# Sigma: A Summation Package for Discovering and Proving

– Symbolic Summation in Difference Fields –

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# A Bonus Problem in “Concrete Mathematics”

Chapter 6. Special Numbers, Bonus problem 69:

Find a closed form for

$$\sum_{k=1}^{n} k^2 \mathrm{H}_{n+k},$$

where $\mathrm{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

Knuth’s answer to the problem is

$$\frac{1}{3}n\big(n+\frac{1}{2}\big)(n+1)(2\mathrm{H}_{2n}-\mathrm{H}_n)-\frac{1}{36}n\left(10n^2+9n-1\right)$$

with the remark

*“It would be nice to automate the derivation of formulas such as this.”*

In[1]:= **<< Sigma‘**

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Sigma -A summation package by Carsten Schneider
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In[2]:= **Problem69 = SigmaSum[k^2 SigmaHNumber[n + k], {k, 1, n}]**

Out[2]= $\sum_{k=1}^{n}\left(k^2\ H_{k+n}\right)$

In[3]:= **SigmaReduce[Problem69]//Simplify**

Out[3]= $-\frac{1}{36}\ n\ (1+n)\ (-1+10\ n+6\ (1+2\ n)\ H_n-12\ (1+2\ n)\ H_{2n})$

- First implementation of Karr’s algorithm in a major computer algebra system

# Calkin's Identity and Variations

Find a closed form for

$$\sum_{k=0}^{n}\Bigg(\sum_{j=0}^{k}\binom{n}{j}\Bigg)^{3}$$

## Case 1:

In[4]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{a}}\Bigg(\sum_{\mathbf{j=0}}^{\mathbf{k}}(\binom{\mathbf{n}}{\mathbf{j}}^{.})\Bigg);$

In[5]:= $\mathbf{SigmaReduce[mySum]}$

Out[5]= $\frac{1}{2}\Bigg((-a+n)\binom{n}{a}^{.} + (2+2\,a-n)\sum_{\iota_1=0}^{a}(\binom{n}{\iota_1}^{.})\Bigg)$

## Case 2:

In[6]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{a}}(\Bigg(\sum_{\mathbf{j=0}}^{\mathbf{k}}(\binom{\mathbf{n}}{\mathbf{j}}^{.})\Bigg)^{\mathbf{2}});$

In[7]:= $\mathbf{SigmaReduce[mySum]}$

Out[7]= $\sum_{\iota_1=0}^{a}(\Bigg(\sum_{\iota_2=0}^{\iota_1}(\binom{n}{\iota_2}^{.})\Bigg)^{2})$

In[8]:= $\mathbf{SigmaReduce[mySum, SimplifyByExt \to Depth]}$

Out[8]= $(-a+n)\binom{n}{a}^{.}\sum_{\iota_1=0}^{a}(\binom{n}{\iota_1}^{.}) + \Big(1+a-\frac{n}{2}\Big)\Bigg(\sum_{\iota_1=0}^{a}(\binom{n}{\iota_1}^{.})\Bigg)^{2} +$

$\sum_{\iota_1=0}^{a}\Big(-\frac{1}{2}\,n\,(\binom{n}{\iota_1}^{.})^{2}\Big)$

# Case 3: (Definite Summation)

In[9]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{n}}\left(\left(\sum_{\mathbf{j=0}}^{\mathbf{k}}\left(\binom{\mathbf{n}}{\mathbf{j}}^{\cdot}\right)\right)^{\mathbf{3}}\right)$

## • Finding a recurrence

In[10]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}][[\mathbf{1}]]$

Out[10]= $-4\ (1+2\ \mathtt{n})\ \mathtt{SUM}[\mathtt{n}] - (12+7\ \mathtt{n})\ \mathtt{SUM}[1+\mathtt{n}]$

$+(1+\mathtt{n})\ \mathtt{SUM}[2+\mathtt{n}] == 2\ \left(-\ 10\ \ +9\ \mathtt{n}\ \right)\left(\sum_{\iota_1=0}^{\mathtt{n}}\left(\binom{\mathtt{n}}{\iota_1}^{\cdot}\right)\right)^{3}$

In[11]:= $\mathbf{rec} = \mathbf{rec}/.\{\sum_{\iota_1=\mathbf{0}}^{\mathbf{n}}\left(\binom{\mathbf{n}}{\iota_1}^{\cdot}\right) \to (\mathbf{2})^{\mathbf{n}.}\}$

Out[11]= $-4\ (1+2\ \mathtt{n})\ \mathtt{SUM}[\mathtt{n}] - (12+7\ \mathtt{n})\mathtt{SUM}[1+\mathtt{n}]$

$+(1+\mathtt{n})\ \mathtt{SUM}[2+\mathtt{n}] == 2\ \left(-\ 10\ ((2)^{\mathtt{n}.})^3 + 9\ \mathtt{n}\ ((2)^{\mathtt{n}.})^3\right)$

## • Solving the recurrence

In[12]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[\mathbf{n}],$

$\mathbf{Tower} \to \{\binom{\mathbf{2n}}{\mathbf{n}}^{\cdot}\}]$

Out[12]= $\left\{\left\{0, \mathtt{n}\ \binom{2\ \mathtt{n}}{\mathtt{n}}^{\cdot}\ (2)^{\mathtt{n}.}\right\}, \left\{1, \frac{1}{2}\ (2+\mathtt{n})\ ((2)^{\mathtt{n}.})^3\right\}\right\}$

## • Finding the linear combination

In[13]:= $\mathbf{FindLinearCombination}[\mathbf{recSol}, \mathbf{mySum}, \mathbf{2}]$

Out[13]= $-\frac{3}{4}\ \mathtt{n}\ \binom{2\ \mathtt{n}}{\mathtt{n}}^{\cdot}\ (2)^{\mathtt{n}.} + \frac{1}{2}\ (2+\mathtt{n})\ ((2)^{\mathtt{n}.})^3$

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3 \qquad \left\{c\,n\binom{2\,n}{n}2^n+\frac{1}{2}(2+n)\,2^{3\,n}\,|\,c\in\mathbb{Q}\right\}$$

find rec ↓      find solutions ↑

$$-4\,(1+2\,n)\,\mathrm{SUM}[n]-(12+7\,n)\mathrm{SUM}[1+n] + (1+n)\,\mathrm{SUM}[2+n]=2\,2^{3\,n}(-10+9\,n) \quad (1)$$

GOAL: Find $c\in\mathbb{Q}$ such that for all $n\geq 0$:

$$\underbrace{\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3}_{=:\,\mathrm{lhs}[n]}=\underbrace{c\,n\binom{2\,n}{n}2^n+\frac{1}{2}(2+n)\,2^{3\,n}}_{=:\,\mathrm{rhs}[n]}$$

ANSATZ: Find $c\in\mathbb{Q}$ s.t.

$$\begin{array}{l} 1=\mathrm{lhs}[0]\overset{!}{=}\mathrm{rhs}[0]=1\\ 9=\mathrm{lhs}[1]\overset{!}{=}\mathrm{rhs}[1]=12+4\,c\end{array}\quad\rightarrow c=-\frac{3}{4}$$

| |
|---|
| Any sequence fulfilling (1) is uniquely determined by the first two entries: |

$$\mathrm{SUM}[2+n]\leftarrow\frac{1}{n+1}4\,(1+2\,n)\,\mathrm{SUM}[n] + (12+7\,n)\mathrm{SUM}[1+n]+2\,2^{3\,n}(-10+9\,n)$$

Hence

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3=\underbrace{-\frac{3}{4}}_{c}\,n\binom{2\,n}{n}2^n+\frac{1}{2}(2+n)\,2^{3\,n}$$

# Calkin's Identity and Variations

**Case 1:**

$$\sum_{k=0}^{a} x^k \sum_{j=0}^{k} \binom{n}{j} y^j = \frac{x^{a+1} \sum_{j=0}^{a} \binom{n}{j} y^j - \sum_{j=0}^{a} \binom{n}{j} x^j y^j}{x-1}$$

specializes to:

$$\sum_{k=0}^{n} x^k \sum_{j=0}^{k} y^k \binom{n}{j} = \frac{x^{n+1} (1+y)^n - (1+x\,y)^n}{x-1}$$

**Case 2, non-alternating:**

$$\sum_{k=0}^{a} \left( \sum_{j=0}^{k} \binom{n}{j} \right)^2 = (n-a) \binom{n}{a} \sum_{j=0}^{a} \binom{n}{j} + (1+a-\frac{n}{2}) \left( \sum_{j=0}^{a} \binom{n}{j} \right)^2 - \frac{n}{2}, \sum_{j=0}^{a} \binom{n}{j}^2$$

specializes to:

$$\sum_{k=0}^{n} \left( \sum_{j=0}^{k} \binom{n}{j} \right)^2 = (n+1)\,4^n - \frac{n}{2}\,4^n - \frac{n}{2} \binom{2\,n}{n}$$

**Case 2, alternating:**

$$\sum_{k=0}^{a}(-1)^k \left(\sum_{j=0}^{k}\binom{n}{j}\right)^2 = 2\,(n-a)\binom{n}{a}(-1)^a \sum_{j=0}^{a}\binom{n}{j} + n\,(-1)^a \left(\sum_{j=0}^{a}\binom{n}{j}\right)^2 - \sum_{j=0}^{a}(n-2\,j)\binom{n}{j}^2 (-1)^j$$

specializes to:

$$\sum_{k=0}^{n}(-1)^k \left(\sum_{j=0}^{k}\binom{n}{j}\right)^2 = \begin{cases} 0 & \text{if } n \text{ is even} \\ -(-1)^{\frac{n-1}{2}}\, n\binom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

**Case 2 for even $n$, interlaced alternating:**

$$\sum_{k=0}^{2\,n}\left(\sum_{j=0}^{k}(-1)^{\frac{1}{2}(j-1)\,j}\binom{2\,n}{j}\right)^2 = \frac{2^{2\,n}}{4}\left(4 + 6\,n - 4\,n\,(-1)^n + 3\,n\sum_{j=2}^{n}\frac{\binom{4\,j}{2\,j}}{(4\,j-3)\,2^{2\,j}} + 3\,n\sum_{j=2}^{n}\frac{\binom{4\,j}{2\,j}}{(4\,j-1)\,2^{2\,j}}\right)$$

**Case 3, Calkin's identity:**

$$\boxed{\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3 = \frac{n}{2}\,8^n + 8^n - \frac{3\,n}{4}\,2^n\binom{2\,n}{n}}$$

**Case 3 for even $n$, alternating:**

$$\sum_{k=0}^{2\,n}(-1)^k\left(\sum_{j=0}^{k}\binom{2\,n}{j}\right)^3 = \frac{64^n}{2} - \frac{(-1)^n}{16\,n}\frac{64^n}{\binom{2\,n}{n}}\sum_{i=0}^{n-1}(3+11\,i)\binom{2\,i}{i}^2\binom{3\,i}{i}64^{-i}$$

# Definite Summation

GOAL: Find a closed form for

$$\sum_{k=1}^{n} \left( \frac{H_k \, (3+k+n)! \, (-1)^{k} \, (-1)^{-1+n}}{(1+k)! \, (2+k)! \, (-k+n)!} \right) - \frac{(n)!}{(3+n)!} \sum_{k=1}^{n} \left( \frac{(3+k+n)! \, (-1)^{k} \, (1-(2+n)\,(-1)^{n})}{k \, (1+k)!^{2} \, (-k+n)!} \right)$$

(The number of rhombus tilings of a symmetric hexagon, Fulmek & Krattenthaler)

In[14]:= $$\mathbf{mySum1} = \sum_{k=1}^{n} \left( \frac{H_k\ (3+k+n)!\ (-1)^{k}\ (-1)^{-1+n}}{(1+k)!\ (2+k)!\ (-k+n)!} \right);$$

## Finding a recurrence

In[15]:= $\mathbf{rec1 = GenerateRecurrence[mySum1][[1]]}$

Out[15]= $$n\ (1+n)\ (2+n)\ (3+n)\ (4+n)\ (-1+n)!\ \Big( -(9+2\ n)\ (8+6\ n+n^2)\ \mathrm{SUM}[n] + (9+2\ n)\ (13+8\ n+n^2)\ \mathrm{SUM}[1+n] + (30+42\ n+17\ n^2+2\ n^3)\ \mathrm{SUM}[2+n] - (3+n)\ (25+15\ n+2\ n^2)\ \mathrm{SUM}[3+n] \Big) == 2\ (-1)^n\ (9+2\ n)\ (35+24\ n+4\ n^2)\ (4+n)!$$

## Solving the recurrence

In[16]:= $\mathbf{recSol1 = SolveRecurrence[rec1, SUM[n], Tower \to \{H_n\}]}$

Out[16]= $$\Big\{\{0,1\}, \Big\{0, \frac{3-n^2+4\ H_n+6\ n\ H_n+2\ n^2\ H_n}{(1+n)\ (2+n)}\Big\}, \Big\{0, \frac{1}{4}\ (2+n)\ (-1)^{n}\Big\}, \Big\{1, \frac{(16-13\ n^2-5\ n^3+32\ H_n+64\ n\ H_n+40\ n^2\ H_n+8\ n^3\ H_n)\ (-1)^{n}}{4\ (1+n)\ (2+n)}\Big\}\Big\}$$

## Finding the linear combination

In[17]:= $\mathbf{solution1 = FindLinearCombination[recSol1, mySum1, 3]}$

Out[17]= $$-1 - \frac{3-n^2+4\ H_n+6\ n\ H_n+2\ n^2\ H_n}{(1+n)\ (2+n)} + \frac{1}{4}\ (2+n)\ (-1)^{n} + \frac{(16-13\ n^2-5\ n^3+32\ H_n+64\ n\ H_n+40\ n^2\ H_n+8\ n^3\ H_n)\ (-1)^{n}}{4\ (1+n)\ (2+n)}$$

In[18]:= $\mathbf{mySum2} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \left( \frac{(\mathbf{3+k+n})!\ (-\mathbf{1})^{\mathbf{k}}\ (\mathbf{1}-(\mathbf{2+n})\ (-\mathbf{1})^{\mathbf{n}})}{\mathbf{k}\ (\mathbf{1+k})!^{\mathbf{2}}\ (-\mathbf{k+n})!} \right);$

## Finding a recurrence

In[19]:= **rec2 = GenerateRecurrence[mySum2, RecOrder → 2][[1]]**

Out[19]= $-n\ (1+n)\ (3+n)\ (1+3\ (-1)^n+(-1)^n\ n)\ (-1+4\ (-1)^n+(-1)^n\ n)\ (28+15\ n+2\ n^2)\ (-1+n)!\ \mathtt{SUM}[n]+6\ n\ (1+n)\ (3+n)^2\ (-1+2\ (-1)^n+(-1)^n\ n)\ (-1+4\ (-1)^n+(-1)^n\ n)\ (-1+n)!\ \mathtt{SUM}[1+n]+n\ (1+n)\ (3+n)\ (-1+2\ (-1)^n+(-1)^n\ n)\ (1+3\ (-1)^n+(-1)^n\ n)\ (10+9\ n+2\ n^2)\ (-1+n)!\ \mathtt{SUM}[2+n] == 2\ (-1+2\ (-1)^n+(-1)^n\ n)\ (1+3\ (-1)^n+(-1)^n\ n)\ (-1+4\ (-1)^n+(-1)^n\ n)\ (35+24\ n+4\ n^2)\ (4+n)!$

## Solving the recurrence

$((-1)^k)^2 = 1$

In[20]:= **recSol2 = SolveRecurrence[rec2, SUM[n], Tower → {$H_n$}, WithMinusPower → True]**

Out[20]= $\{\{0, 2+n-(-1)^n\}, \{0, 16-6\ n^2-n^3+(-1)^n+28\ n\ (-1)^n+23\ n^2\ (-1)^n+8\ n^3\ (-1)^n+n^4\ (-1)^n\}, \{1, -\frac{1}{28}\ (260-150\ n^2-39\ n^3+336\ H_n+616\ n\ H_n+336\ n^2\ H_n+56\ n^3\ H_n-325\ (-1)^n+365\ n^2\ (-1)^n+228\ n^3\ (-1)^n+39\ n^4\ (-1)^n-672\ H_n\ (-1)^n-1568\ n\ H_n\ (-1)^n-1288\ n^2\ H_n\ (-1)^n-448\ n^3\ H_n\ (-1)^n-56\ n^4\ H_n\ (-1)^n)\}\}$

## Finding the linear combination

In[21]:= **solution2 = FindLinearCombination[recSol2, mySum2, 2]**

Out[21]= $(3+n)\ \left(-1+3\ n+2\ n^2-\left(-1+6\ n+7\ n^2+2\ n^3\right)\ (-1)^n+2\ \left(2+3\ n+n^2\right)\ H_n\ (-1+(2+n)\ (-1)^n)\right)$

In[22]:= $\mathbf{solution1 - solution2/((n+1)(n+2)(n+3))//Simplify}$

Out[22]= $-2 + (2 + \mathtt{n})\,(-1)^{\mathtt{n}}.$

# Indefinite Summation in Difference Field

Goal: Find a closed form for

$$\sum_{k=0}^{n} k\, k!$$

## A Difference Field for the Problem

Let $t_1, t_2$ be indeterminates where

$$\begin{array}{rcl} t_1 & \longleftrightarrow & k \\ t_2 & \longleftrightarrow & k! \end{array}$$

Consider the **field automorphism** $\sigma : \mathbb{Q}(t_1, t_2) \to \mathbb{Q}(t_1, t_2)$ canonically defined by

$$\begin{array}{rcl} \sigma(c) = c \quad \forall c \in \mathbb{Q} & & \\ \sigma(t_1) = t_1 + 1 & \qquad & \mathrm{S}\, k = k + 1 \\ \sigma(t_2) = (t_1 + 1)t_2 & \qquad & \mathrm{S}\, k! = (k + 1)! \end{array}$$

$(\mathbb{Q}(t_1, t_2), \sigma)$ is our difference field.

## The Telescoping Problem

Find $g \in \mathbb{Q}(t_1, t_2)$ : $\boxed{\sigma(g) - g = t_1\, t_2}$

$$\downarrow \text{ by Karr}$$

$$g = t_2.$$

## The Closed Form

$$\boxed{(k + 1)! - k! = k\, k!}$$

$$\downarrow$$

$$\sum_{k=0}^{n} k\, k! = (n + 1)! - 1.$$

# A Simple Example

In[23]:= $\mathbf{mySum} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \left(\mathbf{H_k} \binom{\mathbf{n}}{\mathbf{k}}\right);$

## Finding a recurrence

In[24]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}]$

Out[24]= $\{4\ (1+n)\ \mathtt{SUM}[n]-2\ (3+2\ n)\ \mathtt{SUM}[1+n]+(2+n)\ \mathtt{SUM}[2+n] == 1\}$

## Solving the recurrence

In[25]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}[[\mathbf{1}]], \mathbf{SUM}[\mathbf{n}], \mathbf{Tower} \to \{\mathbf{SigmaPower}[\mathbf{2}, \mathbf{n}]\}, \mathbf{NestedSumExt} \to \mathbf{2}]$

Out[25]= $\left\{\{0, (2)^{n}\}, \left\{0, (2)^{n} \sum_{\iota_1=1}^{n} \left(\frac{1}{\iota_1}\right)\right\}, \left\{1, -(2)^{n} \sum_{\iota_1=1}^{n} \left(\frac{1}{\iota_1\ (2)^{\iota_1}}\right)\right\}\right\}$

## Finding the linear combination

In[26]:= $\mathbf{FindLinearCombination}[\mathbf{recSol}, \mathbf{mySum}, \mathbf{2}]$

Out[26]= $(2)^{n} \sum_{\iota_1=1}^{n} \left(\frac{1}{\iota_1}\right) - (2)^{n} \sum_{\iota_1=1}^{n} \left(\frac{1}{\iota_1\ (2)^{\iota_1}}\right)$

**Where is a proof?**

# Z's Creative Telescoping Trick

- GIVEN

$$\mathrm{SUM}(n) := \sum_{k=1}^{n} \underbrace{\mathrm{H}_k \binom{n}{k}}_{=:f(n,k)}$$

- FIND $c_0(n)$, $c_1(n)$, $c_2(n)$ and $g(n,k)$ s.t.

$$\boxed{g(n,k+1) - g(n,k)}$$
$$= \boxed{c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $1 \leq k \leq n$ and all $n \geq 1$

Sigma computes:

$$c_1(n) := 4\,(1+n), \quad c_2(n) := -2\,(3+2\,n), \quad c_3(n) := 2+n$$

and

$$g(n,k) := \frac{(1+n)\,(-2+k-n+(2\,k-2\,k^2+k\,n)\,\mathrm{H}_k)\,\binom{n}{k}}{(1-k+n)\,(2-k+n)}$$

Summing this equation over $k$ from 1 to $n$ gives:

$$\boxed{g(n,n+1) - g(n,1)} =$$

$$\boxed{\begin{array}{l} c_0(n)\,\mathrm{SUM}(n) + \\ c_1(n)\,\big(\mathrm{SUM}(n+1) - f(n+1,n+1)\big) + \\ c_2(n)\,\big(\mathrm{SUM}(n+2) - f(n+2,n+1) - f(n+2,n+2)\big) \end{array}}$$

# Difference Equations and Symbolic Summation

Let $(\mathbb{F}, \sigma)$ be a difference field and

$$\mathbb{K} = \{k \in \mathbb{F} \,|\, \sigma(k) = k\}$$

be the constant field. Assume $\mathbb{Q} \subseteq \mathbb{K}$.

Telescoping

- GIVEN $f \in \mathbb{F}$
- FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$$\downarrow \qquad \qquad \uparrow$$

Parameterized Telescoping

- GIVEN $f_0, \ldots, f_d \in \mathbb{F}$, $a_0, a_1 \in \mathbb{F}$
- FIND ALL $c_0, \ldots, c_d \in \mathbb{K}$, $h \in \mathbb{F}$:

$$\boxed{a_1\, \sigma(h) - a_0\, h = c_0\, f_0 + \cdots + c_d\, f_d}$$

Remark: Z's "Creative Telescoping"

- GIVEN $f_i = \text{summand}(n+i, k) \in \mathbb{F}$
- FIND ALL $c_0, \ldots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0\, f_0 + \cdots + c_d\, f_d}$$

## Linear Difference Equations

- GIVEN $f, a_0, \dots, a_m \in \mathbb{F}$
- FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

## Parameterized Linear Difference Equations

- GIVEN $a_0, \dots, a_m \in \mathbb{F}$,$f_0, \dots, f_d \in \mathbb{F}$.
- FIND ALL $g \in \mathbb{F}$, $c_0, \dots, c_d \in \mathbb{K}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = c_0\,f_0 + \cdots + c_d\,f_d}$$

# My Results

- Streamlining of Karr's ideas result in a simpler algorithm
- Generalization of Karr's algorithm:

  first order $\longrightarrow$ $m$-th order

- New connections:

  indefinite-$\Sigma$ $\longleftrightarrow$ definite-$\Sigma$

# Sum Extensions for Indefinite Summation

In[27]:= $\mathbf{mySum} = \sum_{\iota_1=1}^{\mathbf{N}} \left( \frac{\sum_{\iota_2=1}^{\iota_1} \left( \frac{\sum_{\iota_3=1}^{\iota_2} \left( \frac{1}{\mathbf{K}+\iota_3} \right)}{\mathbf{K}+\iota_2} \right)}{\mathbf{K}+\iota_1} \right);$

In[28]:= $\mathbf{SigmaReduce[mySum]}$

Out[28]= $\sum_{\iota_1=1}^{N} \left( \frac{\sum_{\iota_2=1}^{\iota_1} \left( \frac{\sum_{\iota_3=1}^{\iota_2} \left( \frac{1}{K+\iota_3} \right)}{K+\iota_2} \right)}{K+\iota_1} \right);$

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In[29]:= $\mathbf{SigmaReduce[mySum, SimplifyByExt \to Depth]}$

Out[29]= $\frac{1}{6K^2}\left(6\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right) + 6K\left(\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)^2 + K^2\left(\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)^3 + \left(-3-3K\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)\boxed{\sum_{\iota_1=1}^{N}\left(\frac{K+2\iota_1}{(K+\iota_1)^2}\right)} - K\boxed{\sum_{\iota_1=1}^{N}\left(\frac{K+3\iota_1}{(K+\iota_1)^3}\right)}\right)$

Partial fraction decomposition:

$$\boxed{\frac{K+2i}{(K+i)^2}} = -\frac{K}{(K+i)^2} + \frac{2}{K+i}, \quad \boxed{\frac{K+3i}{(K+i)^2}} = -\frac{2K}{(K+i)^3} + \frac{3}{(K+i)^2}$$

In[30]:= $\mathbf{SigmaReduce[mySum, Tower \to \{\{H_{K+N}, N\}, \{H^{(2)}_{K+N}, N\}, \{H^{(3)}_{K+N}, N\}\}]}$

Out[30]= $\frac{1}{6}\left(-H_K^3 - 3H_K H_{K+N}^2 + H_{K+N}^3 + 3H_K H_K^{(2)} - 3H_K H_{K+N}^{(2)} + H_{K+N}\left(3H_K^2 - 3H_K^{(2)} + 3H_{K+N}^{(2)}\right) - 2H_K^{(3)} + 2H_{K+N}^{(3)}\right)$

## Sum Extensions in the Difference Field Setting

$$\sum_{\iota_1=1}^{\mathtt{N}}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{\mathtt{K}+\iota_3}\right)}{\mathtt{K}+\iota_2}\right)}{\mathtt{K}+\iota_1}\right)$$

The underlying difference field $(\mathbb{Q}(t_1)(t_2)(t_3)(t_4), \sigma)$:

$$\sigma(t_1) = t_1 + 1$$
$$\sigma(t_2) = t_2 + \frac{1}{K + t_1 + 1}$$
$$\sigma(t_3) = t_3 + \sigma(\frac{t_2}{K + t_1})$$
$$\sigma(t_4) = t_4 + \sigma(\frac{t_3}{K + t_1})$$

$$\frac{1}{6}\Big(-\mathtt{H}_{\mathtt{K}}^{3}-3\,\mathtt{H}_{\mathtt{K}}\,\mathtt{H}_{\mathtt{K+N}}^{2}+\mathtt{H}_{\mathtt{K+N}}^{3}+3\,\mathtt{H}_{\mathtt{K}}\,\mathtt{H}_{\mathtt{K}}^{(2)}-3\,\mathtt{H}_{\mathtt{K}}\,\mathtt{H}_{\mathtt{K+N}}^{(2)}$$
$$+\mathtt{H}_{\mathtt{K+N}}\Big(3\,\mathtt{H}_{\mathtt{K}}^{2}-3\,\mathtt{H}_{\mathtt{K}}^{(2)}+3\,\mathtt{H}_{\mathtt{K+N}}^{(2)}\Big)-2\,\mathtt{H}_{\mathtt{K}}^{(3)}+2\,\mathtt{H}_{\mathtt{K+N}}^{(3)}\Big)$$

The underlying difference field $(\mathbb{Q}(t_1)(t_2)(t'_3)(t'_4),\sigma)$:

$$\sigma(t_1)=t_1+1$$
$$\sigma(t_2)=t_2+\frac{1}{K+t_1+1}$$
$$\sigma(t'_3)=t'_3+\frac{1}{(K+t_1+1)^2}$$
$$\sigma(t'_4)=t'_4+\frac{1}{(K+t_1+1)^3}$$

$$\boxed{(\mathbb{Q}(t_1)(t_2)(t_3)(t_4),\sigma)\simeq(\mathbb{Q}(t_1)(t_2)(t'_3)(t'_4),\sigma)}$$

# Sum Extensions for Recurrences

In[31]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{N}} \left( \frac{\binom{\mathbf{N}}{\mathbf{k}} (-\mathbf{1})^{\mathbf{k}}}{(\mathbf{k}+\mathbf{1})^{\mathbf{4}}} \right);$

## Finding a recurrence

In[32]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}]$

Out[32]= $\{(1+N)\ (2+N)\ (3+N)\ (4+N)\ \mathtt{SUM}[N] - 3\ (2+N)\ (3+N)^2\ (4+N)\ \mathtt{SUM}[1+N] + (3+N)\ (4+N)\ (37+21\ N+3\ N^2)\ \mathtt{SUM}[2+N] - (4+N)^4\ \mathtt{SUM}[3+N] == -1\}$

## Solving the recurrence (A First Attempt)

In[33]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}[[\mathbf{1}]], \mathbf{SUM}[\mathbf{N}]]$

Out[33]= $\{\{0, \frac{1}{1+N}\}\}$

The underlying difference field is too small!

## Solving the recurrence (Step I)

In[34]:= $\textbf{recSol} = \textbf{SolveRecurrence}[\textbf{rec}[[\textbf{1}]], \textbf{SUM}[\textbf{N}], \textbf{NestedSumExt} \to \infty]$

$$\text{Out[34]=} \left\{\left\{0, \frac{1}{1+N}\right\}, \left\{0, \frac{\sum_{\iota_1=1}^{N}\left(\frac{1}{1+\iota_1}\right)}{1+N}\right\}, \left\{0, \frac{\sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{1+\iota_2}\right)}{1+\iota_1}\right)}{1+N}\right\}, \left\{1, \frac{\sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{1+\iota_3}\right)}{1+\iota_2}\right)}{1+\iota_1}\right)}{1+N}\right\}\right\}$$

- Inspired by Abramov/Petkovšek and Hendrik/Singer
- My theoretical result:

  We can find <u>all</u> sum extensions over a given **ΠΣ-field** which give more solutions of a homogeneous or **inhomogeneous** recurrence!
- Speed up in computation.
- Further <u>simplification</u> by my **indefinite summation algorithm**

We know:

$$\sum_{i=1}^{N}\frac{\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{K+k}}{K+j}}{K+i} = 3\,\mathrm{H}_K\,\mathrm{H}_{K+N}^{(2)}+\mathrm{H}_{K+N}\left(3\,\mathrm{H}_K^2-3\,\mathrm{H}_K^{(2)}+3\,\mathrm{H}_{K+N}^{(2)}\right)-2\,\mathrm{H}_K^{(3)}+2\,\mathrm{H}_{K+N}^{(3)}\Big)$$

## Solving the recurrence (Step II)

In[35]:= **recSol = SolveRecurrence[rec[[1]], SUM[N], Tower → {H_N, H_N^(2), H_N^(3)}]**

$$\text{Out[35]=}\ \Big\{\Big\{0,\frac{1}{(1+N)^3}\left(2+2\,H_N+2\,N\,H_N+H_N^2+2\,N\,H_N^2+N^2\,H_N^2+H_N^{(2)}+2\,N\,H_N^{(2)}+N^2\,H_N^{(2)}\right)\Big\},$$

$$\Big\{0,\frac{1}{(1+N)^3}\left(-4\,N-2\,N^2+2\,H_N+2\,N\,H_N+H_N^2+2\,N\,H_N^2+N^2\,H_N^2+H_N^{(2)}+2\,N\,H_N^{(2)}+N^2\,H_N^{(2)}\right)\Big\},$$

$$\Big\{0,\frac{1}{(1+N)^3}\left(-N+N^2-H_N-4\,N\,H_N-3\,N^2\,H_N+H_N^2+2\,N\,H_N^2+N^2\,H_N^2+H_N^{(2)}+2\,N\,H_N^{(2)}+N^2\,H_N^{(2)}\right)\Big\},$$

$$\Big\{1,\frac{1}{6\,(1+N)^4}\Big(-6\,N-6\,N\,H_N-6\,N^2\,H_N-3\,N\,H_N^2-6\,N^2\,H_N^2-3\,N^3\,H_N^2+H_N^3+3\,N\,H_N^3+3\,N^2\,H_N^3+N^3\,H_N^3-3\,N\,H_N^{(2)}-6\,N^2\,H_N^{(2)}-3\,N^3\,H_N^{(2)}+3\,H_N\,H_N^{(2)}+9\,N\,H_N\,H_N^{(2)}+9\,N^2\,H_N\,H_N^{(2)}+3\,N^3\,H_N\,H_N^{(2)}+2\,H_N^{(3)}+6\,N\,H_N^{(3)}+6\,N^2\,H_N^{(3)}+2\,N^3\,H_N^{(3)}\Big)\Big\}\Big\}$$

## Finding the linear combination

In[36]:= **FindLinearCombination[recSol, defSum, 3]//Simplify**

$$\text{Out[36]=}\ \frac{1}{6\,(1+N)^4}\Big(3\,(1+N)^2\,H_N^2+(1+N)^3\,H_N^3+3\,(1+N)^2\,H_N^{(2)}+3\,(1+N)\,H_N\left(2+(1+N)^2\,H_N^{(2)}\right)+2\left(3+H_N^{(3)}+3\,N\,H_N^{(3)}+3\,N^2\,H_N^{(3)}+N^3\,H_N^{(3)}\right)\Big)$$

# A Series of Identities (S. Ahlgren)

PROBLEM: Find a closed form for

$$\sum_{j=0}^{n}(1 - a\ j\ H_j + a\ j\ H_{-j+n})\binom{n}{j}^a,\ a \geq 1$$

$$\sum_{j=0}^{n}(1 - j\,\mathrm{H}_j + j\,\mathrm{H}_{n-j})\binom{n}{j} = 1$$

$$\sum_{j=0}^{n}(1 - 2\,j\,\mathrm{H}_j + 2\,j\,\mathrm{H}_{n-j})\binom{n}{j}^2 = 0$$

$$\sum_{j=0}^{n}(1 - 3\,j\,\mathrm{H}_j + 3\,j\,\mathrm{H}_{n-j})\binom{n}{j}^3 = (-1)^n$$

$$\sum_{j=0}^{n}(1 - 4\,j\,\mathrm{H}_j + 4\,j\,\mathrm{H}_{n-j})\binom{n}{j}^4 = (-1)^n\sum_{j=0}^{n}\binom{n}{j}^2 = (-1)^n\binom{2\,n}{n}$$

$$\sum_{j=0}^{n}(1 - 5\,j\,\mathrm{H}_j + 5\,j\,\mathrm{H}_{n-j})\binom{n}{j}^5 = (-1)^n\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{j}$$

# - Case 4 -

## Naive summation

In[37]:= $\mathbf{mySum4} = \sum_{\mathbf{j=0}}^{\mathbf{n}}(\mathbf{1 - 4\ j\ H_j + 4\ j\ H_{-j+n}})\,(((\binom{\mathbf{n}}{\mathbf{j}}))^{\mathbf{.\ 4}})_{\mathbf{j.}}\ ;$

In[38]:= **GenerateRecurrence[mySum4]**

Out[38]= $\{-8\ (1+n)\ (1+2\ n)\ (3+4\ n)\ (5+4\ n)\ (129+193\ n+94\ n^2+15\ n^3)\ \mathrm{SUM}[n] - 4\ (7560+39369\ n+82597\ n^2+92434\ n^3+60256\ n^4+23024\ n^5+4792\ n^6+420\ n^7)\ \mathrm{SUM}[1+n] - 2\ (2+n)\ (1425+6187\ n+9949\ n^2+7891\ n^3+3314\ n^4+706\ n^5+60\ n^6)\ \mathrm{SUM}[2+n] + (2+n)^2\ (3+n)^2\ (15+50\ n+49\ n^2+15\ n^3)\ \mathrm{SUM}[3+n] == 0\}$

## Creative summation

By

$$\sum_{j=0}^{n} j\,\mathrm{H}_{n-j}\binom{n}{j}^4 = \sum_{j=0}^{n}(n-j)\,\mathrm{H}_j\binom{n}{n-j}^4 = \sum_{j=0}^{n}(n-j)\,\mathrm{H}_j\binom{n}{j}^4,$$

we obtain

$$\sum_{j=0}^{n}(1-4\,j\,\mathrm{H}_j+4\,j\,\mathrm{H}_{n-j})\binom{n}{j}^4 = \sum_{j=0}^{n}(1-4\,(n-2\,j)\mathrm{H}_j)\binom{n}{j}^4.$$

In[39]:= $\mathbf{mySum4} = \sum_{\mathbf{j=0}}^{\mathbf{n}}(\mathbf{1} - \mathbf{4\ j\ H_j} + \mathbf{4}\ (-\mathbf{j}+\mathbf{n})\ \mathbf{H_j})\ ((\binom{\mathbf{n}}{\mathbf{j}})^{\mathbf{4}})_{\mathbf{j}}$;

In[40]:= **GenerateRecurrence[mySum4]**

```
Order:  1
Order:  2
Solution!
```

Out[40]= $\{4\ (1+2\ \mathtt{n})^2\ (11+8\ \mathtt{n})\ \mathtt{SUM[n]} + 2\ (29+110\ \mathtt{n}+108\ \mathtt{n}^2+32\ \mathtt{n}^3)\ \mathtt{SUM}[1+\mathtt{n}] + (2+\mathtt{n})^2\ (3+8\ \mathtt{n})\ \mathtt{SUM}[2+\mathtt{n}] == 0\}$

# A Recurrence with minimal order

In[41]:= **rec = GenerateRecurrence[mySum4,**
**SimplifyByExt → DepthNumber]**

```
Order:  1
Solution!
```

Out[41]= $\{2\,(1+n)\,(1+2\,n)\,\mathrm{SUM}[n]+(1+n)^2\,\mathrm{SUM}[1+n]==$

$$2\left(n\,(3+8\,n)+(-3-8\,n)\,\sum_{\iota_1=0}^{n}\frac{(2+n-2\,\iota_1)\,\iota_1^4\,\left(\left(\binom{n}{\iota_1}\right)\right)^4}{(1+n-\iota_1)^4}\right)\}$$

The rhs vanishes (computer proof!), hence

$$2\,(1+n)\,(1+2\,n)\,\mathbf{SUM}[n]+(1+n)^2\,\mathbf{SUM}[1+n]=0.$$

Remember:

$$\sum_{j=0}^{n}(1-4\,j\,\mathrm{H}_j+4\,j\,\mathrm{H}_{n-j})\binom{n}{j}^4=(-1)^n\binom{2\,n}{n}$$

# Challenging Sums (A. Weideman)

In[42]:= $\mathbf{Sum3a} = \sum_{k=0}^{m} \left(3\ (-H_k + H_{-k+m})^2 + H_k^{(2)} + H_{-k+m}^{(2)}\right) \left(\left(\binom{m}{k}\right)^3 (-1)^k\right)_k$;

In[43]:= GenerateRecurrence[Sum3a, RecOrder− > 2]

```
50.42 Second
```

Out[43]= $\{3\ (2 + 3\ m)\ (4 + 3\ m)\ \mathrm{SUM}[m] + (2 + m)^2\ \mathrm{SUM}[2 + m] == 0\}$

In[44]:= $\mathbf{Sum3b} = \sum_{k=0}^{n} 3\ k\ H_k^2 + 2\ H_{-k+n} - 2\ H_k\ (1 + 3\ k\ H_{-k+n}) + k\left(3\ H_{-k+n}^2 + H_k^{(2)} + H_{-k+n}^{(2)}\right) \left(\left(\binom{n}{k}\right)^3 (-1)^k\right)_k$;

In[45]:= GenerateRecurrence[Sum3b]

```
87.33 Second
```

Out[45]= $\{3\ (1 + 3\ n)\ (2 + 3\ n)\ (11 + 16\ n + 6\ n^2)\ \mathrm{SUM}[n] - 6\ (2 + 6\ n + 3\ n^2)\ \mathrm{SUM}[1 + n] + (1 + n)\ (2 + n)\ (1 + 4\ n + 6\ n^2)\ \mathrm{SUM}[2 + n] == 0\}$

In[46]:= $\mathbf{Sum4} = \sum_{k=0}^{m} \left(3\ \left(4\ (H_k - H_{-k+m})^2 + H_k^{(2)} + H_{-k+m}^{(2)}\right) + 2\ k\ \left(8\ (-H_k + H_{-k+m})^3 + 6\ (-H_k + H_{-k+m})\left(H_k^{(2)} + H_{-k+m}^{(2)}\right) - H_k^{(3)} + H_{-k+m}^{(3)}\right)\right) \left(\left(\binom{m}{k}\right)^4\right)_k$;

In[47]:= GenerateRecurrence[Sum4, RecOrder− > 3]

```
1190.07 Second
```

Out[47]= $\{-8\ (1 + m)\ (1 + 2\ m)\ (3 + 4\ m)\ (5 + 4\ m)\ (129 + 193\ m + 94\ m^2 + 15\ m^3)\ \mathrm{SUM}[m] - 4\ (7560 + 39369\ m + 82597\ m^2 + 92434\ m^3 + 60256\ m^4 + 23024\ m^5 + 4792\ m^6 + 420\ m^7)\ \mathrm{SUM}[1 + m] - 2\ (2 + m)\ (1425 + 6187\ m + 9949\ m^2 + 7891\ m^3 + 3314\ m^4 + 706\ m^5 + 60\ m^6)\ \mathrm{SUM}[2 + m] + (2 + m)^2\ (3 + m)^2\ (15 + 50\ m + 49\ m^2 + 15\ m^3)\ \mathrm{SUM}[3 + m] == 0\}$