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PARAMETRIC CURVES

Abstract

Algebraic curves (and surfaces) play an essential role in various application areas such as geometric design or cryptography. We need efficient algorithms for representing algebraic curves in suitable ways for these applications. In particular, we need to be able to switch efficiently between an implicit representation by a defining polynomial to an explicit presentation by a rational parametrization, if such a parametrization exists. We report on algorithms which give optimal results for such representations.

Key words

Computer algebra, algebraic curves, parametrization

Algebraic curves and surfaces play an important and ever increasing role in computer aided geometric design, computer vision, and computer aided manufacturing. Consequently, theoretical results need to be adapted to practical needs. We need efficient algorithms for generating, representing, manipulating, analyzing, rendering algebraic curves and surfaces. In the last years there has been dramatic progress in all areas of algebraic computation. In particular, the application of computer algebra to the design and analysis of algebraic curves and surfaces has been extremely successful. In this lecture we report on some of these developments.

One interesting subproblem in algebraic geometric computation is the rational parametrization of curves. The tacnode curve defined by $f(x, y) = 2x^4 - 3x^2y + y^4 - 2y^3 + y^2$ in the real plane has the rational parametrization

$$x(t) = \frac{t^3 - 6t^2 + 9t - 2}{2t^4 - 16t^3 + 40t^2 - 32t + 9}, \quad y(t) = \frac{t^2 - 4t + 4}{2t^4 - 16t^3 + 40t^2 - 32t + 9}$$

over $\mathbb{Q}$. The criterion for parametrizability is the genus. Only curves of genus 0 have a rational parametrization. We have described a fully symbolic method for the rational parametrization of plane algebraic curves in [SeWi91].

Computing parametrizations essentially requires the full analysis of singularities (either by successive blow-ups, or by Puiseux expansion) and the determination of regular points on the curve. Whereas the singular points do not contribute to the complexity of the field of parametrization, we can control the quality of the resulting parametrization by controlling the field over which we choose this regular point. Thus, finding a regular curve point over a minimal field extension on a curve of genus 0 is one of the central problems in rational parametrization of curves, compare [SeWi97], [SeWi99]. Such regular points with coordinates in an optimal extension field can be determined by a reduction process due to Hilbert and Hurwitz [HiHu90] which we have analyzed further and improved for computational purposes. The quality of parametrizations can be measured by the necessary field extension and also by the number of times the variety is traced by the parametrization, compare [SeWi01a,b].

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