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When is 0.999... equal to 1?

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From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously like it comes out to exactly 1. Is there a way I can automatically decide this? The sum may be written in many ways, but one is:

$$\sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}; \qquad H_j := \sum_{i=1}^{j} \frac{1}{i}.$$

Of course you can expand out the H's and get a quadruple sum. There are zillions of ways to play with it, summing by parts, but I have never managed to get rid of all the summations.

Robin

From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC: Carsten Schneider

Robin and Herb,

I am willing to bet that Castern Schneider's SIGMA package for handling sums with harmonic numbers (among others) can do it in a jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the answer to me.

-Doron

Dear Doron,

Finally, I managed to compute the limit of the sum in a jiffy.

According to my computations the sum

$$\boxed{S := \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}}$$

is not 1!
More precisely, with my Sigma package I obtain as its value

$$-4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5) \simeq 0.9992228377638300876$$

where $\zeta(r) = \sum_{i=1}^{\infty} \frac{1}{i^r}$.

Take the truncated version: $$S(a,b) = \sum_{k=1}^{b} \frac{H_{k+1}-1}{k(k+1)} \sum_{j=1}^{a} \frac{H_j}{j(j+k)},$$

i.e.,

$$\lim_{a,b\to\infty} S(a,b) = S.$$

Sigma simplifies the inner sum to

$$\sum_{j=1}^{a} \frac{H_j}{j(j+k)} = \frac{kH_k^2 - 2H_k + kH_k^{(2)} + 2kH_a^{(2)}}{2k^2} - \frac{(kH_a-1)}{k^2} \sum_{i=1}^{k} \frac{1}{a+i} - \frac{1}{k} \sum_{i=1}^{k} \frac{1}{i} \sum_{j=1}^{i} \frac{1}{a+j}$$

where $H_k^{(r)} = \sum_{i=1}^{k} \frac{1}{i^r}$.

Hence, for

$$\boxed{S'(a,b) := \sum_{k=1}^{b} \frac{H_{k+1}-1}{k(k+1)} \frac{kH_k^2 - 2H_k + kH_k^{(2)} + 2kH_a^{(2)}}{2k^2}},$$

we have

$$\lim_{a,b\to\infty} S'(a,b) = S.$$

Sigma simplifies $S'(a,b)$ to

$$S'(a,b) = A(a,b) + B(a,b) + C(a,b)$$

where

$$\begin{aligned} A(a,b) :=& \frac{1}{2(b+1)^2}\Big(6H_b + 4bH_b + 4H_b^2 + 3bH_b^2 + H_b^3 + bH_b^3 - 6bH_a^{(2)} \\ &+ 2H_bH_a^{(2)} + 2bH_bH_a^{(2)} - 2H_b^{(2)} - 7bH_b^{(2)} + H_bH_b^{(2)} + bH_bH_b^{(2)}\Big), \\ B(a,b) :=& -\frac{2b^2}{(b+1)^2}\Big(H_a^{(2)} + H_b^{(2)}\Big), \\ C(a,b) :=& (H_a^{(2)} - 1)\sum_{i=1}^{b}\frac{H_i}{i^2} - \sum_{i=1}^{b}\frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{b}\frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{b}\frac{H_iH_i^{(2)}}{i^2}. \end{aligned}$$

By

$$\lim_{a,b\to\infty} A(a,b) = 0 \qquad \text{and} \qquad \lim_{a,b\to\infty} B(a,b) = -4\zeta(2)$$

we get

$$S = \lim_{a,b\to\infty} S'(a,b) = -4\zeta(2) + \lim_{a,b\to\infty} C(a,b).$$

ζ-relations by [Borwein, Girgensohn] and [Flajolet, Salvy] give

$$\sum_{i=1}^{\infty} \frac{H_i}{i^2} = 2\zeta(3), \qquad \sum_{i=1}^{\infty} \frac{H_i^2}{i^3} = -\zeta(2)\zeta(3) + \frac{7}{2}\zeta(5),$$

$$\sum_{i=1}^{\infty} \frac{H_i^3}{i^2} = \zeta(2)\zeta(3) + 10\zeta(5), \qquad \sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^2} = \zeta(2)\zeta(3) + \zeta(5).$$

This shows that

$$S = \lim_{a,b\to\infty} S'(a,b) = -4\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5).$$

J.M. Borwein and R. Girgensohn. Evaluation of triple Euler sums. *Electron. J. Combin.*, 3:1–27, 1996.
P. Flajolet and B. Salvy. Euler sums and contour integral representations. *Experim. Math.*, 7(1):15–35, 1998.

From: Doron Zeilberger
To: Carsten Schneider
CC: Robin Pemantle, Herbert Wilf

Wow, you (and your computer!) are wizhes!

⋮

Anyway, even though the bet was one sided, I still feel that Robin and/or Herb owe me a free lunch (and they owe Carsten, and his computer, a free dinner).

Best wishes
Doron

Find the closed form for

$$\mathrm{SUM}(a,k)=\sum_{j=1}^{a}\frac{H_j}{j(j+k)}.$$

1. Compute a recurrence (creative telescoping)

$$k^2\mathrm{SUM}(a,k)-(k+1)(2k+1)\mathrm{SUM}(a,k+1)+(k+1)(k+2)\mathrm{SUM}(a,k+2)=\frac{a(a+k+2)-(a+1)(k+1)H_a}{(k+1)(a+k+1)(a+k+2)}.$$

2. Solve the recurrence (d'Alembertian solutions)

$$h_1=\frac{1}{k},\quad h_2=\frac{1}{k}\sum_{j=2}^{k}\frac{1}{j-1},\quad p=\frac{1}{k}\sum_{j=2}^{k}\frac{\sum_{i=2}^{j}-(\frac{-a^2-H_a-aH_a-ai+H_ai+aH_ai}{(-1+i)(-1+a+i)(a+i)})}{j-1}.$$

3. Simplification (indefinite summation)

$$h_2=\frac{kH_k-1}{k^2},\qquad p=\frac{kH_a-H_k-aH_k+kH_k{}^2+akH_k{}^2}{(1+a)k^2}-\frac{kH_a-1}{k^2}\sum_{i=1}^{k}\frac{1}{i+a}-\frac{1}{k}\sum_{j=1}^{k}\frac{\sum_{i=1}^{j}\frac{1}{a+i}}{j}.$$

4. Find the linear combination

$$\sum_{j=1}^{a}\frac{H_j}{j(j+k)}=\frac{-H_a+(1+a)H_a{}^2}{1+a}h_1+0\,h_2+p.$$

A Session with Sigma

In[1]:= $$\mathbf{rec} = \mathbf{GenerateRecurrence}\Big[\sum_{j=1}^{a}\frac{H_j}{j\,(j+k)}, k\Big][[1]]$$

Out[1]= $$k^2\,(1+k)\,(1+a+k)\,(2+a+k)\,\mathtt{SUM}[k] - (1+k)^2\,(1+a+k)\,(2+a+k)\,(1+2\,k)\,\mathtt{SUM}[1+k] + (1+k)^2\,(2+k)\,(1+a+k)\,(2+a+k)\,\mathtt{SUM}[2+k] == a\,(2+a+k) + (-1-a)\,(1+k)\,H_a$$

In[2]:= $$\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[k], \mathbf{NestedSumExt} \to \infty]$$

Out[2]= $$\Big\{\Big\{0, \frac{1}{k}\Big\}, \Big\{0, \frac{1}{k}\sum_{\iota_1=2}^{k}\frac{1}{-1+\iota_1}\Big\}, \Big\{1, \frac{1}{k}\sum_{\iota_1=2}^{k}\frac{\sum_{\iota_2=2}^{\iota_1}\frac{a^2+a\,\iota_2+H_a\,(1+a-\iota_2-a\,\iota_2)}{(-1+\iota_2)\left(-a+a^2-\iota_2+2\,a\,\iota_2+\iota_2^2\right)}}{-1+\iota_1}\Big\}\Big\}$$

In[3]:= $$\mathbf{recSol} = \mathbf{SigmaReduce}[\mathbf{recSol}, k, \mathbf{SimpleSumRepresentation} \to \mathbf{True}]$$

Out[3]= $$\Big\{\Big\{0, \frac{1}{k}\Big\}, \Big\{0, \frac{-1+k\,\sum_{\iota_1=1}^{k}\frac{1}{\iota_1}}{k^2}\Big\}, \Big\{1, \frac{1}{2\,(1+a)\,k^2}\Bigg(2\,k\,H_a\Bigg(1-(1+a)\sum_{\iota_1=1}^{k}\frac{1}{a+\iota_1}\Bigg) + (1+a)\Bigg(-2\sum_{\iota_1=1}^{k}\frac{1}{\iota_1} + 2\sum_{\iota_1=1}^{k}\frac{1}{a+\iota_1} + k\Bigg(\sum_{\iota_1=1}^{k}\frac{1}{\iota_1^2} + \Bigg(\sum_{\iota_1=1}^{k}\frac{1}{\iota_1}\Bigg)^2 - 2\sum_{\iota_1=1}^{k}\frac{\sum_{\iota_2=1}^{\iota_1}\frac{1}{a+\iota_2}}{\iota_1}\Bigg)\Bigg)\Bigg)\Big\}\Big\}$$

In[4]:= $$\mathbf{FindLinearCombination}\Big[\mathbf{recSol}, \sum_{j=1}^{a}\frac{H_j}{j\,(j+k)}, 2\Big]$$

Out[4]= $$\frac{-2\,\sum_{\iota_1=1}^{k}\frac{1}{\iota_1} + 2\,\sum_{\iota_1=1}^{k}\frac{1}{a+\iota_1} + k\left(2\,H_a^{(2)} + \sum_{\iota_1=1}^{k}\frac{1}{\iota_1^2} + \left(\sum_{\iota_1=1}^{k}\frac{1}{\iota_1}\right)^2 - 2\left(H_a\,\sum_{\iota_1=1}^{k}\frac{1}{a+\iota_1} + \sum_{\iota_1=1}^{k}\frac{\sum_{\iota_2=1}^{\iota_1}\frac{1}{a+\iota_2}}{\iota_1}\right)\right)}{2\,k^2}$$

Indefinite Summation in Difference Fields

Goal: Find a closed form for

$$\sum_{k=0}^{n} H_k$$

where $H_k = \sum_{i=1}^{k} \frac{1}{i}$.

A difference field for the problem

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(t)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, \\ \sigma(k) &= k+1, & \mathcal{S}\, k &= k+1, \\ \sigma(t) &= t + \frac{1}{k+1}, & \mathcal{S}\, H_k &= H_k + \frac{1}{k+1}. \end{aligned}$$

$(\mathbb{Q}(k)(t), \sigma)$ is our difference field.

The telescoping problem

$$\text{Find } g \in \mathbb{Q}(k)(t) : \ \boxed{\sigma(g) - g = t}$$

$$\downarrow$$

$$g = (t-1)\,k.$$

The closed form

$$\boxed{\mathcal{S}\,(H_k - 1)(k) - (H_k - 1)k = H_k}$$

$$\downarrow$$

$$\sum_{k=0}^{n} H_k = (H_{n+1} - 1)(n+1).$$

FIND $g \in \mathbb{Q}(k)(t)$:

$$\sigma(g) - g = t.$$

Denominator bounding: COMPUTE a polynomial $d \in \mathbb{Q}(k)[t]^*$ s.t. $d = 1$

$$\forall g \in \mathbb{Q}(k)(t): \quad \sigma(g) - g = t \quad \Rightarrow \quad g\, d \in \mathbb{Q}(k)[t].$$

FIND $g' \in \mathbb{Q}(k)[t]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = t.$$

Degree bounding: COMPUTE $b \geq 0$ s.t. $b = 2$

$$\forall g \in \mathbb{Q}(k)[t] \quad \sigma(g) - g = t \quad \Rightarrow \quad \deg(g) \leq b.$$

COMPUTE coefficients of $g = g_2\,t^2 + g_1\,t + g_0$.

$g = k\,t - k$

$$\big[\sigma(g_2)\big(t + \frac{1}{k+1}\big)^2 + \sigma(g_1 t + g_0)\big] - \big[g_2\,t^2 + g_1 t + g_0\big] = t$$

coeff. comp. $\rightarrow$ $\boxed{\sigma(g_2) - g_2 = 0}$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,t + g_0) - (g_1\,t + g_0) = t - c\Big[\frac{2t(k+1)+1}{(k+1)^2}\Big]$$

coeff comp. $\rightarrow$ $\boxed{\sigma(g_1) - g_1 = 1 - c\frac{2}{k+1}}$

$c = 0,\ \ g_1 = k + d,$
$d \in \mathbb{Q}$

$g_0 = -k, d = 0 \rightarrow$ $\boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$

Zeilberger's Creative Telescoping Paradigm

- GIVEN

$$\mathrm{SUM}(k) := \sum_{j=1}^{a} \underbrace{\frac{H_j}{j(j+k)}}_{=:\, f(k,j)}$$

- FIND $c_0(k)$, $c_1(k)$, $c_2(k)$, and $g(k,j)$ s.t.

$$\boxed{g(k,j+1) - g(k,j)} = \boxed{c_0(k)\, f(k,j) + c_1(k)\, f(k+1,j) + c_2(k)\, f(k+2,j)}$$

for all $j, k \geq 1$.

Sigma computes:

$$c_0(k) := k^2,\ \ c_1(k) := -(k+1)(2k+1),\ \ c_2(k) := (k+1)(k+2),$$

$$g(k,j) := -\frac{jH_j + k + j}{(k+j)(k+j+1)},$$
$$g(k,j+1) := -\frac{(1+j)H_j + k + j + 2}{(k+j+1)(k+j+2)}.$$

Summing this equation over j from 1 to a gives:

$$\boxed{g(k, a+1) - g(k, 1)} = \boxed{c_0(k)\,\mathrm{SUM}(k) + c_1(k)\,\mathrm{SUM}(k+1) + c_2(k)\,\mathrm{SUM}(k+2).}$$

Summation and Linear Difference Equations

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- $\mathbb{F} = \mathbb{K}(t_1)\dots(t_e)$ rational function field,
- for all $1 \leq i \leq e$:

$$\sigma(t_i) = \alpha_i\, t_i + \beta_i, \quad \alpha_i, \beta_i \in \mathbb{K}(t_1)\dots(t_{i-1}),$$

- $\text{const}_\sigma \mathbb{K}(t_1)\dots(t_e) = \mathbb{K}$ plus other constraints. $[\mathbb{K} = \mathbb{Q}$
  $\mathbb{K} = \mathbb{Q}(a,k)]$

Telescoping

- GIVEN $f \in \mathbb{F}$
- FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$$\downarrow \qquad \uparrow$$

Parameterized Telescoping

- GIVEN $f_0, \dots, f_d \in \mathbb{F}$, $a_0, a_1 \in \mathbb{F}$
- FIND ALL $c_0, \dots, c_d \in \mathbb{K}$, $h \in \mathbb{F}$:

$$\boxed{a_1\, \sigma(h) - a_0\, h = c_0\, f_0 + \dots + c_d\, f_d}$$

Remark: Z's "Creative Telescoping"

- GIVEN $f_i = \text{summand}(n+i, k) \in \mathbb{F}$
- FIND ALL $c_0, \dots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0\, f_0 + \dots + c_d\, f_d}$$

Linear Difference Equations

- GIVEN $f, a_0, \dots, a_m \in \mathbb{F}$
- FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\, \sigma^m(g) + \cdots + a_0\, g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

Parameterized Linear Difference Equations

- GIVEN $a_0, \dots, a_m \in \mathbb{F}, f_0, \dots, f_d \in \mathbb{F}$.
- FIND ALL $g \in \mathbb{F}$, $c_0, \dots, c_d \in \mathbb{K}$:

$$\boxed{a_m\, \sigma^m(g) + \cdots + a_0\, g = c_0\, f_0 + \cdots + c_d\, f_d}$$