ISSAC 2005

Finding Telescopers with Minimal Depth for Indefinite Nested Sum and Product Expressions

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Reducing the depth

Karr's algorithm (1981):

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{H_j}{j} = n-(n+1)H_n+(n+1)\sum_{j=1}^{n}\frac{H_j}{j}, \qquad H_j=\sum_{i=1}^{j}\frac{1}{i}.$$

3–nested sum $\longrightarrow$ 2–nested sum expression

ISSAC'05:

$$\sum_{k=1}^{n}\frac{1}{k^3}\sum_{j=1}^{k}\frac{H_j}{j^2} = \boxed{H_n^{(3)}}\sum_{j=1}^{n}\frac{H_j}{j^2}-\boxed{\sum_{j=1}^{n}\frac{H_j(j^3\boxed{H_j^{(3)}}-1)}{j^5}}, \qquad H_j^{(3)}=\sum_{i=1}^{j}\frac{1}{i^3}.$$

FIND a closed form for $\sum_{k=1}^{n} \frac{1}{k^3} \sum_{j=1}^{k} \frac{H_j}{j^2}.$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)(t)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

		nested depth
$\sigma(c) = c \quad \forall c \in \mathbb{Q},$		
$\sigma(k) = k + 1,$	$\mathcal{S}\, k = k + 1,$	$\delta(k) = 1$
$\sigma(h) = h + \frac{1}{k+1},$	$\mathcal{S}\, H_k = H_k + \frac{1}{k+1},$	$\delta(h) = 2$
$\sigma(t) = t + \frac{(k+1)h + 1}{(k+1)^3},$	$\mathcal{S} \sum_{j=1}^{k} \frac{H_j}{j^2} = \sum_{j=1}^{k} \frac{H_j}{j^2} + \frac{(k+1)H_k + 1}{(k+1)^3},$	$\delta(t) = 3.$

The refined telescoping problem:

FIND $g \in \mathbb{E} \geq_{\mathbf{3}} \mathbb{Q}(k)(h)(t)$ s.t.

$$\boxed{\sigma(g) - g = \frac{t}{k^3}.}$$

Our algorithm computes

$$g = \frac{h_3 k^3 - 1}{k^3} t - x \in \mathbb{E}$$

where $\mathbb{E} := \mathbb{Q}(k)(h)(t)(h_3)(x)$ with

$$\sigma(h_3) = h_3 + \frac{1}{(k+1)^3}, \qquad \delta(h_3) = 2,$$

$$\sigma(x) = x + \frac{h_3(1 + (k+1)h)}{(k+1)^3}, \quad \delta(x) = 3.$$

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$\rightarrow$

We get for

$$g(k+1) - g(k) = \frac{1}{k^3} \sum_{j=1}^{k} \frac{H_j}{j^2}$$

the solution

$$g(k) = \frac{H_k^{(3)} k^3 - 1)}{k^3} \sum_{j=1}^{k} \frac{H_j}{j^2} - \sum_{j=1}^{k} \frac{H_j(j^3 H_j^{(3)} - 1)}{j^5}.$$

$\downarrow$

$$H_n^{(3)} \sum_{j=1}^{n} \frac{H_j}{j^2} - \sum_{j=1}^{n} \frac{H_j(j^3 H_j^{(3)} - 1)}{j^5} = g(n+1) - g(1) = \sum_{k=1}^{n} \frac{1}{k^3} \sum_{j=1}^{k} \frac{H_j}{j^2}.$$

FIND $g \in \mathbb{Q}(k)(h)(t)$ s.t.

$$\sigma(g) - g = \frac{t}{k^3}.$$

Denominator bounding: COMPUTE $d \in \mathbb{Q}(k)(h)[t]$ s.t. $d = 1$

$$\forall g \in \mathbb{Q}(k)(h)(t): \quad \sigma(g) - g = \frac{t}{k^3} \quad \Longrightarrow \quad g = \frac{g'}{d} \quad \text{with} \quad g' \in \mathbb{Q}(k)(h)[t].$$

FIND numerator $g' \in \mathbb{Q}(k)(h)[t]$.

Degree bounding: COMPUTE $b \geq 0$ s.t. $\deg(g') \leq b$. b=2

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FIND $g = g_2\,t^2 + g_1\,t + g_0 \in \mathbb{Q}(k)(h)[t]$: ?

$$\Big[\sigma(g_2)\big(t + \tfrac{1+(k+1)h}{(k+1)^3}\big)^2 + \sigma(g_1 t + g_0)\Big] - \Big[g_2\,t^2 + g_1 t + g_0\Big] = \tfrac{t}{k^3}$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

$$\sigma(g_1\,t + g_0) - (g_1\,t + g_0) = \tfrac{t}{k^3} - c\Big[\tfrac{(1+(k+1)h)^2}{(k+1)^6} + \tfrac{2(1+(k+1)h)}{(k+1)^3}t\Big]$$

coeff. comp.

$$\sigma(g_1) - g_1 = \frac{1}{k^3} - c\,\frac{2(1+(k+1)h)}{(k+1)^3}$$

?

$$\sigma(h_3) = h_3 + \frac{1}{(k+1)^3}$$

FIND $g \in \mathbb{Q}(k)(h)(t)$ s.t.

$$\sigma(g) - g = \frac{t}{k^3}.$$

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coeff. comp. → $\sigma(g_2) - g_2 = 0$ → $g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,t + g_0) - (g_1\,t + g_0) = \tfrac{t}{k^3} - c\left[\tfrac{(1+(k+1)h)^2}{(k+1)^6} + \tfrac{2(1+(k+1)h)}{(k+1)^3}t\right]$$

coeff. comp. → $\sigma(g_1) - g_1 = \frac{1}{k^3} - c\,\frac{2(1+(k+1)h)}{(k+1)^3}$

→ $c = 0$, $g_1 = \frac{h_3 k^3 - 1}{k^3} + d$, $d \in \mathbb{Q}$

$$? \leftarrow \sigma(g_0) - g_0 = -\frac{h_3(1+(k+1)h)}{(k+1)^3} + d\frac{1+(k+1)h}{(k+1)^3}$$

$$\sigma(h_3) = h_3 + \frac{1}{(k+1)^3}$$

$$\sigma(x) = x + \frac{h_3(1+(k+1)h)}{(k+1)^3}$$

FIND $g \in \mathbb{Q}(k)(h)(t)$ s.t.

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FIND numerator $g' \in \mathbb{Q}(k)(h)[t]$.

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FIND $g = g_2 t^2 + g_1 t + g_0 \in \mathbb{Q}(k)(h)(h_3)(x)[t]$

$$\boxed{g = \frac{(h_3k^3 - 1)}{k^3} t - x}$$

$$\left[\sigma(g_2)\left(t + \tfrac{1+(k+1)h}{(k+1)^3}\right)^2 + \sigma(g_1 t + g_0)\right] - \left[g_2 t^2 + g_1 t + g_0\right] = \tfrac{t}{k^3}$$

coeff. comp. →

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$$g_2 = c \in \mathbb{Q}$$

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coeff. comp. →

$$\boxed{\sigma(g_1) - g_1 = \frac{1}{k^3} - c\,\tfrac{2(1+(k+1)h)}{(k+1)^3}}$$

$$c = 0, \quad g_1 = \tfrac{h_3k^3-1}{k^3} + d$$

$$d \in \mathbb{Q}$$

$$\boxed{\sigma(g_0) - g_0 = -\frac{h_3(1+(k+1)h)}{(k+1)^3} + d\frac{1+(k+1)h}{(k+1)^3}}$$

←

$$g_0 = -x$$

$$d = 0$$

The general case

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- $\mathbb{F} = \mathbb{K}(t_1)\dots(t_e)$ rational function field,
- for all $1 \leq i \leq e$:

$$\sigma(t_i) = \alpha_i\, t_i + \beta_i, \quad \alpha_i, \beta_i \in \mathbb{K}(t_1)\dots(t_{i-1}),$$

- $\text{const}_\sigma \mathbb{K}(t_1)\dots(t_e) = \mathbb{K}$ plus other constraints.

GIVEN: $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$.

FIND: $\mathrm{V}(\mathbf{f}, \mathbb{F}) := \{(c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{F} :$

$$\sigma(g) - g = c_1\, f_1 + \dots + c_n\, f_n \}.$$

(Karr)

This is a vector space over $\mathbb{K}$ with dimension $\leq n + 1$.

FIND: $\mathbb{E} \geq_d \mathbb{F}$ with $d := \delta(\mathbb{F})$ s.t.

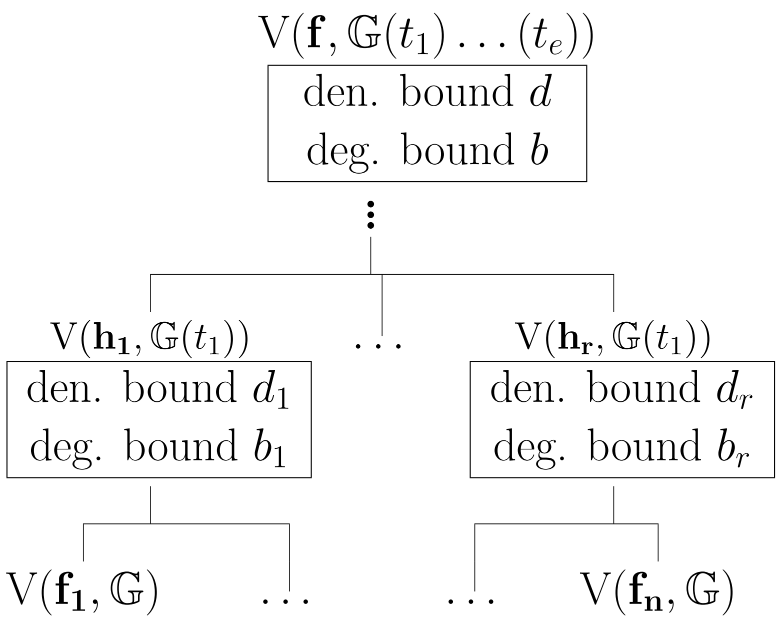
$$\forall \mathbb{E}' \geq_d \mathbb{E} : \quad \mathrm{V}(\mathbf{f}, \mathbb{E}') = \mathrm{V}(\mathbf{f}, \mathbb{E}).$$

(New)

Write $(\mathbb{G}(t_1)\dots(t_e),\sigma)$, $\quad \delta(\mathbb{G}) = d-1$, $\delta(t_i) = d$; $\quad \mathbf{f} \in \mathbb{G}(t_1)\dots(t_e)^n$.

(Karr) FIND a basis of $\mathrm{V}(\mathbf{f},\mathbb{G}(t_1)\dots(t_e))$.

by a reduction:



Write $(\mathbb{G}(t_1)\dots(t_e),\sigma)$, $\delta(\mathbb{G}) = d-1$, $\delta(t_i) = d$; $\mathbf{f} \in \mathbb{G}(t_1)\dots(t_e)^n$.

(Karr) FIND a basis of $V(\mathbf{f}, \mathbb{G}(t_1)\dots(t_e))$.

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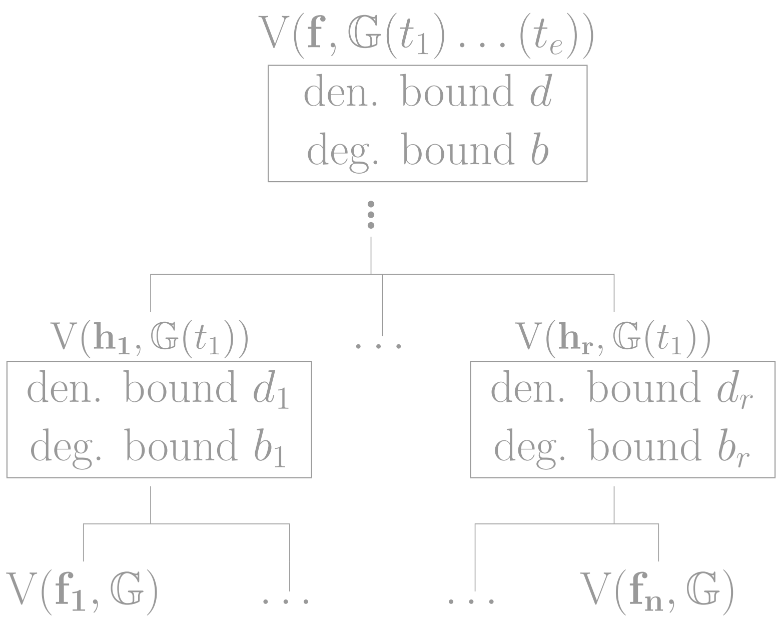
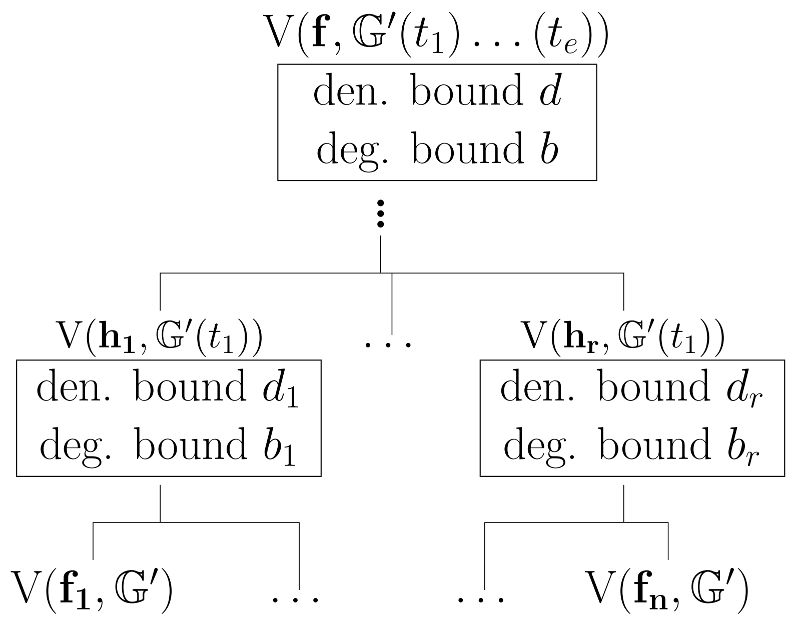


(New) FIND $\mathbb{E} \geq_d \mathbb{G}(t_1)\dots(t_e)$ s.t.

$$\forall \mathbb{E}' \geq_d \mathbb{E}: \quad V(\mathbf{f}, \mathbb{E}') = V(\mathbf{f}, \mathbb{E}).$$

Step 1: COMPUTE $\mathbb{G}' \geq_{d-1} \mathbb{G}$ s.t.

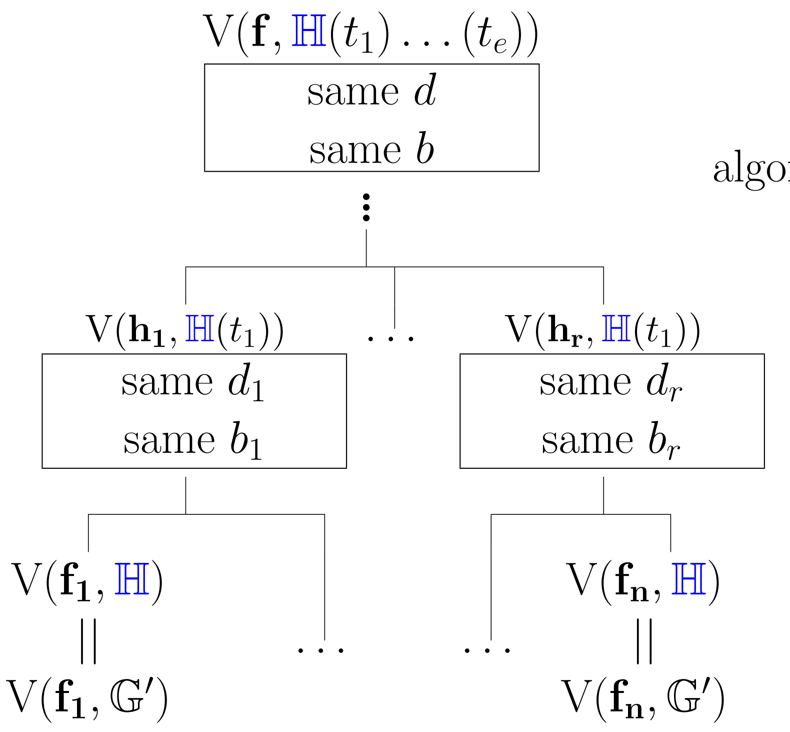


$$\forall \mathbb{H} \geq_{d-1} \mathbb{G}': \ V(\mathbf{f_i}, \mathbb{H}) = V(\mathbf{f_i}, \mathbb{G}')$$

Write $(\mathbb{G}(t_1)\dots(t_e),\sigma)$, $\delta(\mathbb{G}) = d-1$, $\delta(t_i) = d$; $\mathbf{f} \in \mathbb{G}(t_1)\dots(t_e)^n$.

Take $\mathbb{H}(s) \geq \mathbb{G}'$, $\delta(\mathbb{H}) = d-1$, $\delta(s) = d$.

CONSEQUENCE A:

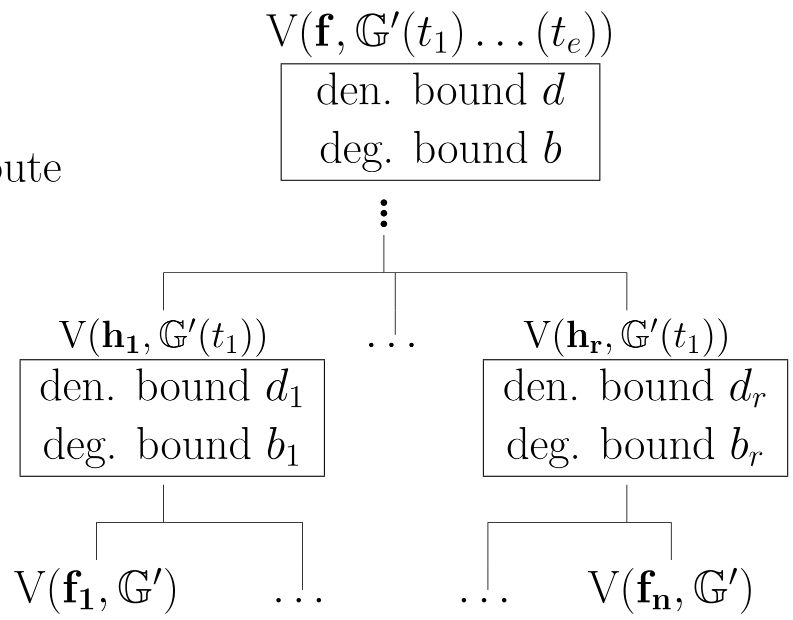


algorithms compute

(New) FIND $\mathbb{E} \geq_d \mathbb{G}(t_1)\dots(t_e)$ s.t.

$\forall \mathbb{E}' \geq_d \mathbb{E}: \quad V(\mathbf{f}, \mathbb{E}') = V(\mathbf{f}, \mathbb{E})$.

Step 1: COMPUTE $\mathbb{G}' \geq_{d-1} \mathbb{G}$ s.t.



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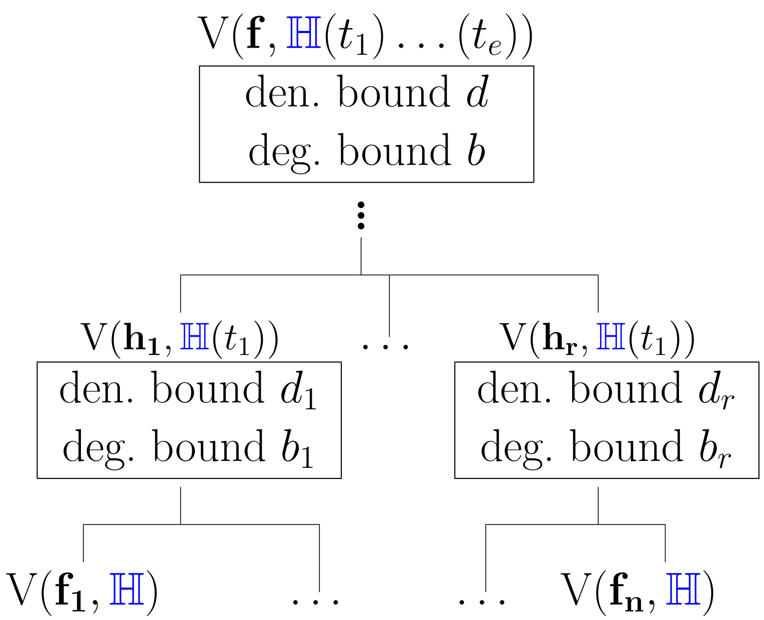
Write $(\mathbb{G}(t_1)\dots(t_e),\sigma)$, $\delta(\mathbb{G}) = d-1$, $\delta(t_i) = d$; $\mathbf{f} \in \mathbb{G}(t_1)\dots(t_e)^n$.

Take $\mathbb{H}(s) \geq \mathbb{G}'$, $\delta(\mathbb{H}) = d-1$, $\delta(s) = d$.

We get a reduction for $\mathbb{H}$:



Write $(\mathbb{G}(t_1)\ldots(t_e),\sigma)$, $\delta(\mathbb{G}) = d-1$, $\delta(t_i) = d$; $\mathbf{f} \in \mathbb{G}(t_1)\ldots(t_e)^n$.

Take $\mathbb{H}(s) \geq \mathbb{G}'$, $\delta(\mathbb{H}) = d-1$, $\delta(s) = d$.

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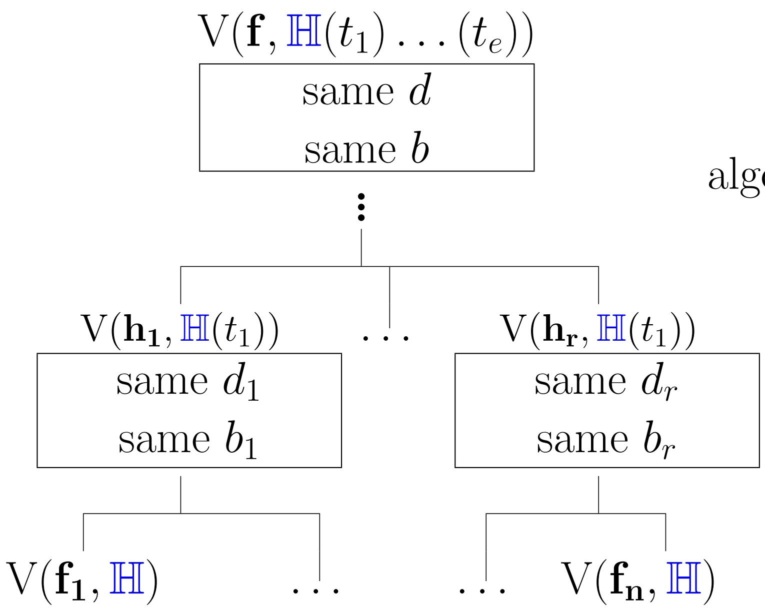
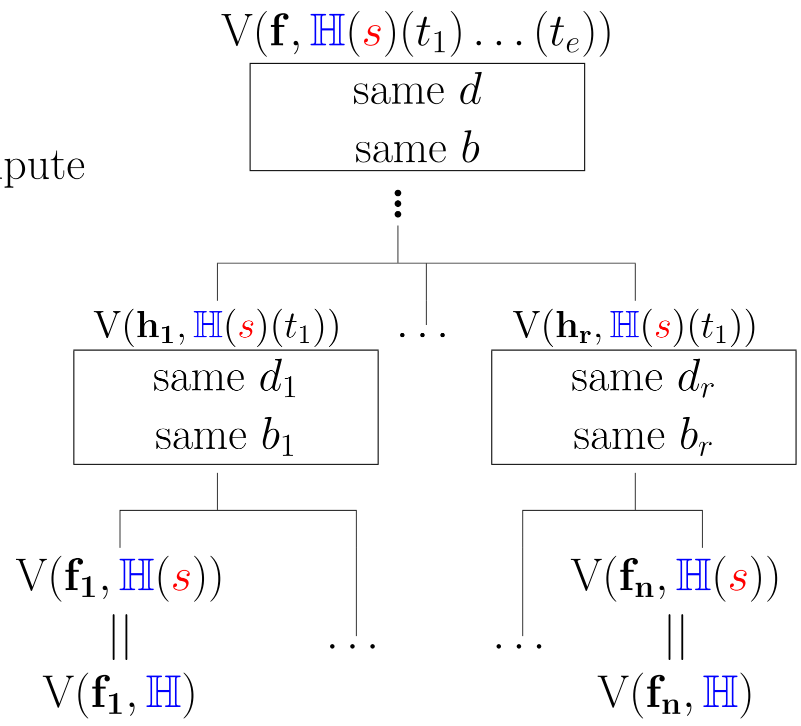


algorithms compute

D

CONSEQUENCE B:



Step II: Find $\mathbb{H}(s)$ by Completion

ASSUMPTION: There is $\mathbb{H}(s) \geq \mathbb{H}$ with

$$\mathrm{V}(\mathbf{f}, \mathbb{G}') = \mathrm{V}(\mathbf{f}, \mathbb{H}) \subsetneq \mathrm{V}(\mathbf{f}, \mathbb{H}(s))$$

where

$$\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{G}'^n \subseteq \mathbb{H}^n.$$

COMPLETION:

f_1

> If there is no $s_1 \in \mathbb{G}'$ s.t.
> $$\sigma(s_1) = s_1 + f_1,$$
> adjoin such an s_1, i.e., $\mathbb{G}_1 := \mathbb{G}'(s_1)$;
> otherwise $\mathbb{G}_1 := \mathbb{G}'$.

f_2

> If there is no $s_2 \in \mathbb{G}_1$ s.t.
> $$\sigma(s_2) = s_2 + f_2,$$
> adjoin such an s_2, i.e., $\mathbb{G}_2 := \mathbb{G}(s_2)$;
> otherwise $\mathbb{G}_2 := \mathbb{G}_1$.

$\vdots$

We get a $\Pi\Sigma$-field $(\mathbb{G}'(s_1)\dots(s_r), \sigma)$ with

$$\dim \mathrm{V}(\mathbf{f}, \mathbb{G}'(s_1)\dots(s_r)) = n + 1.$$

$$\Rightarrow \qquad \mathbb{H}(s) \text{ “} \subseteq \text{” } \mathbb{G}'(s_1)\dots(s_r).$$

RESULTS

"We cannot extend more":

GIVEN $(\mathbb{F}, \sigma)$ with $d := \delta(\mathbb{F})$ and $\mathbf{f} \in \mathbb{F}^n$.
COMPUTE $\mathbb{D} \geq_d \mathbb{F}$ s.t.

$$\forall \mathbb{H} \geq_d \mathbb{D} : \; \mathrm{V}(\mathbf{f}, \mathbb{H}) = \mathrm{V}(\mathbf{f}, \mathbb{D})$$

Is there a better extension $\mathbb{E} \geq_d \mathbb{F}$?

RESULTS

"We cannot extend more":

GIVEN $(\mathbb{F},\sigma)$ with $d:=\delta(\mathbb{F})$ and $\mathbf{f}\in\mathbb{F}^n$.
COMPUTE $\mathbb{D}\geq_d\mathbb{F}$ s.t.

$$\forall\mathbb{E}\geq_d\mathbb{D}:\ \mathrm{V}(\mathbf{f},\mathbb{E})=\mathrm{V}(\mathbf{f},\mathbb{D})$$

Let $\mathbb{E}\geq_d\mathbb{F}$ be any $\Pi\Sigma$-extension;
let $(\mathbb{D},\sigma)$ be a **depth-optimal** $\Pi\Sigma$-extension:

IF

$\sigma(g)-g=c_1f_1+\dots c_nf_n$ for some $c_i\in\mathbb{K}$, $g\in\mathbb{E}$

THEN

$\sigma(h)-h=c_1f_1+\dots c_nf_n$ for some $h\in\mathbb{D}$

with

$$\delta(h)\leq\delta(g).$$

Do completion

$$\mathbb{F}(s), \quad \sigma(s) = s + f$$

not with

$$\Sigma\text{-extension} \;:\Leftrightarrow \nexists g \in \mathbb{F} : \sigma(g) = g + f$$

but with

$$\delta\text{-optimal } \Sigma\text{-extension} \;:\Leftrightarrow \forall \mathbb{E} \geq_{\delta(f)} \mathbb{F} \;\; \nexists g \in \mathbb{E} : \sigma(g) = g + f$$

(Constructive with algorithms shown before)