AMS 2005

Algebraic Combinatorics

16.–19. June, Mainz

When is 0.999... equal to 1?

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From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously like it comes out to exactly 1. Is there a way I can automatically decide this? The sum may be written in many ways, but one is:

$$\sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}; \qquad H_j := \sum_{i=1}^{j} \frac{1}{i}.$$

Of course you can expand out the H's and get a quadruple sum. There are zillions of ways to play with it, summing by parts, but I have never managed to get rid of all the summations.

Robin

From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC: Carsten Schneider

Robin and Herb,

I am willing to bet that Castern Schneider's SIGMA package for handling sums with harmonic numbers (among others) can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the answer to me.

-Doron

Dear Doron,

Finally, I managed to compute the limit of the sum in a jiffy.

According to my computations the sum

$$\boxed{S := \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}}$$

is not 1!
More precisely, with my Sigma package I obtain as its value

$$-4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5) \simeq 0.9992228377638300876$$

where $\zeta(r) = \sum_{i=1}^{\infty} \frac{1}{i^r}$.

Take the truncated version: $$S(a,b)=\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{a}\frac{H_j}{j(j+k)},$$

i.e.,

$$\lim_{a,b\to\infty} S(a,b)=S.$$

Sigma simplifies the inner sum to

$$\sum_{j=1}^{a}\frac{H_j}{j(j+k)}=\frac{kH_k^2-2H_k+kH_k^{(2)}+2kH_a^{(2)}}{2k^2}-\frac{(kH_a-1)}{k^2}\sum_{i=1}^{k}\frac{1}{a+i}-\frac{1}{k}\sum_{i=1}^{k}\frac{1}{i}\sum_{j=1}^{i}\frac{1}{a+j}$$

where $H_k^{(r)}=\sum_{i=1}^{k}\frac{1}{i^r}$.

Hence, for

$$\boxed{S'(a,b) := \sum_{k=1}^{b} \frac{H_{k+1}-1}{k(k+1)} \frac{kH_k^2 - 2H_k + kH_k^{(2)} + 2kH_a^{(2)}}{2k^2}},$$

we have

$$\lim_{a,b\to\infty} S'(a,b) = S.$$

Sigma simplifies $S'(a,b)$ to

$$S'(a,b) = A(a,b) + B(a,b) + C(a,b)$$

where

$$\begin{aligned} A(a,b) :=& \frac{1}{2(b+1)^2}\Big(6H_b + 4bH_b + 4H_b^2 + 3bH_b^2 + H_b^3 + bH_b^3 - 6bH_a^{(2)} \\ &+ 2H_bH_a^{(2)} + 2bH_bH_a^{(2)} - 2H_b^{(2)} - 7bH_b^{(2)} + H_bH_b^{(2)} + bH_bH_b^{(2)}\Big), \\ B(a,b) :=& -\frac{2b^2}{(b+1)^2}\Big(H_a^{(2)} + H_b^{(2)}\Big), \\ C(a,b) :=& (H_a^{(2)} - 1)\sum_{i=1}^{b}\frac{H_i}{i^2} - \sum_{i=1}^{b}\frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{b}\frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{b}\frac{H_iH_i^{(2)}}{i^2}. \end{aligned}$$

By

$$\lim_{a,b\to\infty} A(a,b) = 0 \qquad \text{and} \qquad \lim_{a,b\to\infty} B(a,b) = -4\zeta(2)$$

we get

$$S = \lim_{a,b\to\infty} S'(a,b) = -4\zeta(2) + \lim_{a,b\to\infty} C(a,b).$$

ζ-relations by [Borwein, Girgensohn] and [Flajolet, Salvy] give

$$\sum_{i=1}^{\infty} \frac{H_i}{i^2} = 2\zeta(3), \qquad \sum_{i=1}^{\infty} \frac{H_i^2}{i^3} = -\zeta(2)\zeta(3) + \frac{7}{2}\zeta(5),$$

$$\sum_{i=1}^{\infty} \frac{H_i^3}{i^2} = \zeta(2)\zeta(3) + 10\zeta(5), \qquad \sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^2} = \zeta(2)\zeta(3) + \zeta(5).$$

This shows that

$$S = \lim_{a,b\to\infty} S'(a,b) = -4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5).$$

J.M. Borwein and R. Girgensohn. Evaluation of triple Euler sums. *Electron. J. Combin.*, 3:1–27, 1996.
P. Flajolet and B. Salvy. Euler sums and contour integral representations. *Experim. Math.*, 7(1):15–35, 1998.

From: Doron Zeilberger
To: Carsten Schneider
CC: Robin Pemantle, Herbert Wilf

Wow, you (and your computer!) are wizhes!

⋮

Anyway, even though the bet was one sided, I still feel that Robin and/or Herb owe me a free lunch (and they owe Carsten, and his computer, a free dinner).

Best wishes
Doron

Find the closed form for

$$\mathrm{SUM}(a,k) = \sum_{j=1}^{a} \frac{H_j}{j(j+k)}.$$

1. Compute a recurrence (creative telescoping)

$$k^2\mathrm{SUM}(a,k) - (k+1)(2k+1)\mathrm{SUM}(a,k+1) + (k+1)(k+2)\mathrm{SUM}(a,k+2) = \frac{a(a+k+2)-(a+1)(k+1)H_a}{(k+1)(a+k+1)(a+k+2)}.$$

2. Solve the recurrence (d'Alembertian solutions)

$$h_1 = \frac{1}{k},\quad h_2 = \frac{kH_k - 1}{k^2},\quad p = \frac{kH_a - H_k - aH_k + k{H_k}^2 + ak{H_k}^2}{(1+a)k^2} - \frac{kH_a - 1}{k^2}\sum_{i=1}^{k}\frac{1}{i+a} - \frac{1}{k}\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{a+i}}{j}.$$

i.e.,

$$\text{Solution Space} = \{c_1\, h_1 + c_2\, h_2 + p \mid c_1, c_2 \in \mathbb{C}\}$$

3. Find the linear combination

$$\sum_{j=1}^{a} \frac{H_j}{j(j+k)} = \frac{-H_a + (1+a){H_a}^2}{1+a} h_1 + 0\, h_2 + p.$$

A Session with

In[1]:= $<<$ **Sigma‘**

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Sigma - A summation package by Carsten Schneider © RISC-Linz
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In[2]:= $\mathbf{mySum} = \sum_{j=1}^{a} \frac{\mathbf{H_j}}{\mathbf{j\,(j+k)}}$

1. Compute a recurrence

In[3]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}, \mathbf{k}][[\mathbf{1}]]$

Out[3]= $k^2\,(1+k)\,(1+a+k)\,(2+a+k)\,\mathrm{SUM}[k] - (1+k)^2\,(1+a+k)\,(2+a+k)\,(1+2\,k)\,\mathrm{SUM}[1+k] + (1+k)^2\,(2+k)\,(1+a+k)\,(2+a+k)\,\mathrm{SUM}[2+k] == a\,(2+a+k) + (-1-a)\,(1+k)\,\mathrm{H_a}$

2. Solve the recurrence

In[4]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[\mathbf{k}], \mathbf{NestedSumExt} \to \infty]$

Out[4]= $\Big\{\Big\{0, \frac{1}{k}\Big\}, \Big\{0, \frac{-1 + k\,\sum_{\iota_1=1}^{k}\frac{1}{\iota_1}}{k^2}\Big\}, \Big\{1, \frac{1}{2\,(1+a)\,k^2}\Big(2\,k\,\mathrm{H_a}\Big(1-(1+a)\sum_{\iota_1=1_2}^{k}\frac{1}{a+\iota_1}\Big) + (1+a)\Big(-2\sum_{\iota_1=1}^{k}\frac{1}{\iota_1} + 2\sum_{\iota_1=1}^{k}\frac{1}{a+\iota_1} + k\Big(\sum_{\iota_1=1}^{k}\frac{1}{\iota_1^2} + \Big(\sum_{\iota_1=1}^{k}\frac{1}{\iota_1}\Big) - 2\sum_{\iota_1=1}^{k}\frac{\sum_{\iota_2=1}^{\iota_1}\frac{1}{a+\iota_2}}{\iota_1}\Big)\Big)\Big)\Big\}\Big\}$

3. Find the linear combination

In[5]:= $\mathbf{FindLinearCombination}\big[\mathbf{recSol}, \mathbf{mySum}, \mathbf{2}\big]$

Out[5]= $\frac{-2\sum_{\iota_1=1}^{k}\frac{1}{\iota_1} + 2\sum_{\iota_1=1}^{k}\frac{1}{a+\iota_1} + k\Big(2\,\mathrm{H}_a^{(2)} + \sum_{\iota_1=1}^{k}\frac{1}{\iota_1^2} + \Big(\sum_{\iota_1=1}^{k}\frac{1}{\iota_1}\Big)^2 - 2\Big(\mathrm{H_a}\sum_{\iota_1=1}^{k}\frac{1}{a+\iota_1} + \sum_{\iota_1=1}^{k}\frac{\sum_{\iota_2=1}^{\iota_1}\frac{1}{a+\iota_2}}{\iota_1}\Big)\Big)}{2\,k^2}$

Zeilberger's Creative Telescoping Paradigm

- GIVEN

$$\mathrm{SUM}(k) := \sum_{j=1}^{a} \underbrace{\frac{H_j}{j(j+k)}}_{=:\, f(k,j)}$$

- FIND $c_0(k)$, $c_1(k)$, $c_2(k)$, and $g(k,j)$ s.t.

$$\boxed{g(k,j+1) - g(k,j)} = \boxed{c_0(k)\, f(k,j) + c_1(k)\, f(k+1,j) + c_2(k)\, f(k+2,j)}$$

for all $j, k \geq 1$.

Sigma computes:

$$c_0(k) := k^2,\ \ c_1(k) := -(k+1)(2k+1),\ \ c_2(k) := (k+1)(k+2),$$

$$g(k,j) := -\frac{jH_j + k + j}{(k+j)(k+j+1)},$$

$$g(k,j+1) := -\frac{(1+j)H_j + k + j + 2}{(k+j+1)(k+j+2)}.$$

Summing this equation over j from 1 to a gives:

$$\boxed{g(k, a+1) - g(k, 1)} = \boxed{c_0(k)\,\mathrm{SUM}(k) + c_1(k)\,\mathrm{SUM}(k+1) + c_2(k)\,\mathrm{SUM}(k+2).}$$