (2)^{\mathtt{n}\cdot}}{256\ (-5+2\ \mathtt{n})\ (3-8\ \mathtt{n}+4\ \mathtt{n}^2)^2}$$

3 Karr's Method and an Example

Goal: Find a closed form for

$$\sum_{k=0}^{n} k\,k!$$

A Difference Field for the Problem

Let t_1, t_2 be indeterminates where

$$\begin{array}{ccc} t_1 & \longleftrightarrow & k \\ t_2 & \longleftrightarrow & k! \end{array}$$

Consider the **field automorphism** $\sigma : \mathbb{Q}(t_1, t_2) \to \mathbb{Q}(t_1, t_2)$ canonically defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q} & & \\ \sigma(t_1) &= t_1 + 1 & \qquad \mathrm{S}\,k &= k+1 \\ \sigma(t_2) &= (t_1+1)t_2 & \qquad \mathrm{S}\,k! &= (k+1)! \end{aligned}$$

$(\mathbb{Q}(t_1, t_2), \sigma)$ is our difference field.

The Telescoping Problem

Find $g \in \mathbb{Q}(t_1, t_2)$: $\boxed{\sigma(g) - g = t_1\,t_2}$

$$\downarrow \text{ by Karr}$$

$$g = t_2.$$

The Closed Form

$$\boxed{(k+1)! - k! = k\,k!}$$

$$\downarrow$$

$$\sum_{k=0}^{n} k\,k! = (n+1)! - 1.$$

4 Sum Extensions for Indefinite Summation

In[9]:= $\mathbf{mySum} = \sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{K+\iota_3}\right)}{K+\iota_2}\right)}{K+\iota_1}\right);$

In[10]:= **Sigma[mySum]**

Out[10]= $\sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{K+\iota_3}\right)}{K+\iota_2}\right)}{K+\iota_1}\right);$

In[11]:= **Sigma[mySum, SimplifyByExt → Depth]**

Out[11]= $\frac{1}{6K^2}\left(6\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)+6K\left(\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)^2+K^2\left(\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)^3+\left(-3-3K\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)\boxed{\sum_{\iota_1=1}^{N}\left(\frac{K+2\iota_1}{(K+\iota_1)^2}\right)}-K\boxed{\sum_{\iota_1=1}^{N}\left(\frac{K+3\iota_1}{(K+\iota_1)^3}\right)}\right)$

Partial fraction decomposition:

$$\boxed{\frac{K+2i}{(K+i)^2}}=-\frac{K}{(K+i)^2}+\frac{2}{K+i},\quad \boxed{\frac{K+3i}{(K+i)^2}}=-\frac{2K}{(K+i)^3}+\frac{3}{(K+i)^2}$$

In[12]:= $\mathbf{Sigma}\left[\mathbf{mySum}, \mathbf{Tower} \to \{\{\mathbf{H_{K+N}}, \mathbf{N}\}, \{\mathbf{H^{(2)}_{K+N}}, \mathbf{N}\}, \{\mathbf{H^{(3)}_{K+N}}, \mathbf{N}\}\}\right]$

Out[12]= $\frac{1}{6}\left(-H_K^3-3H_KH_{K+N}^2+H_{K+N}^3+3H_KH_K^{(2)}-3H_KH_{K+N}^{(2)}+H_{K+N}\left(3H_K^2-3H_K^{(2)}+3H_{K+N}^{(2)}\right)-2H_K^{(3)}+2H_{K+N}^{(3)}\right)$

New Insights in Indefinite Summation

$$\sum_{\iota_1=1}^{\mathtt{N}}\left(\frac{\displaystyle\sum_{\iota_2=1}^{\iota_1}\left(\frac{\displaystyle\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{\mathtt{K}+\iota_3}\right)}{\mathtt{K}+\iota_2}\right)}{\mathtt{K}+\iota_1}\right)$$

The underlying difference field $(\mathbb{Q}(t_1)(t_2)(t_3)(t_4), \sigma)$:

$$\sigma(t_1) = t_1 + 1$$
$$\sigma(t_2) = t_2 + \frac{1}{K + t_1 + 1}$$
$$\sigma(t_3) = t_3 + \sigma(\frac{t_2}{K + t_1})$$
$$\sigma(t_4) = t_4 + \sigma(\frac{t_3}{K + t_1})$$

$$\frac{1}{6}\Big(-\mathtt{H}_{\mathtt{K}}^{3}-3\,\mathtt{H}_{\mathtt{K}}\,\mathtt{H}_{\mathtt{K+N}}^{2}+\mathtt{H}_{\mathtt{K+N}}^{3}+3\,\mathtt{H}_{\mathtt{K}}\,\mathtt{H}_{\mathtt{K}}^{(2)}-3\,\mathtt{H}_{\mathtt{K}}\,\mathtt{H}_{\mathtt{K+N}}^{(2)}$$

$$+\mathtt{H}_{\mathtt{K+N}}\Big(3\,\mathtt{H}_{\mathtt{K}}^{2}-3\,\mathtt{H}_{\mathtt{K}}^{(2)}+3\,\mathtt{H}_{\mathtt{K+N}}^{(2)}\Big)-2\,\mathtt{H}_{\mathtt{K}}^{(3)}+2\,\mathtt{H}_{\mathtt{K+N}}^{(3)}\Big)$$

The underlying difference field $(\mathbb{Q}(t_1)(t_2)(t_3')(t_4'),\sigma)$:

$$\begin{aligned}
\sigma(t_1) &= t_1+1\\
\sigma(t_2) &= t_2+\frac{1}{K+t_1+1}\\
\sigma(t_3') &= t_3'+\frac{1}{(K+t_1+1)^2}\\
\sigma(t_4') &= t_4'+\frac{1}{(K+t_1+1)^3}
\end{aligned}$$

$$\boxed{(\mathbb{Q}(t_1)(t_2)(t_3)(t_4),\sigma)\simeq(\mathbb{Q}(t_1)(t_2)(t'_3)(t'_4),\sigma)}$$

5 Definite Summation

GOAL: Find a closed form for

$$\sum_{k=1}^{n}\left(\frac{H_k\,(3+k+n)!\,(-1)^{k}\,(-1)^{-1+n}}{(1+k)!\,(2+k)!\,(-k+n)!}\right) - \frac{(n)!}{(3+n)!}\sum_{k=1}^{n}\left(\frac{(3+k+n)!\,(-1)^{k}\,\left(1-(2+n)\,(-1)^{n}\right)}{k\,(1+k)!^{2}\,(-k+n)!}\right)$$

(The number of rhombus tilings of a symmetric hexagon, Fulmek & Krattenthaler)

In[13]:= $\mathbf{mySum1} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \left(\frac{\mathbf{H_k\ (3+k+n)!_{.}\ (-1)^{k_{.}}\ (-1)^{-1+n_{.}}}}{\mathbf{(1+k)!_{.}\ (2+k)!_{.}\ (-k+n)!_{.}}} \right);$

Finding a recurrence

In[14]:= $\mathbf{rec1 = GenerateRecurrence[mySum1][[1]]}$

Out[14]= $\mathtt{n\ (1+n)\ (2+n)\ (3+n)\ (4+n)\ (-1+n)!}$
$\left(-(9+2\,n)\left(8+6\,n+n^2\right)\,\mathrm{SUM}[n]+ (9+2\,n)\left(13+8\,n+n^2\right)\,\mathrm{SUM}[1+n]+ \left(30+42\,n+17\,n^2+2\,n^3\right)\,\mathrm{SUM}[2+n]- (3+n)\left(25+15\,n+2\,n^2\right)\,\mathrm{SUM}[3+n]\right) ==$
$2\,(-1)^n\,(9+2\,n)\left(35+24\,n+4\,n^2\right)(4+n)!$

Solving the recurrence

In[15]:= $\mathbf{recSol1 = SolveRecurrence[rec1, SUM[n], Tower \to \{H_n\}]}$

Out[15]= $\left\{\{0,1\}, \left\{0, \frac{3-n^2+4\,H_n+6\,n\,H_n+2\,n^2\,H_n}{(1+n)\,(2+n)}\right\}, \left\{0, \frac{1}{4}\,(2+n)\,(-1)^{n_{.}}\right\}, \left\{1, \frac{\left(16-13\,n^2-5\,n^3+32\,H_n+64\,n\,H_n+40\,n^2\,H_n+8\,n^3\,H_n\right)(-1)^{n_{.}}}{4\,(1+n)\,(2+n)}\right\}\right\}$

Finding the linear combination

In[16]:= $\mathbf{solution1 = FindLinearCombination[recSol1, mySum1, 3]}$

Out[16]= $-1-\frac{3-n^2+4\,H_n+6\,n\,H_n+2\,n^2\,H_n}{(1+n)\,(2+n)}+\frac{1}{4}\,(2+n)\,(-1)^{n_{.}}+ \frac{\left(16-13\,n^2-5\,n^3+32\,H_n+64\,n\,H_n+40\,n^2\,H_n+8\,n^3\,H_n\right)(-1)^{n_{.}}}{4\,(1+n)\,(2+n)}$

In[17]:= $\mathbf{mySum2} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \left(\frac{\mathbf{(3+k+n)!_{.}\ (-1)^{k.}\ (1-(2+n)\ (-1)^{n.})}}{\mathbf{k\ (1+k)!_{.}^{2}\ (-k+n)!_{.}}} \right);$

Finding a recurrence

In[18]:= $\mathbf{rec2 = GenerateRecurrence[mySum2, RecOrder \to 2][[1]]}$

Out[18]= $-n\ (1+n)\ (3+n)\ (1+3\ (-1)^{n}+(-1)^{n}\ n)$
$(-1+4\ (-1)^{n}+(-1)^{n}\ n)\ \left(28+15\ n+2\ n^{2}\right)\ (-1+n)!\ \mathtt{SUM}[n]+$
$6\ n\ (1+n)\ (3+n)^{2}\ (-1+2\ (-1)^{n}+(-1)^{n}\ n)$
$(-1+4\ (-1)^{n}+(-1)^{n}\ n)\ (-1+n)!\ \mathtt{SUM}[1+n]+$
$n\ (1+n)\ (3+n)\ (-1+2\ (-1)^{n}+(-1)^{n}\ n)$
$(1+3\ (-1)^{n}+(-1)^{n}\ n)\ \left(10+9\ n+2\ n^{2}\right)\ (-1+n)!\ \mathtt{SUM}[2+n] ==$
$2\ (-1+2\ (-1)^{n}+(-1)^{n}\ n)\ (1+3\ (-1)^{n}+(-1)^{n}\ n)$
$(-1+4\ (-1)^{n}+(-1)^{n}\ n)\ \left(35+24\ n+4\ n^{2}\right)\ (4+n)!$

Solving the recurrence

$$\boxed{((-1)^{k})^{2} = 1}$$

In[19]:= $\mathbf{recSol2} =$
$\mathbf{SolveRecurrence[rec2, SUM[n], Tower \to \{H_n\},}$
$\boxed{\mathbf{WithMinusPower \to True}}\mathbf{]}$

Out[19]= $\{\{0, 2+n-(-1)^{n.}\}, \{0, 16-6\ n^{2}-n^{3}+$
$(-1)^{n.}+28\ n\ (-1)^{n.}+23\ n^{2}\ (-1)^{n.}+8\ n^{3}\ (-1)^{n.}+n^{4}\ (-1)^{n.}\},$
$\{1, -\frac{1}{28}\ (260-150\ n^{2}-39\ n^{3}+336\ H_{n}+$
$616\ n\ H_{n}+336\ n^{2}\ H_{n}+56\ n^{3}\ H_{n}-325\ (-1)^{n.}+365\ n^{2}\ (-1)^{n.}+$
$228\ n^{3}\ (-1)^{n.}+39\ n^{4}\ (-1)^{n.}-672\ H_{n}\ (-1)^{n.}-1568\ n\ H_{n}\ (-1)^{n.}-$
$1288\ n^{2}\ H_{n}\ (-1)^{n.}-448\ n^{3}\ H_{n}\ (-1)^{n.}-56\ n^{4}\ H_{n}\ (-1)^{n.})\}\}$

Finding the linear combination

In[20]:= $\mathbf{solution2 = FindLinearCombination[recSol2, mySum2, 2]}$

Out[20]= $(3+n)\ \left(-1+3\ n+2\ n^{2}-\left(-1+6\ n+7\ n^{2}+2\ n^{3}\right)\ (-1)^{n.}+\right.$
$\left.2\ \left(2+3\ n+n^{2}\right)\ H_{n}\ (-1+(2+n)\ (-1)^{n.})\right)$

In[21]:= **solution1 − solution2/((n + 1)(n + 2)(n + 3))//Simplify**

Out[21]= $-2 + (2 + n)\ (-1)^{n}$.

6 Difference Equations and Symbolic Summation

Let $(\mathbb{F}, \sigma)$ be a difference field and

$$\mathbb{K} = \{k \in \mathbb{F} \,|\, \sigma(k) = k\}$$

be the constant field.

Telescoping

- GIVEN $f \in \mathbb{F}$
- FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

Extended Telescoping

- GIVEN $f_0, \ldots, f_d \in \mathbb{F}$, $a_0, a_1 \in \mathbb{F}$
- FIND ALL $c_0, \ldots, c_d \in \mathbb{K}$, $h \in \mathbb{F}$:

$$\boxed{a_1\, \sigma(h) - a_0\, h = c_0\, f_0 + \cdots + c_d\, f_d}$$

Remark: Z's "Creative Telescoping"

- GIVEN $f_i = \text{summand}(n+i, k) \in \mathbb{F}$
- FIND ALL $c_0, \ldots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0\, f_0 + \cdots + c_d\, f_d}$$

m-th Order Linear Difference Equations

- GIVEN $f, a_0, \dots, a_m \in \mathbb{F}$
- FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

The General Problem

- GIVEN $a_0, \dots, a_m \in \mathbb{F}$,$f_0, \dots, f_d \in \mathbb{F}$.
- FIND ALL $g \in \mathbb{F}$, $c_0, \dots, c_d \in \mathbb{K}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = c_0\,f_0 + \cdots + c_d\,f_d}$$

My Results

- Streamlining of Karr's ideas result in a simpler algorithm
- Generalization of Karr's algorithm:

$$\text{first order} \longrightarrow m\text{-th order}$$

- New connections:

$$\text{indefinite-}\Sigma \longleftrightarrow \text{definite-}\Sigma$$

7 Sum Extensions for Recurrences

An Alternating Sum (P. Kirschenhofer)

In[22]:= $\mathbf{mySum} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{N}} \left(\frac{\binom{\mathbf{N}}{\mathbf{k}} \cdot (-\mathbf{1})^{\mathbf{k}}}{(\mathbf{k}+\mathbf{1})^{\mathbf{4}}} \right);$

Finding a recurrence

In[23]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}]$

Out[23]= $\{(1+\mathtt{N})\ (2+\mathtt{N})\ (3+\mathtt{N})\ (4+\mathtt{N})\ \mathtt{SUM}[\mathtt{N}] - 3\ (2+\mathtt{N})\ (3+\mathtt{N})^2\ (4+\mathtt{N})\ \mathtt{SUM}[1+\mathtt{N}] + (3+\mathtt{N})\ (4+\mathtt{N})\ (37+21\ \mathtt{N}+3\ \mathtt{N}^2)\ \mathtt{SUM}[2+\mathtt{N}] - (4+\mathtt{N})^4\ \mathtt{SUM}[3+\mathtt{N}] == -1\}$

Solving the recurrence (A First Attempt)

In[24]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}[[\mathbf{1}]], \mathbf{SUM}[\mathbf{N}]]$

Out[24]= $\{\{0, \frac{1}{1+\mathtt{N}}\}\}$

Solving the recurrence (Step I)

In[25]:= **recSol** = **SolveRecurrence**[**rec**[[**1**]], **SUM**[**N**],
NestedSumExt → ∞]

$$\text{Out[25]}= \left\{\left\{0, \frac{1}{1+N}\right\}, \left\{0, \frac{\sum_{\iota_1=1}^{N}\left(\frac{1}{1+\iota_1}\right)}{1+N}\right\}, \left\{0, \frac{\sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{1+\iota_2}\right)}{1+\iota_1}\right)}{1+N}\right\},\right.$$

$$\left.\left\{1, \frac{\sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{1+\iota_3}\right)}{1+\iota_2}\right)}{1+\iota_1}\right)}{1+N}\right\}\right\}$$

- Inspired by Abramov/Petkovšek and Hendrik/Singer
- My theoretical result:

  We can find all sum extensions over a given **ΠΣ-field** which give more solutions of a homogeneous or **inhomogeneous** recurrence!

- Speed up in computation.
- Further simplification by my **indefinite summation algorithm**

We know:

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{K+k}}{K+j}}{K+i} = 3\,\mathrm{H}_K\,\mathrm{H}_{K+N}^{(2)} + \mathrm{H}_{K+N}\left(3\,\mathrm{H}_K^2 - 3\,\mathrm{H}_K^{(2)} + 3\,\mathrm{H}_{K+N}^{(2)}\right) - 2\,\mathrm{H}_K^{(3)} + 2\,\mathrm{H}_{K+N}^{(3)}\right)$$

Solving the recurrence (Step II)

In[26]:= $\mathbf{recSol} =$
$\mathbf{SolveRecurrence[rec[[1]], SUM[N], Tower \to \{H_N, H_N^{(2)}, H_N^{(3)}\}]}$

Out[26]= $\Big\{\Big\{0, \frac{1}{(1+N)^3}\left(2 + 2\,H_N + 2\,N\,H_N + H_N^2 + 2\,N\,H_N^2 + N^2\,H_N^2 + H_N^{(2)} + 2\,N\,H_N^{(2)} + N^2\,H_N^{(2)}\right)\Big\},$

$\Big\{0, \frac{1}{(1+N)^3}\big(-4\,N - 2\,N^2 + 2\,H_N +$
$2\,N\,H_N + H_N^2 + 2\,N\,H_N^2 + N^2\,H_N^2 + H_N^{(2)} + 2\,N\,H_N^{(2)} + N^2\,H_N^{(2)}\big)\Big\},$

$\Big\{0, \frac{1}{(1+N)^3}\big(-N + N^2 - H_N - 4\,N\,H_N -$
$3\,N^2\,H_N + H_N^2 + 2\,N\,H_N^2 + N^2\,H_N^2 + H_N^{(2)} + 2\,N\,H_N^{(2)} + N^2\,H_N^{(2)}\big)\Big\},$

$\Big\{1, \frac{1}{6\,(1+N)^4}\big(-6\,N - 6\,N\,H_N - 6\,N^2\,H_N - 3\,N\,H_N^2 - 6\,N^2\,H_N^2 - 3\,N^3\,H_N^2 + H_N^3 +$
$3\,N\,H_N^3 + 3\,N^2\,H_N^3 + N^3\,H_N^3 - 3\,N\,H_N^{(2)} - 6\,N^2\,H_N^{(2)} - 3\,N^3\,H_N^{(2)} +$
$3\,H_N\,H_N^{(2)} + 9\,N\,H_N\,H_N^{(2)} + 9\,N^2\,H_N\,H_N^{(2)} + 3\,N^3\,H_N\,H_N^{(2)} + 2\,H_N^{(3)} +$
$6\,N\,H_N^{(3)} + 6\,N^2\,H_N^{(3)} + 2\,N^3\,H_N^{(3)}\big)\Big\}\Big\}$

Finding the linear combination

In[27]:= $\mathbf{FindLinearCombination[recSol, defSum, 3]//Simplify}$

Out[27]= $\frac{1}{6\,(1+N)^4}\big(3\,(1+N)^2\,H_N^2 +$
$(1+N)^3\,H_N^3 + 3\,(1+N)^2\,H_N^{(2)} + 3\,(1+N)\,H_N\left(2 + (1+N)^2\,H_N^{(2)}\right) +$
$2\left(3 + H_N^{(3)} + 3\,N\,H_N^{(3)} + 3\,N^2\,H_N^{(3)} + N^3\,H_N^{(3)}\right)\big)$

Back to Krattenthaler: A manual sum extension

In[28]:= **recSol1 = SolveRecurrence[rec1, SUM[n], Tower → {H_n}]**

$$\text{Out[28]}= \Big\{\{0,1\},\Big\{0,\frac{3-n^2+4\,H_n+6\,n\,H_n+2\,n^2\,H_n}{(1+n)\,(2+n)}\Big\},\Big\{0,\frac{1}{4}\,(2+n)\,(-1)^{n}\Big\},\Big\{1,\frac{\left(16-13\,n^2-5\,n^3+32\,H_n+64\,n\,H_n+40\,n^2\,H_n+8\,n^3\,H_n\right)(-1)^{n}}{4\,(1+n)\,(2+n)}\Big\}\Big\}$$

Solving the recurrence automatically

In[29]:= **SolveRecurrence[rec1, SUM[n], NestedSumExt → ∞]**

$$\text{Out[29]}= \Big\{\{0,1\},\{0,(2+n)\,(-1)^{n}\},\Big\{0,-\frac{2-n+6\,\sum_{\iota_1=1}^{n}\left(\frac{1+\iota_1}{\iota_1\,(2+\iota_1)}\right)+6\,n\,\sum_{\iota_1=1}^{n}\left(\frac{1+\iota_1}{\iota_1\,(2+\iota_1)}\right)}{6\,(1+n)}\Big\},$$
$$\Big\{1,\frac{1}{(1+n)\,(2+n)}\left((-1)^{n}\left(3+3\,n+n^2+8\,\boxed{\sum_{\iota_1=1}^{n}\left(\frac{1+\iota_1}{\iota_1\,(2+\iota_1)}\right)}+16\,n\sum_{\iota_1=1}^{n}\left(\frac{1+\iota_1}{\iota_1\,(2+\iota_1)}\right)+10\,n^2\sum_{\iota_1=1}^{n}\left(\frac{1+\iota_1}{\iota_1\,(2+\iota_1)}\right)+2\,n^3\sum_{\iota_1=1}^{n}\left(\frac{1+\iota_1}{\iota_1\,(2+\iota_1)}\right)\right)\right)\Big\}\Big\}$$

Solving the recurrence without simplification

In[30]:= **SolveRecurrence[rec1, SUM[n], NestedSumExt → ∞,**
AlgebraicRelationInSumSolutions → True]

$$\text{Out[30]}= \Big\{\{0,1\},\{0,(2+n)\,(-1)^{n}\},\Big\{0,-\sum_{\iota_1=1}^{n}\left((3+2\,\iota_1)\,(-1)^{\iota_1}\sum_{\iota_2=1}^{\iota_1}\left(\frac{(-1)^{\iota_2}}{\iota_2\,(2+\iota_2)}\right)\right)\Big\},$$
$$\Big\{1,2\sum_{\iota_1=1}^{n}\left((3+2\,\iota_1)\,(-1)^{\iota_1}\boxed{\sum_{\iota_2=1}^{\iota_1}\left(\frac{1+\iota_2}{\iota_2\,(2+\iota_2)}\right)}\right)\Big\}\Big\}$$

8 Finding a Recurrence and Sum Extensions

In[31]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{n}} \left(\mathbf{H_k} \binom{\mathbf{n}}{\mathbf{k}}\right);$

"Creative telescoping"

In[32]:= **GenerateRecurrence[mySum]**

Out[32]= $\{4\ (1+n)\ \mathtt{SUM}[n] - 2\ (3+2\ n)\ \mathtt{SUM}[1+n] + (2+n)\ \mathtt{SUM}[2+n] == 1\}$

Indefinite summation

In[33]:= $\mathbf{Sigma}\left[\mathbf{mySum}, \mathbf{Tower} \to \{\sum_{\mathbf{k=0}}^{\mathbf{a}} \left(\mathbf{H_k} \binom{\mathbf{n+1}}{\mathbf{k}}\right), \sum_{\mathbf{k=0}}^{\mathbf{a}} \left(\mathbf{H_k} \binom{\mathbf{2+n}}{\mathbf{k}}\right)\}\right]$

Out[33]= $$\frac{1}{4\ (1+n)}\left(-n - n\ (1+n)\ H_n + 2\ (3+2\ n) \sum_{\iota_1=0}^{n}\left(H_{\iota_1}\binom{1+n}{\iota_1}\right) + (-2-n) \sum_{\iota_1=0}^{n}\left(\frac{(2+n)\ H_{\iota_1}\binom{1+n}{\iota_1}}{2+n-\iota_1}\right)\right)$$

$$\mathrm{SUM}[n] = \sum_{k=0}^{n} \mathrm{H}_k \binom{n}{k}$$

$$\mathrm{SUM}[n+1] = \sum_{k=0}^{n+1} \mathrm{H}_k \binom{n+1}{k} = \sum_{k=0}^{n} \mathrm{H}_k \binom{n+1}{k} + \mathrm{H}_{n+1}$$

$$\mathrm{SUM}[n+2] = \sum_{k=0}^{n+2} \mathrm{H}_k \binom{n+2}{k}$$

$$= \sum_{k=0}^{n} \mathrm{H}_k \binom{n+2}{k} + (n+2)\,\mathrm{H}_{n+1} + \mathrm{H}_{n+2}$$

Indefinite summation with sum extensions

In[34]:= $\mathbf{Sigma}\Big[\mathbf{mySum}, \mathbf{Tower} \to \Big\{\sum_{\mathbf{k=0}}^{\mathbf{a}}\Big(\mathbf{H_k}\binom{\mathbf{n+1}}{\mathbf{k}}\Big), \sum_{\mathbf{k=0}}^{\mathbf{a}}\Big(\mathbf{H_k}\binom{\mathbf{2+n}}{\mathbf{k}}\Big)\Big\},$

$\mathbf{SimplifyByExt} \to \mathbf{DepthNumber}\Big]$

Out[34]= $\frac{1}{4\ (1+n)}\Bigg(1-n+2\ (1+n)\ H_n+2\ (1+n)\ \sum_{\iota_1=0}^{n}\Big(H_{\iota_1}\binom{1+n}{\iota_1}\Big)+ (-2-n)\ \sum_{\iota_1=0}^{n}\left(\frac{\binom{1+n}{\iota_1}}{2+n-\iota_1}\right)\Bigg)$

"Creative telescoping" with sum extensions

In[35]:= $\mathbf{GenerateRecurrence}[\mathbf{mySum}, \mathbf{SimplifyByExt} \to \mathbf{DepthNumber}]$

Out[35]= $\Big\{-2\ \mathtt{SUM}[n]+\mathtt{SUM}[1+n] == \sum_{\iota_1=0}^{n}\left(\frac{\binom{n}{\iota_1}}{1+n-\iota_1}\right)\Big\}$

9 How One Can Play with Sums

Problem: Find a closed form for

In[36]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{2\ n}} \left(\mathbf{k\ H_k} \left(\left(\binom{\mathbf{2\ n}}{\mathbf{k}}\right)^{\mathbf{3}} (-\mathbf{1})^{\mathbf{k}}\right)_{\mathbf{k}}\right);$

Creative telescoping

In[37]:= **rec = GenerateRecurrence[mySum, RecOrder → 4]**

```
1841.48 Second
```

Out[37]=
```
{81 (1 + n) (10 + 117 n + 441 n^2 + 648 n^3 + 324 n^4)^2 (579023679111696+
      6203096595284292 n + 30574972749055508 n^2+
      92475481987210701 n^3 + 192864735750636284 n^4+
      295166120513347017 n^5 + 344113220933469194 n^6+
      312890401572444600 n^7 + 225181229898272112 n^8+
      129339961859979540 n^9 + 59474372437202472 n^10+
      21854565707771808 n^11 + 6372893337871680 n^12+
      1455288215784768 n^13 + 254598040577664 n^14+
      32934777209856 n^15 + 2967155877888 n^16+
      166161051648 n^17 + 4353564672 n^18) SUM[n]+
  108 ( - 8911086594732000+
      595686855250231800 n + 16380227867435099780 n^2+
      185672492904312930710 n^3 + 1271723758536088957353 n^4+
      6026151073985872712073 n^5 + 21197749937538020891079 n^6+
      57793321639546981142298 n^7 + 125693551925945528389705 n^8+
      222521457681141044963341 n^9+
      325368258856450491542511 n^10+
      397108616509050749048718 n^11+
      407622807225028518763356 n^12+
      353729663174629500044400 n^13+
      260330393614389288503220 n^14+
      162709980603775713128520 n^15+
      86335405854765454150272 n^16 + 38809363531072919958144 n^17+
      14720133478715210657664 n^18 + 4681642828665855843072 n^19+
      1237296059054356451328 n^20 + 268300933294762027008 n^21+
      46890597952821408768 n^22 + 6437495043769780224 n^23+
      668002856934260736 n^24 + 49220844925353984 n^25+
      2293562354761728 n^26 + 50779978334208 n^27) SUM[1 + n]+
  18 (2 + n) (3 + 2 n) ( - 8228295571986000 - 29467353203684820 n+
      138151839326711 6428 n^2 + 1997813992219 1293573 n^3+
      13914438797197 1638219 n^4 + 625542630805627460455 n^5+
      2017285686440215860490 n^6 + 4933055970124372861135 n^7+
      9465689765373655917267 n^8 + 14579998008141370748253 n^9+
      18312629998410321364656 n^10 + 18961209332586432771048 n^11+
      16304139557770332127212 n^12 + 11695416700671314908740 n^13+
      7013537868185350191792 n^14 + 3515617464514069708512 n^15+
      1469465760759532649280 n^16 + 509652781805658910464 n^17+
      145518011266651170048 n^18 + 33806212169624059392 n^19+
      6282436535103246336 n^20 + 910948598145469440 n^21+
      99231835717287936 n^22 + 7633845045411840 n^23+
      369565397876736 n^24 + 8463329722368 n^25)
    SUM[2 + n] + 12 (2 + n) (3 + n)
    (3 + 2 n) (5 + 2 n) ( - 64001714143920 - 503422860673228 n+
      4002975025720952 n^2 + 79747990756043705 n^3+
      565678480977551301 n^4 + 2447100392628223047 n^5+
      7404218627394040182 n^6 + 16709317348234374364 n^7+
      29191436701822318447 n^8 + 40425384732611573230 n^9+
      45074461215631426464 n^10 + 40878463232569911732 n^11+
      30338483534960452020 n^12 + 18477110572629289128 n^13+
      9232514580951306000 n^14 + 3772738135947714336 n^15+
      1252587607610477760 n^16 + 334329670014178176 n^17+
      70597472266909440 n^18 + 11513259270314496 n^19+
      1397288190984192 n^20 + 118711550287872 n^21+
      6295254515712 n^22 + 156728328192 n^23) SUM[3 + n]+
  (2 + n) (3 + n)^2 (4 + n)^2 (3 + 2 n) (5 + 2 n)
    (7 + 2 n)^3 ( - 945554940 - 7607976456 n + 35254988575 n^2+
      756814949687 n^3 + 4816720182041 n^4 + 17947420546069 n^5+
      45372683784936 n^6 + 83005099177032 n^7 + 113701841575020 n^8+
      118788006388788 n^9 + 95405698339488 n^10+
      58876332512544 n^11 + 27669385543104 n^12+
      9716847158592 n^13 + 2466213765120 n^14 + 426750114816 n^15+
      44986834944 n^16 + 2176782336 n^17) SUM[4 + n] ==
 0}
```

Creative telescoping with sum extensions

In[38]:= **rec = GenerateRecurrence[mySum, SimplifyByExt → DepthNumber]**

```
71.52 Second
```

Out[38]= $\{-36\,(1+n)\,(2+n)\,(1+2\,n)\,(3+2\,n)\,(1+3\,n)\,(2+3\,n)\,(1+6\,n)\,(5+6\,n)\,\left(1+6\,n+6\,n^2\right)\,\left(13+18\,n+6\,n^2\right)\,\mathrm{SUM}[n]-24\,(1+n)\,(2+n)\,(1+2\,n)\,(3+2\,n)\,\left(1+6\,n+6\,n^2\right)\,\left(90+814\,n+2543\,n^2+3864\,n^3+3126\,n^4+1296\,n^5+216\,n^6\right)\,\mathrm{SUM}[1+n]-4\,(1+n)^2\,(2+n)^2\,(1+2\,n)\,(3+2\,n)^3\,\left(1+6\,n+6\,n^2\right)^2\,\mathrm{SUM}[2+n]==$

$$2\Big(\big(-19512-448728\,n-4422462\,n^2-24996138\,n^3-91349700\,n^4-227427644\,n^5-376226464\,n^6-308925516\,n^7+319086320\,n^8+1617697256\,n^9+3088351728\,n^{10}+3851758512\,n^{11}+3453843392\,n^{12}+2288224320\,n^{13}+1119909888\,n^{14}+396032256\,n^{15}+96095232\,n^{16}+14349312\,n^{17}+995328\,n^{18}\big)\,(-1)^{2\,n.}+\big(-14802-376587\,n-3834063\,n^2-21159534\,n^3-71496792\,n^4-157297032\,n^5-232167060\,n^6-231571656\,n^7-153801504\,n^8-65046240\,n^9-15816384\,n^{10}-1679616\,n^{11}\big)\sum_{\iota_1=1}^{2\,n}\left(\frac{(-1+\iota_1)\,\iota_1^3\,\left(\left(\binom{2\,n}{\iota_1}\right)^{.\,3}\,(-1)^{\iota_1.}\right)_{\iota_1.}}{(1+2\,n-\iota_1)^3}\right)+\big(-26016-715824\,n-8970272\,n^2-68124912\,n^3-352009200\,n^4-1316397856\,n^5-3697583664\,n^6-7984118976\,n^7-13441452832\,n^8-17772262080\,n^9-18480846528\,n^{10}-15046225664\,n^{11}-9482866944\,n^{12}-4533055488\,n^{13}-1588110336\,n^{14}-384380928\,n^{15}-57397248\,n^{16}-3981312\,n^{17}\big)\sum_{\iota_1=1}^{2\,n}\left(\frac{(-1+\iota_1)\,\iota_1^3\,\left(\left(\binom{2\,n}{\iota_1}\right)^{.\,3}\,(-1)^{\iota_1.}\right)_{\iota_1.}}{(1+2\,n-\iota_1)^3\,(2+2\,n-\iota_1)^3\,(3+2\,n-\iota_1)^3}\right)\Big)\}$$

Simplification of the recurrence

In[39]:= $$\mathbf{mySumA} = \sum_{\iota_1=1}^{2\,n}\left(\frac{\left((-1+\iota_1)\,\iota_1^3\,\left(\left(\binom{2\,n}{\iota_1}\right)^3\,(-1)^{\iota_1}\right)\right)_{\iota_1}}{(1+2\,n-\iota_1)^3}\right);$$

⋮

In[40]:= $$\mathbf{resultA} = 2\,n - n\left(\frac{(3\,n)!\,(-1)^{n}}{((n)!)^3}\right)_{n};$$

In[41]:= $$\mathbf{mySumB} = \sum_{\iota_1=1}^{2\,n}\left(\frac{\left((-1+\iota_1)\,\iota_1^3\,\left(\left(\binom{2\,n}{\iota_1}\right)^3\,(-1)^{\iota_1}\right)\right)_{\iota_1}}{(1+2\,n-\iota_1)^3\,(2+2\,n-\iota_1)^3\,(3+2\,n-\iota_1)^3}\right);$$

⋮

In[42]:= $$\mathbf{resultB} = \frac{-3-15\,n-12\,n^2+17\,n^3+38\,n^4+28\,n^5+8\,n^6}{4\,(1+n)^2\,(1+2\,n)^3} + \frac{3\,(1+3\,n)\,(2+3\,n)\left(\frac{(3\,n)!\,(-1)^{n}}{((n)!)^3}\right)_{n}}{8\,(1+n)^4\,(1+2\,n)^3};$$

In[43]:= $$\mathbf{rec} = \mathbf{rec}/.\{\mathbf{mySumA}\to\mathbf{resultA},\mathbf{mySumB}\to\mathbf{resultB},(-1)^{2\,n}\to 1\}$$

Out[43]= $$\{4\,(1+n)\,(2+n)\,(1+2\,n)\,(3+2\,n)\,(1+6\,n+6\,n^2)\,\big(-9\,(1+3\,n)\,(2+3\,n)\,(1+6\,n)\,(5+6\,n)\,(13+18\,n+6\,n^2)\,\mathrm{SUM}[n]-6\,(90+814\,n+2543\,n^2+3864\,n^3+3126\,n^4+1296\,n^5+216\,n^6)\,\mathrm{SUM}[1+n]-(1+n)\,(2+n)\,(3+2\,n)^2\,(1+6\,n+6\,n^2)\,\mathrm{SUM}[2+n]\big) == -\frac{1}{1+n}\Big(6\,(6504+144754\,n+1384851\,n^2+7537254\,n^3+26070977\,n^4+60620448\,n^5+97542252\,n^6+109802520\,n^7+86051628\,n^8+45881424\,n^9+15822864\,n^{10}+3172608\,n^{11}+279936\,n^{12})\left(\frac{(3\,n)!\,(-1)^{n}}{((n)!)^3}\right)_{n}\Big)\}$$

Solving the recurrence without simplification

In[44]:= $\mathbf{SolveRecurrence}\big[\mathbf{rec}[[1]], \mathbf{SUM}[\mathbf{n}], \mathbf{NestedSumExt} \to \infty,$

$$\mathbf{Tower} \to \{\Big(\frac{((\mathbf{n})!.)^{3}\ (\mathbf{6\ n})!.\ (-\mathbf{1})^{\mathbf{n}.}}{((\mathbf{2\ n})!.)^{3}\ (\mathbf{3\ n})!.}\Big)_{\mathbf{n}.}\},$$

$\mathbf{AlgebraicRelationInSumSolutions} \to \mathbf{True}\big]$

Out[44]= $$\Big\{\Big\{0, \Big(\frac{(n)!.^{3}\ (6\ n)!.\ (-1)^{n.}}{(2\ n)!.^{3}\ (3\ n)!.}\Big)_{n.}\Big\},$$

$$\Big\{0, n\ \Big(\frac{(3\ n)!.\ (-1)^{n.}}{(n)!.^{3}}\Big)_{n.}\Big\}, \Big\{1, \frac{1}{6}\ \Big(\frac{(n)!.^{3}\ (6\ n)!.\ (-1)^{n.}}{(2\ n)!.^{3}\ (3\ n)!.}\Big)_{n.}$$

$$\sum_{\iota_1=1}^{n}\Bigg(\Bigg(\iota_1\ \big(1-6\ \iota_1+6\ \iota_1^2\big)\ \Big(\frac{(3\ \iota_1)!.\ (-1)^{\iota_1.}}{(\iota_1)!.^{3}}\Big)_{\iota_1.}$$

$$\sum_{\iota_2=2}^{\iota_1}\Big(\big(9360-64710\ \iota_2+63189\ \iota_2^2+413410\ \iota_2^3-$$

$$1436799\ \iota_2^4+2117172\ \iota_2^5-1737846\ \iota_2^6+$$

$$826740\ \iota_2^7-213840\ \iota_2^8+23328\ \iota_2^9\big)/$$

$$\big((-1+\iota_2)\ \iota_2\ (-3+2\ \iota_2)\ (-5+3\ \iota_2)$$

$$(-4+3\ \iota_2)\ \big(13-18\ \iota_2+6\ \iota_2^2\big)\ \big(1-6\ \iota_2+6\ \iota_2^2\big)\big)\Big)\Bigg)\Big/$$

$$\Bigg((-1+2\ \iota_1)^2\ (-2+3\ \iota_1)\ (-1+3\ \iota_1)$$

$$\Big(\frac{(\iota_1)!.^{3}\ (6\ \iota_1)!.\ (-1)^{\iota_1.}}{(2\ \iota_1)!.^{3}\ (3\ \iota_1)!.}\Big)_{\iota_1.}\Bigg)\Bigg)\Big\}\Big\}$$

Solving the recurrence with simplification

In[45]:= **recSol =**

$$\textbf{SolveRecurrence}[\textbf{rec}[[1]],\textbf{SUM}[n],\textbf{NestedSumExt}\to\infty,\ \textbf{Tower}\to\{\left(\frac{((n)!.)^3\,(6\,n)!.\,(-1)^{n.}}{((2\,n)!.)^3\,(3\,n)!.}\right)_{n.}\}]$$

Out[45]=
$$\{\{0,\left(\frac{((n)!.)^3\,(6\,n)!.\,(-1)^{n.}}{((2\,n)!.)^3\,(3\,n)!.}\right)_{n.}\},\{0,n\left(\frac{(3\,n)!.\,(-1)^{n.}}{((n)!.)^3}\right)_{n.}\},\{1,-\Bigg(n\left(\frac{(3\,n)!.\,(-1)^{n.}}{((n)!.)^3}\right)_{n.}\Bigg(60-887\,n+4948\,n^2-13599\,n^3+19512\,n^4-13932\,n^5+3888\,n^6+\left(2-25\,n+117\,n^2-258\,n^3+270\,n^4-108\,n^5\right)\sum_{\iota_1=2}^{n}\Big(\big(9360-64710\,\iota_1+63189\,\iota_1^2+413410\,\iota_1^3-1436799\,\iota_1^4+2117172\,\iota_1^5-1737846\,\iota_1^6+826740\,\iota_1^7-213840\,\iota_1^8+23328\,\iota_1^9\big)/\big((-1+\iota_1)\,\iota_1\,(-3+2\,\iota_1)\,(-5+3\,\iota_1)\,(-4+3\,\iota_1)\,(13-18\,\iota_1+6\,\iota_1^2)\,(1-6\,\iota_1+6\,\iota_1^2)\big)\Big)\Bigg)\Bigg)/\left(12\,(-1+2\,n)\,(-2+3\,n)\,(-1+3\,n)\,(1-6\,n+6\,n^2)\right)\}\}$$

Solving the recurrence with simple sums

In[46]:= **SolveRecurrence[rec, SUM[n], NestedSumExt → ∞, Tower → tower, SimpleSumRepresentation → True]**

Out[46]=
$$\{\{0,\left(\frac{(n)!.^3\,(6\,n)!.\,(-1)^{n.}}{(2\,n)!.^3\,(3\,n)!.}\right)_{n.}\},\{0,n\left(\frac{(3\,n)!.\,(-1)^{n.}}{(n)!.^3}\right)_{n.}\},\{1,\frac{1}{6\,(-1+2\,n)(-2+3\,n)(-1+3\,n)}\Bigg(\left(\frac{(3\,n)!.\,(-1)^{n.}}{(n)!.^3}\right)_{n.}\Bigg(-12+91\,n-245\,n^2+261\,n^3-90\,n^4+\left(-10\,n+65\,n^2-135\,n^3+90\,n^4\right)\sum_{\iota_1=2}^{n}\left(\frac{1}{-1+\iota_1}\right)+\left(-12\,n+78\,n^2-162\,n^3+108\,n^4\right)\sum_{\iota_1=2}^{n}\left(\frac{1}{-3+2\,\iota_1}\right)+\left(6\,n-39\,n^2+81\,n^3-54\,n^4\right)\sum_{\iota_1=2}^{n}\left(\frac{1}{-5+3\,\iota_1}\right)+\left(6\,n-39\,n^2+81\,n^3-54\,n^4\right)\sum_{\iota_1=2}^{n}\left(\frac{1}{-4+3\,\iota_1}\right)\Bigg)\Bigg)\}\}$$

Solving the recurrence with standard objects

In[47]:= $\mathbf{tower} = \left\{\left(\frac{(\mathbf{3\ n})!_{.}\ (-\mathbf{1})^{\mathbf{n}.}}{((\mathbf{n})!_{.})^{\mathbf{3}}}\right)_{\mathbf{n}.}, \left(\frac{((\mathbf{n})!_{.})^{\mathbf{3}}\ (\mathbf{6\ n})!_{.}\ (-\mathbf{1})^{\mathbf{n}.}}{((\mathbf{2\ n})!_{.})^{\mathbf{3}}\ (\mathbf{3\ n})!_{.}}\right)_{\mathbf{n}.}, \mathbf{H_n}, \mathbf{H_{2\ n}}, \mathbf{H_{3\ n}}\right\};$

In[48]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[\mathbf{n}], \mathbf{NestedSumExt} \to \infty, \mathbf{Tower} \to \mathbf{tower}]$

Out[48]= $\left\{\left\{0, \left(\frac{(n)!_{.}^{\,3}\ (6\ n)!_{.}\ (-1)^{n.}}{(2\ n)!_{.}^{\,3}\ (3\ n)!_{.}}\right)_{n.}\right\}, \left\{0, n\left(\frac{(3\ n)!_{.}\ (-1)^{n.}}{(n)!_{.}^{\,3}}\right)_{n.}\right\}, \left\{1, \frac{1}{6}\left(1 + 3\ n\ H_n + 6\ n\ H_{2\ n} - 3\ n\ H_{3\ n}\right)\left(\frac{(3\ n)!_{.}\ (-1)^{n.}}{(n)!_{.}^{\,3}}\right)_{n.}\right\}\right\}$

The closed form

In[49]:= **FindLinearCombination[recSol, mySum, 2]**

Out[49]= $$\frac{1}{6}\left(\left(1+3\,n\,H_n+6\,n\,H_{2\,n}-3\,n\,H_{3\,n}\right)\left(\frac{(3\,n)!\;(-1)^{n}}{((n)!)^{3}}\right)_{n}-\left(\frac{((n)!)^{3}\;(6\,n)!\;(-1)^{n}}{((2\,n)!)^{3}\;(3\,n)!}\right)_{n}\right)$$