The CreaComp Project: Theorema used in Computer−Supported Teaching and Learning of Mathematics

Wolfgang Windsteiger

RISC-Linz

A-4232 Hagenberg, Austria

Saarbrücken, November 14, 2005

What is CreaComp?

- *CreaComp* is a project at the University of Linz, which aims at
  - producing *computer-supported learning units* for mathematics,
  - combining *MeetMath* and *Theorema*,
  - *stimulating the students' creativity* during the process of learning mathematics by providing an environment, in which computer experiments support the understanding of mathematical concepts.

MeetMath

- Mathematical course-material based on *Mathematica*.
- Java-based navigation
- Didactical concept:
  - self-paced learning, i.e. not necessarily linear structure of the content
  - motivation ⟶ aquisition ⟶ strenghtening ⟶ assessment

Main Aspects

- *Mathematica algorithms available as black-box* (in appropriate phases according to white-box/black-box principle)
- *Learning through experiments* (graphics, animation, interaction)

Theorema: known

3 of 17

4 of 17

The Combination of MeetMath & Theorema

- Computer-supported teaching of mathematics “traditionally” supports visualization and computation (both symbolic and numeric).
- Formal mathematics ↗ skipped ↘ done by hand
- Experimental mathematics *versus* formal mathematics

5 of 17

The Combination of MeetMath & Theorema

- Computer-supported teaching of mathematics “traditionally” supports visualization and computation (both symbolic and numeric).
- Formal mathematics ↗ skipped ↘ done by hand
- Experimental mathematics versus formal mathematics

CreaComp Approach

- Bring computer-support also to formal mathematics
- “*Experimental formal mathematics*”

6 of 17

An Example: Teaching Convergence of Real-Valued Sequences

Definition["convergence", any[f, a, ϵ, N],

converges[f, a] :⟺ $\underset{\substack{\epsilon \\ \epsilon>0}}{\forall} \underset{N\in\mathbb{N}}{\exists}$ is-closer[f, a, ϵ, N]

is-closer[f, a, ϵ, N] :⟺ $\underset{\substack{n\in\mathbb{N} \\ n\geq N}}{\forall} |f_n - a| < \epsilon$]

We communicate: converges[*f*, *a*] means that “f_n comes close to *a* for large *n*”

Definition["example", any[n],

$g_n := \frac{2}{n^2+3n}$

$h_n := \mathrm{Cos}[n\,\pi] + \frac{1}{n}$]

Visual Exploration of the New Concepts

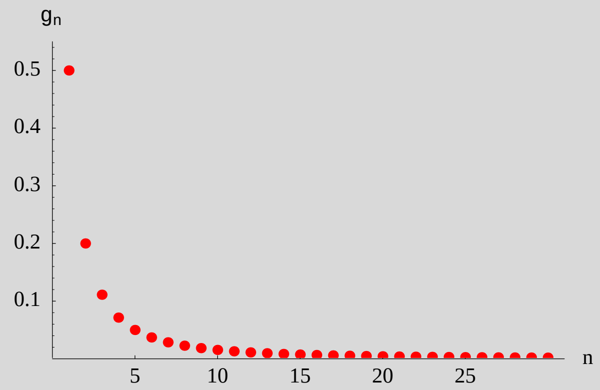
PrintSequence[g, {20, 30}]

n	g_n
20	0.00434783
21	0.00396825
22	0.00363636
23	0.00334448
24	0.00308642
25	0.00285714
26	0.00265252
27	0.00246914
28	0.00230415
29	0.00215517
30	0.0020202

PrintSequence[h, {20, 30}]

n	h_n
20	1.05
21	−0.952381
22	1.04545
23	−0.956522
24	1.04167
25	−0.96
26	1.03846
27	−0.962963
28	1.03571
29	−0.965517
30	1.03333

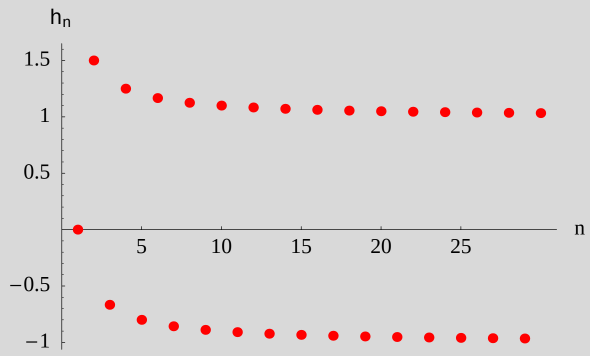
PlotGraphSequence[g, {1, 30}]



- Graphics -

Comes close to 0 ???

PlotGraphSequence[h, {1, 30}]



- Graphics -

Comes close to both 1 and -1 ???

Limited if example becomes more complicated

Definition["example:compl", any[n],

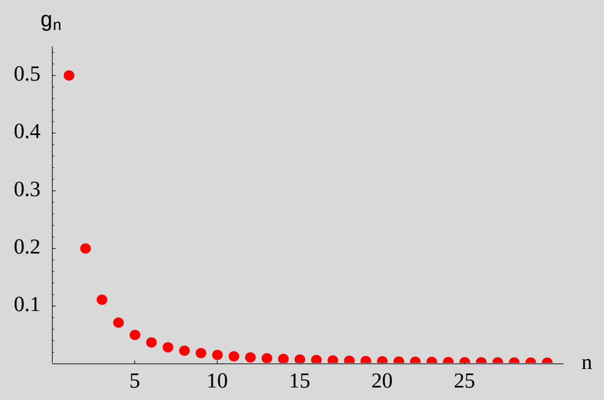
$$s_n := \frac{n^2 + 222\,n + 13059}{2004\,n^2 + 17\,n}]$$

UseAlso[⟨Definition["example:compl"]⟩]

```
PrintSequence[g, {20, 30}]
```

n	g_n
20	0.00434783
21	0.00396825
22	0.00363636
23	0.00334448
24	0.00308642
25	0.00285714
26	0.00265252
27	0.00246914
28	0.00230415
29	0.00215517
30	0.0020202

```
PlotGraphSequence[g, {1, 30}]
```



- Graphics -

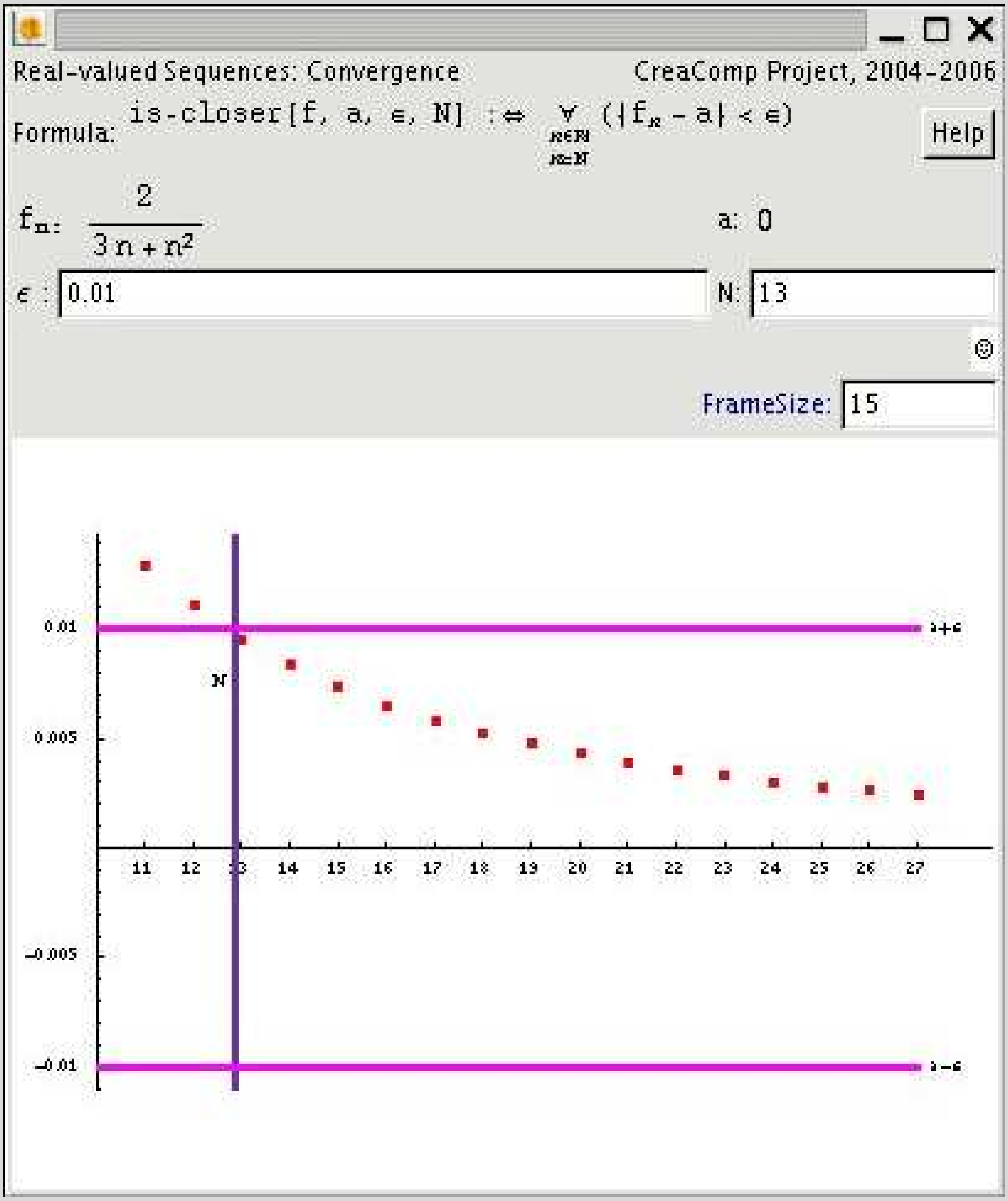
Check convergence of g

Guess $a = 0$. We need to find out whether

$$\underset{\substack{\epsilon \\ \epsilon>0}}{\forall} \ \underset{N\in\mathbb{N}}{\exists} \ \text{is-closer}[g, 0, \epsilon, N]$$

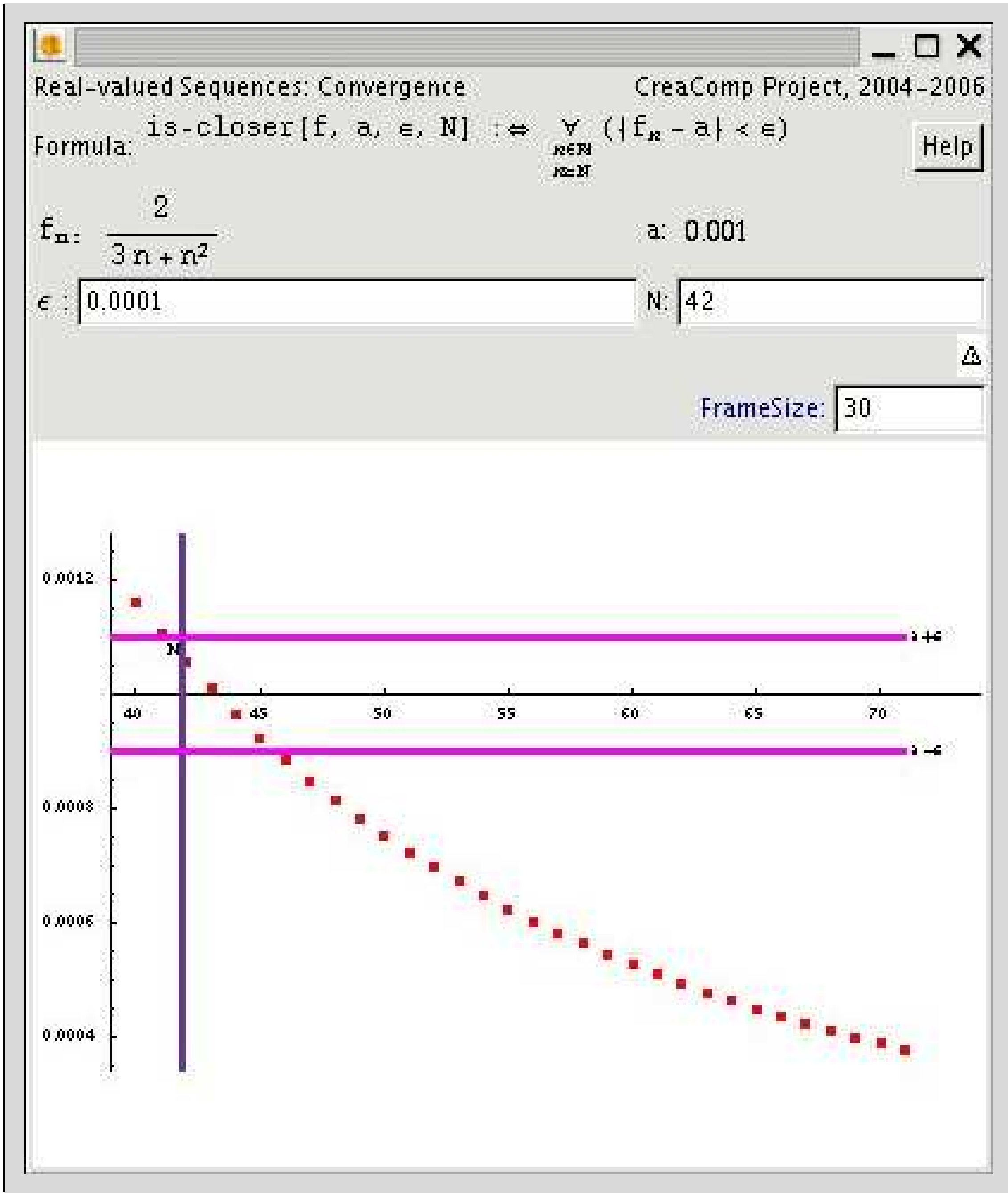
$\text{ChooseEpsFindN}\left[\frac{2}{n^2+3\,n}, 0\right]$

Interactive graphics widget:



$\text{ChooseEpsFindN}\left[\frac{2}{n^2+3\,n}, 0.001\right]$

Interactive graphics widget:



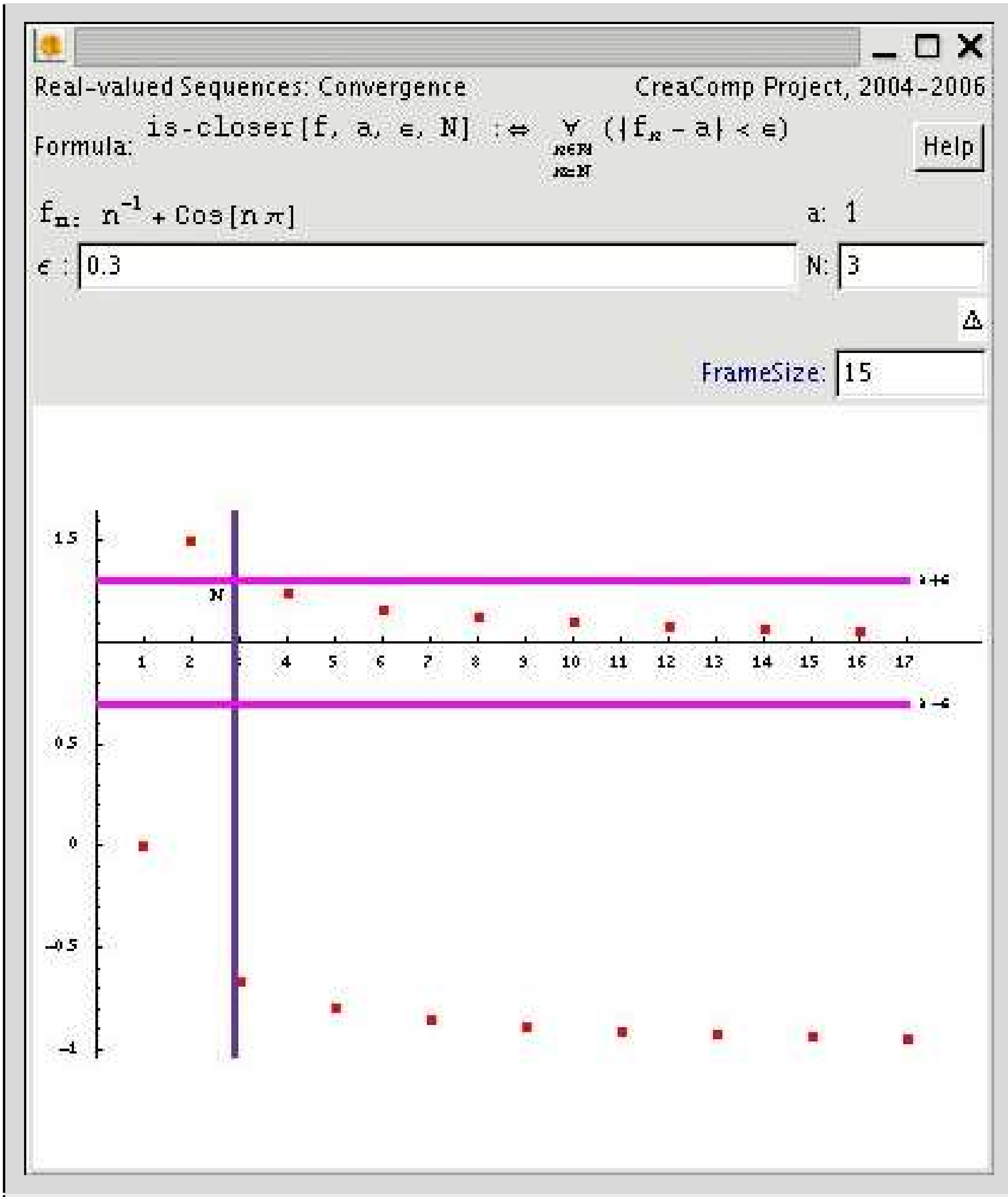
Check convergence of h

Guess $a = 1$. We need to find out whether

$$\underset{\substack{\epsilon \\ \epsilon>0}}{\forall} \; \underset{N\in\mathbb{N}}{\exists} \; \text{is-closer}[h, 1, \epsilon, N]$$

```
ChooseEpsFindN[Cos[n π] + 1/n, 1]
```

Interactive graphics widget:



- Graphics -

Guess $a = -1$. We need to find out whether

$\underset{\epsilon > 0}{\underset{\epsilon}{\forall}} \; \underset{N \in \mathbb{N}}{\exists}$ is-closer$[h, -1, \epsilon, N]$

ChooseEpsFindN$\left[\text{Cos}[n\,\pi] + \frac{1}{n}, -1\right]$

Interactive graphics widget:

- Structure of learning units:

— hyperlinked documents

— one “main document” describing the “main path” through a topic

— links to related/similar/other topics

8 of 17

9 of 17

Formal Exploration: First Level

- Visual exploration: inspect finite cases
- The same can be done by computation: *substitute* infinite ranges by *finite ranges*
- One quantifier at the time
- Introduce additional parameters to adjust the range

10 of 17

Finitary Version of Convergence

$$\text{converges}[f, a] :\Longleftrightarrow \underset{\substack{\epsilon \\ \epsilon>0}}{\forall} \underbrace{\underset{N\in\mathbb{N}}{\exists} \text{is-closer}[f, a, \epsilon, N]}_{\text{stays-closer}[f,a,\epsilon]}$$

- One quantifier at the time, reasonable also in automated proving!

11 of 17

Finitary Version of Convergence

$$\text{converges}[f, a] :\Longleftrightarrow \underset{\substack{\epsilon \\ \epsilon>0}}{\forall} \underbrace{\underset{N\in\mathbb{N}}{\exists} \text{is-closer}[f, a, \epsilon, N]}_{\text{stays-closer}[f,a,\epsilon]}$$

↓

$$\text{converges}[f, a] :\Longleftrightarrow \underset{\substack{\epsilon \\ \epsilon>0}}{\forall} \text{stays-closer}[f, a, \epsilon]$$

12 of 17

Finitary Version of Convergence

$$\text{converges}[f, a] :\Longleftrightarrow \underset{\substack{\epsilon \\ \epsilon>0}}{\forall} \underbrace{\underset{N\in\mathbb{N}}{\exists} \text{is-closer}[f, a, \epsilon, N]}_{\text{stays-closer}[f,a,\epsilon]}$$

↓

$$\text{converges}[f, a] :\Longleftrightarrow \underset{\substack{\epsilon \\ \epsilon>0}}{\forall} \text{stays-closer}[f, a, \epsilon]$$

↓

converges#[f, a, B] :⟺ $\underset{\epsilon \in B}{\forall}$ stays-closer[f, a, ϵ]

- Computational experiments with converges#[f, a, B] for *concrete f, a, B* … not too interesting

Finitary Version of Convergence

stays-closer[f, a, ϵ] :⟺ $\underset{N \in \mathbb{N}}{\exists}$ is-closer[f, a, ϵ, N]

↓

stays-closer#[f, a, ϵ, max] :⟺ $\underset{N=1,\ldots,max}{\exists}$ is-closer[f, a, ϵ, N]

↓

Definition["stays closer on finite segment", any[f, a, ϵ, N, max],
stays-closer#[f, a, ϵ, max] :⟺ $\underset{N=1,\ldots,max}{\exists}$ is-closer#[f, a, ϵ, N]
is-closer#[f, a, ϵ, N] :⟺ $\underset{n=N,\ldots,N+30}{\forall}$ $|f_n - a| < \epsilon$]

Definition["threshold on finite segment", any[f, a, ϵ, max],
threshold#[f, a, ϵ, max] := $\underset{N=1,\ldots,max}{\ni}$ is-closer#[f, a, ϵ, N]]

UseAlso[
⟨Definition["stays closer on finite segment"], Definition["threshold on finite segment"]⟩]

Compute[stays-closer#[g, 0, 0.1, 200]]

True

Compute[⟨⟨0.1^i, stays-closer#[g, 0, 0.1^i, 200]⟩ $\underset{i=1,\ldots,5}{|}$⟩]

⟨⟨0.1, True⟩, ⟨0.01, True⟩, ⟨0.001, True⟩, ⟨0.0001, True⟩, ⟨0.00001, False⟩⟩

$\text{Compute}\left[\left\langle \left\langle 0.1^{i}, \text{stays-closer\#}\left[g, 0, 0.1^{i}, 500\right]\right\rangle \underset{i=1,\ldots,5}{|} \right\rangle\right]$

⟨⟨0.1, True⟩, ⟨0.01, True⟩, ⟨0.001, True⟩, ⟨0.0001, True⟩, ⟨0.00001, True⟩⟩

$\text{Compute}\left[\left\langle \left\langle 0.1^{i}, \text{threshold\#}\left[g, 0, 0.1^{i}, 500\right]\right\rangle \underset{i=1,\ldots,5}{|} \right\rangle\right]$

⟨⟨0.1, 4⟩, ⟨0.01, 13⟩, ⟨0.001, 44⟩, ⟨0.0001, 140⟩, ⟨0.00001, 446⟩⟩

$\text{Compute}\left[\left\langle \left\langle 0.1^{i}, \text{threshold\#}\left[g, 0.001, 0.1^{i}, 1000\right]\right\rangle \underset{i=1,\ldots,5}{|} \right\rangle\right]$

— *Theorema::impossible : "such that" does not make any sense if there is no such thing.*

— *Theorema::impossible : "such that" does not make any sense if there is no such thing.*

⟨⟨0.1, 4⟩, ⟨0.01, 13⟩, ⟨0.001, 31⟩, ⟨0.0001, Null⟩, ⟨0.00001, Null⟩⟩

$\text{Compute}\left[\left\langle \left\langle 0.1^{i}, \text{stays-closer\#}\left[h, 1, 0.1^{i}, 500\right]\right\rangle \underset{i=1,\ldots,5}{|} \right\rangle\right]$

⟨⟨0.1, False⟩, ⟨0.01, False⟩, ⟨0.001, False⟩, ⟨0.0001, False⟩, ⟨0.00001, False⟩⟩

$\text{Compute}\left[\left\langle \left\langle 0.1^{i}, \text{stays-closer\#}\left[h, -1, 0.1^{i}, 500\right]\right\rangle \underset{i=1,\ldots,5}{|} \right\rangle\right]$

⟨⟨0.1, False⟩, ⟨0.01, False⟩, ⟨0.001, False⟩, ⟨0.0001, False⟩, ⟨0.00001, False⟩⟩

Formal Exploration: Second Level — Threshold Computation

- In the computations above: Find the threshold for *concrete* ϵ by computation
- Now: Could we find the threshold for *arbitrary* ϵ ???
- Need to find an N, such that

$$\left|\frac{2}{n^2+3n}-0\right|<\epsilon$$

holds *for all* $n \geq N$ ⟶ inequality solving (link)

$$\text{InequalitySolve}\left[n>0 \bigwedge \left|\frac{2}{n^2+3\,n}-0\right| < \left(\frac{1}{100}\right)^2, n\right]$$

$$n > \frac{1}{2}\left(-3+\sqrt{80009}\right)$$

$$\text{Table}\left[\text{InequalitySolve}\left[n>0 \bigwedge \left|\frac{2}{n^2+3\,n}-0\right| < \left(\frac{1}{100}\right)^i, n\right], \{i, 1, 5\}\right] \text{ // TableForm}$$

$$n > \tfrac{1}{2}\left(-3+\sqrt{809}\right)$$
$$n > \tfrac{1}{2}\left(-3+\sqrt{80009}\right)$$
$$n > \tfrac{1}{2}\left(-3+\sqrt{8000009}\right)$$
$$n > \tfrac{1}{2}\left(-3+\sqrt{800000009}\right)$$
$$n > \tfrac{1}{2}\left(-3+\sqrt{80000000009}\right)$$

$$\text{InequalitySolve}\left[n>0 \bigwedge \left|\frac{2}{n^2+3\,n}-0\right| < \epsilon, \{\epsilon, n\}\right]$$

$$\epsilon > 0 \,\&\&\, n > -\frac{3}{2}+\frac{1}{2}\sqrt{\frac{8+9\,\epsilon}{\epsilon}}$$

$$\text{InequalitySolve}\left[n>0 \bigwedge \left|\frac{2}{n^2+3\,n}-\frac{1}{10^3}\right| < \epsilon, \{\epsilon, n\}\right]$$

$$\left(0<\epsilon<\frac{1}{1000} \,\&\&\, -\frac{3}{2}+\frac{1}{2}\sqrt{\frac{8009+9000\,\epsilon}{1+1000\,\epsilon}} < n < -\frac{3}{2}+\frac{1}{2}\sqrt{\frac{-8009+9000\,\epsilon}{-1+1000\,\epsilon}}\right) \,||$$
$$\left(\epsilon \geq \frac{1}{1000} \,\&\&\, n > -\frac{3}{2}+\frac{1}{2}\sqrt{\frac{8009+9000\,\epsilon}{1+1000\,\epsilon}}\right)$$

14 of 17

15 of 17

Formal Exploration: Third Level — A Proof

Proposition["g converges to 0",
converges[g, 0]]

Definition["g", any[n],

$g_n := \frac{2}{n^2 + 3\,n}$]

Definition["convergence", any[f, a, ϵ, N],

converges[f, a] :⟺ $\underset{\substack{\epsilon \\ \epsilon>0}}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{\substack{n\in\mathbb{N} \\ n\geq N}}{\forall} |f_n - a| < \epsilon$]

Use[]

Prove[Proposition["g converges to 0"],
using → {Definition["g"], Definition["convergence"]}, by → PCS]

- ProofObject -

◆ Inspect failing proof

Formula (4), using (Definition (g)), is implied by:

(5) $N^* \in \mathbb{N} \bigwedge \underset{n}{\forall} (n \in \mathbb{N} \wedge n \geq N^* \Rightarrow |\frac{2}{n^2+3*n} - 0| < \epsilon_0)$.

The proof of (5) fails. (The prover "QR" was unable to transform the proof situation.)

15 of 17

16 of 17

Formal Exploration: Third Level — A Proof

◆ Inspect failing proof: We can use the knowledge found with InequalitySolve

Lemma["inequality:abs", any[n, ϵ],

$n > -\frac{3}{2} + \frac{1}{2}\sqrt{\frac{8+9\,\epsilon}{\epsilon}} \Rightarrow \left|\frac{2}{n^2 + 3\,n} - 0\right| < \epsilon$]

Prove[Proposition["g converges to 0"],
using → {Definition["g"], Definition["convergence"], Lemma["inequality:abs"]}, by → PCS]

- ProofObject -

◆ Inspect failing proof: What kind of knowledge would be of help?

Formula (4), using (Definition (g)), is implied by:

$N^* \in \mathbb{N} \bigwedge \underset{n}{\forall} (n \in \mathbb{N} \wedge n \geq N^* \Rightarrow |\frac{2}{n^2+3*n} - 0| < \epsilon_0),$

which, using (Lemma (inequality:abs)), is implied by:

(5) $N^* \in \mathbb{N} \bigwedge \underset{n}{\forall} \left(n \in \mathbb{N} \wedge n \geq N^* \Rightarrow n > (-\frac{3}{2}) + \frac{1}{2} * \left(\sqrt{\frac{8+9*\epsilon_0}{\epsilon_0}}\right)\right).$

The proof of (5) fails. (The prover "QR" was unable to transform the proof situation.)

Lemma["ceiling", any[x, y],
 x ≥ ⌈y + 1⌉ ⇒ x > y "ineq"
 ⌈y⌉ ∈ ℕ "nat"]

Prove[Proposition["g converges to 0"], using → {Definition["g"],
 Definition["convergence"], Lemma["inequality:abs"], Lemma["ceiling"]}, by → PCS,
 transformBy → ProofSimplifier,
 TransformerOptions → {steps → Useful, branches → Proved}]

- ProofObject -

Prove:

(Proposition (g converges to 0)) converges[g, 0],

under the assumptions:

(Definition (g)) $\underset{n}{\forall} (g_n := \frac{2}{n^2+3*n}),$

(Definition (convergence)) $\underset{f,a}{\forall} \left(\text{converges}[f, a] :\Leftrightarrow \underset{\substack{\epsilon \\ \epsilon>0}}{\forall} \underset{N\in\mathbb{N}}{\exists} \underset{\substack{n\in\mathbb{N} \\ n\geq N}}{\forall} (|f_n - a| < \epsilon)\right),$

(Lemma (inequality:abs)) $\underset{n,\epsilon}{\forall} \left(n > (-\frac{3}{2}) + \frac{1}{2} * \left(\sqrt{\frac{8+9*\epsilon}{\epsilon}}\right) \Rightarrow |\frac{2}{n^2+3*n} - 0| < \epsilon\right),$

(Lemma (ceiling): ineq) $\underset{x,y}{\forall} (x \geq \lceil y + 1 \rceil \Rightarrow x > y),$

(Lemma (ceiling): nat) $\underset{y}{\forall} (\lceil y \rceil \in \mathbb{N}).$

Formula (Proposition (g converges to 0)), using (Definition (convergence)), is implied by:

(1) $\underset{\substack{\epsilon \\ \epsilon>0}}{\forall} \underset{N\in\mathbb{N}}{\exists} \underset{\substack{n\in\mathbb{N} \\ n\geq N}}{\forall} (|g_n - 0| < \epsilon).$

We assume

(2) $\epsilon_0 > 0,$

and show

(3) $\underset{N\in\mathbb{N}}{\exists} \underset{\substack{n\in\mathbb{N} \\ n\geq N}}{\forall} (|g_n - 0| < \epsilon_0).$

We have to find N^* such that

(4) $N^* \in \mathbb{N} \bigwedge \forall_n (n \in \mathbb{N} \wedge n \geq N^* \Rightarrow |g_n - 0| < \epsilon_0)$.

Formula (4), using (Definition (g)), is implied by:

$N^* \in \mathbb{N} \bigwedge \forall_n (n \in \mathbb{N} \wedge n \geq N^* \Rightarrow |\frac{2}{n^2 + 3 * n} - 0| < \epsilon_0)$,

which, using (Lemma (inequality:abs)), is implied by:

$N^* \in \mathbb{N} \bigwedge \forall_n \left(n \in \mathbb{N} \wedge n \geq N^* \Rightarrow n > (-\frac{3}{2}) + \frac{1}{2} * \left(\sqrt{\frac{8 + 9 * \epsilon_0}{\epsilon_0}}\right)\right)$,

which, using (Lemma (ceiling): ineq), is implied by:

(5) $N^* \in \mathbb{N} \bigwedge \forall_n \left(n \in \mathbb{N} \wedge n \geq \left\lceil \left((-\frac{3}{2}) + \frac{1}{2} * \left(\sqrt{\frac{8 + 9 * \epsilon_0}{\epsilon_0}}\right)\right) + 1 \right\rceil\right)$.

Partially solving it, formula (5) is implied by

(6) $\left\lceil \left((-\frac{3}{2}) + \frac{1}{2} * \left(\sqrt{\frac{8 + 9 * \epsilon_0}{\epsilon_0}}\right)\right) + 1 \right\rceil \in \mathbb{N} \bigwedge \left(N^* = \left\lceil \left((-\frac{3}{2}) + \frac{1}{2} * \left(\sqrt{\frac{8 + 9 * \epsilon_0}{\epsilon_0}}\right)\right) + 1 \right\rceil\right)$.

We can partially solve (6). By taking $N^* \leftarrow \left\lceil \left((-\frac{3}{2}) + \frac{1}{2} * \left(\sqrt{\frac{8 + 9 * \epsilon_0}{\epsilon_0}}\right)\right) + 1 \right\rceil$, formula (6) is implied by

(7) $\left\lceil \left((-\frac{3}{2}) + \frac{1}{2} * \left(\sqrt{\frac{8 + 9 * \epsilon_0}{\epsilon_0}}\right)\right) + 1 \right\rceil \in \mathbb{N}$.

Formula (7) is proved because it is an instance of (Lemma (ceiling): nat).

□

⏮ ◀ ▶ ⏭ 16 of 17

⏮ ◀ ▶ ⏭ 17 of 17

Conclusions

- CreaComp: Give computer-support to graphics and animation (compared to static graphics and pre-computed animations ⟶ Web)
- CreaComp: Give computer-support additionally to “formal mathematics”