When is $0.999\ldots$ equal to 1?

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Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously like it comes out to exactly 1. Is there a way I can automatically decide this? The sum may be written in many ways, but one is:

$$\sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1} - 1)}{jk(k+1)(j+k)}; \quad H_j := \sum_{i=1}^{j} \frac{1}{i}.$$  

Of course you can expand out the H’s and get a quadruple sum. There are zillions of ways to play with it, summing by parts, but I have never managed to get rid of all the summations.

Robin
Robin and Herb,

I am willing to bet that Carsten Schneider’s SIGMA package for handling sums with harmonic numbers (among others) can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the answer to me.
-Doron
Dear Doron,

Finally, I managed to compute the limit of the sum in a jiffy.

According to my computations the sum

\[ S := \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1} - 1)}{jk(k + 1)(j + k)} \]

is not 1!

More precisely, with my Sigma package I obtain as its value

\[-4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5) \simeq 0.9992228377638300876\]

where \( \zeta(r) = \sum_{i=1}^{\infty} \frac{1}{i^r} \).
Take the truncated version: \[ S(a, b) = \sum_{k=1}^{b} \frac{H_{k+1} - 1}{k(k + 1)} \sum_{j=1}^{a} \frac{H_j}{j(j + k)}, \]
i.e.,
\[ \lim_{a,b \to \infty} S(a, b) = S. \]

{\textbf{Sigma}} simplifies the inner sum to
\[ \sum_{j=1}^{a} \frac{H_j}{j(j + k)} = \frac{kH_k^2 - 2H_k + kH_k^{(2)} + 2kH_a^{(2)}}{2k^2} - \frac{(kH_a - 1)}{k^2} \sum_{i=1}^{k} \frac{1}{a + i} - \frac{1}{k} \sum_{i=1}^{k} \frac{1}{i} \sum_{j=1}^{a} \frac{1}{a + j} \]
where \( H_k^{(r)} = \sum_{i=1}^{k} \frac{1}{i^r}. \)
Hence, for

\[ S'(a, b) := \sum_{k=1}^{b} \frac{H_{k+1} - 1}{k(k + 1)} \frac{kH_k^2 - 2H_k + kH_k^{(2)} + 2kH_a^{(2)}}{2k^2}, \]

we have

\[ \lim_{a,b \to \infty} S'(a, b) = S. \]
Sigma simplifies $S'(a, b)$ to

$$S'(a, b) = A(a, b) + B(a, b) + C(a, b)$$

where

$$A(a, b) := \frac{1}{2(b+1)^2} \left( 6H_b + 4bH_b + 4H_b^2 + 3bH_b^2 + H_b^3 + bH_b^3 - 6bH_b^{(2)} ight. \left. + 2H_bH_a^{(2)} + 2bH_bH_a^{(2)} - 2H_b^{(2)} - 7bH_b^{(2)} + H_bH_b^{(2)} + bH_bH_b^{(2)} \right),$$

$$B(a, b) := -\frac{2b^2}{(b+1)^2} \left( H_a^{(2)} + H_b^{(2)} \right),$$

$$C(a, b) := (H_a^{(2)} - 1) \left( \sum_{i=1}^{b} \frac{H_i}{i^2} - \sum_{i=1}^{b} \frac{H_i^2}{i^3} + \frac{1}{2} \sum_{i=1}^{b} \frac{H_i^3}{i^2} + \frac{1}{2} \sum_{i=1}^{b} \frac{H_i H_i^{(2)}}{i^2} \right).$$
By

\[
\lim_{a,b \to \infty} A(a, b) = 0 \quad \text{and} \quad \lim_{a,b \to \infty} B(a, b) = -4\zeta(2)
\]

we get

\[
S = \lim_{a,b \to \infty} S'(a, b) = -4\zeta(2) + \lim_{a,b \to \infty} C(a, b).
\]
ζ-relations by [Borwein, Girgensohn] and [Flajolet, Salvy] give

\[
\sum_{i=1}^{\infty} \frac{H_i}{i^2} = 2\zeta(3),
\]

\[
\sum_{i=1}^{\infty} \frac{H_i^3}{i^2} = \zeta(2)\zeta(3) + 10\zeta(5),
\]

\[
\sum_{i=1}^{\infty} \frac{H_i^2}{i^3} = -\zeta(2)\zeta(3) + \frac{7}{2}\zeta(5),
\]

\[
\sum_{i=1}^{\infty} \frac{H_iH_i^{(2)}}{i^2} = \zeta(2)\zeta(3) + \zeta(5).
\]

This shows that

\[
S = \lim_{a,b \to \infty} S''(a, b) = -4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5).
\]

From: Doron Zeilberger  
To: Carsten Schneider  
CC: Robin Pemantle, Herbert Wilf  

Wow, you (and your computer!) are wizhes!  

Anyway, even though the bet was one sided, I still feel that Robin and/or Herb owe me a free lunch (and they owe Carsten, and his computer, a free dinner).  

Best wishes  
Doron
Find the closed form for

\[ \text{SUM}(a, k) = \sum_{j=1}^{a} \frac{H_j}{j(j+k)}. \]

1. Compute a recurrence (creative telescoping)

\[ k^2 \text{SUM}(a, k) - (k+1)(2k+1) \text{SUM}(a, k+1) + (k+1)(k+2) \text{SUM}(a, k+2) = \frac{a(a + k + 2) - (a + 1)(k + 1)H_a}{(k + 1)(a + k + 1)(a + k + 2)}. \]

2. Solve the recurrence (d’Alembertian solutions)

\[
\begin{align*}
    h_1 &= \frac{1}{k}, \\
    h_2 &= \frac{kH_k - 1}{k^2}, \\
    p &= \frac{kH_a - H_k - aH_k + kH_k^2 + akH_k^2}{(1 + a)k^2} - \frac{kH_a - 1}{k^2} \sum_{i=1}^{k} \frac{1}{i + a} - \frac{1}{k} \sum_{j=1}^{j} \frac{1}{a + i}.
\end{align*}
\]

i.e.,

\[ \text{Solution Space} = \{c_1 h_1 + c_2 h_2 + p \mid c_1, c_2 \in \mathbb{C}\} \]

3. Find the linear combination

\[
\sum_{j=1}^{a} \frac{H_j}{j(j+k)} = \frac{-H_a + (1 + a)H_a^2}{1 + a} h_1 + 0 h_2 + p.
\]
A Session with

Sigma - A summation package by Carsten Schneider © RISC-Linz

\[ \text{In}[1] := \text{<<Sigma}'\]

\[ \text{In}[2] := \text{mySum} = \sum_{j=1}^{a} \frac{H_j}{j(j+k)} \]

1. Compute a recurrence

\[ \text{In}[3] := \text{rec} = \text{GenerateRecurrence}[\text{mySum}, k][[1]] \]

\[ \text{Out}[3] = k^2(1+k)(1+a+k)(2+a+k) \text{SUM}[k] - (1+k)^2(1+a+k)(2+a+k)(1+2k) \text{SUM}[1+k] + (1+k)^2(2+k)(1+a+k)(2+a+k) \text{SUM}[2+k] = a(2a+k) + (-1-a)(1+k)H_a \]

2. Solve the recurrence

\[ \text{In}[4] := \text{recSol} = \text{SolveRecurrence}[\text{rec}, \text{SUM}[k], \text{NestedSumExt} \to \infty] \]

\[ \text{Out}[4] = \{\{0, \frac{1}{k}\}, \{0, \frac{-1+k \sum_{\ell_1=1}^{k} \frac{1}{\ell_1}}{k^2}\}, \{1, \frac{1}{2(1+a)k^2}(2kH_a(1-(1+a)\sum_{\ell_1=1}^{k} \frac{1}{a+\ell_1}) + (1+a)\left(-2 \sum_{\ell_1=1}^{k} \frac{1}{\ell_1} + 2 \sum_{\ell_1=1}^{k} \frac{1}{a+\ell_1} + k\left(\sum_{\ell_1=1}^{k} \frac{1}{\ell_1^2} + \left(\sum_{\ell_1=1}^{k} \frac{1}{\ell_1}\right) - 2 \sum_{\ell_1=1}^{k} \frac{\sum_{\ell_2=1}^{k} \frac{1}{a+\ell_2}}{\ell_1}\right)\right)\})\} \]

3. Find the linear combination

\[ \text{In}[5] := \text{FindLinearCombination}[\text{recSol}, \text{mySum}, 2] \]

\[ \text{Out}[5] = \frac{-2 \sum_{\ell_1=1}^{k} \frac{1}{\ell_1} + 2 \sum_{\ell_1=1}^{k} \frac{1}{a+\ell_1} + k\left(2H_a(2) + \sum_{\ell_1=1}^{k} \frac{1}{\ell_1^2} + \left(\sum_{\ell_1=1}^{k} \frac{1}{\ell_1}\right)^2 - 2 \left(H_a \sum_{\ell_1=1}^{k} \frac{1}{a+\ell_1} + \sum_{\ell_1=1}^{k} \frac{\sum_{\ell_2=1}^{k} \frac{1}{a+\ell_2}}{\ell_1}\right)\right)}{2k^2} \]

\[=\]
• GIVEN

\[ \text{SUM}(k) := \sum_{j=1}^{a} H_j \]

\[
\frac{H_j}{j(j+k)} = f(k,j)
\]

• FIND \( c_0(k), c_1(k), c_2(k) \), and \( g(k,j) \) s.t.

\[
g(k,j+1) - g(k,j) = c_0(k) f(k,j) + c_1(k) f(k+1,j) + c_2(k) f(k+2,j)
\]

for all \( j, k \geq 1 \).
\[ c_0(k) := k^2, \ c_1(k) := -(k+1)(2k+1), \ c_2(k) := (k+1)(k+2). \]

\[ g(k, j) := \frac{jH_j + k + j}{(k+j)(k+j+1)(k+j+2)}, \]

\[ g(k, j+1) := \frac{(1+j)H_j + k + j + 2}{(k+j+1)(k+j+2)}. \]
Summing this equation over $j$ from 1 to $a$ gives:

$$g(k, a + 1) - g(k, 1) = c_0(k) \text{SUM}(k) + c_1(k) \text{SUM}(k + 1) + c_2(k) \text{SUM}(k + 2).$$