Combinatorial Sequences: Non-Holonomicity and Inequalities

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Abstract

Holonomic functions (respectively sequences) satisfy linear ordinary differential equations (respectively recurrences) with polynomial coefficients. This class can be generalized to functions of several continuous or discrete variables, thus encompassing most special functions that occur in applications, for instance in mathematical physics. In particular, all hypergeometric functions are holonomic.

This work makes several contributions to the theory of holonomic functions and sequences. In the first part, new methods are introduced to show that a given function or sequence is *not* holonomic. First, number-theoretic methods are applied, and connections to the theory of transcendental numbers are pointed out. A new application of the saddle point method from asymptotic analysis to a concrete function is given, which proves its non-holonomicity.

The second part addresses questions of positivity of holonomic (and more general) sequences. First, two new methods for proving positivity of sequences algorithmically are presented. The first one is limited to holonomic sequences and is based on the signs of the recurrence coefficients. The second method is applicable to a class much larger than the holonomic sequences. Its main idea is the construction of an inductive proof. To perform the induction step, the involved sequences and their shifts are replaced by real variables. The induction step is thus reduced to a (sufficient) system of polynomial equations and inequalities over the reals. Its satisfiability is known to be decidable by Cylindrical Algebraic Decomposition. Our procedure does not terminate in general, but succeeds in automatically proving numerous non-trivial examples from standard textbooks on inequalities.

Finally, solutions of linear recurrences with constant coefficients are considered from the viewpoint of positivity. We show that such sequences, called C-finite, oscillate in certain non-trivial cases, i.e., are neither eventually positive nor eventually negative. To this end, a result from Diophantine geometry, viz. about lattice points in certain regions of the plane, is provided. Furthermore, we investigate the asymptotic density of the positivity set of an arbitrary C-finite sequence. Its existence is established, and its possible values are determined. The methods we use for this belong to the theory of equidistributed sequences.

Zusammenfassung

Holonomische Funktionen (bzw. Folgen) sind dadurch charakterisiert, dass sie lineare gewöhnliche Differentialgleichungen (bzw. Rekurrenzen) mit polynomiellen Koeffizienten erfüllen. Diese Klasse kann auf Funktionen mehrerer diskreter oder stetiger Variablen verallgemeinert werden und umfasst sodann die meisten speziellen Funktionen, die in Anwendungen, etwa in der mathematischen Physik, auftreten, insbesondere alle hypergeometrischen Funktionen.

Diese Arbeit leistet mehrere Beiträge zur Theorie der holonomischen Funktionen und Folgen. Im ersten Teil werden neue Methoden vorgestellt, um zu beweisen, dass eine gegebene Funktion oder Folge *nicht* holonomisch ist. Zunächst werden zahlentheoretische Methoden angewandt und Zusammenhänge zur Theorie der transzendenten Zahlen herausgearbeitet. Dann wird eine neue Anwendung der Methode der Sattelpunkt-Asymptotik auf eine konkrete Funktion präsentiert, welche diese Funktion als nicht holonomisch erweist.

Der zweite Teil behandelt Fragen zur Positivität holonomischer (und allgemeinerer) Folgen. Zunächst werden zwei neue Methoden vorgestellt, um Positivität von Folgen algorithmisch zu beweisen. Die erste beschränkt sich auf holonomische Folgen und basiert auf den Vorzeichen der Koeffizienten der Rekurrenz. Die zweite Methode geht weit über die Klasse der holomischen Folgen hinaus. Ihre Grundidee ist die Konstruktion eines Induktionsbeweises, wobei für den Induktionsschritt die auftretenden Folgen und ihre Shifts durch reelle Variablen ersetzt werden. Der Induktionsschritt wird somit auf ein (hinreichendes) System von reellen polynomiellen Gleichungen und Ungleichungen reduziert, dessen Erfüllbarkeit bekanntlich entscheidbar ist (durch Cylindrical Algebraic Decomposition). Die Prozedur terminiert im allgemeinen nicht, liefert aber automatische Beweise von zahlreichen nichttrivialen Beispielen aus Standardwerken über Ungleichungen.

Schließlich wird das Vorzeichen von Lösungen linearer Rekurrenzen mit konstanten Koeffizienten betrachtet. Es wird gezeigt, dass derartige Folgen in gewissen nichttrivialen Fällen oszillieren, also weder schließlich positiv noch negativ sind. Zu diesem Zweck wird ein Ergebnis aus der diophantischen Geometrie, genauer über Gitterpunkte in speziellen Teilmengen der reellen Ebene, bereitgestellt. Weiters wird die asymptotische Dichte der Positivitätsmenge einer solchen Folge untersucht. Ihre Existenz wird bewiesen, und ihre möglichen Werte werden ermittelt. Die angewandten Methoden stammen aus der Theorie der Gleichverteilung.

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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Contents

Introduction

The present work is concerned with questions that arose in the area of automated identity proving. The starting point of this subject, a subfield of computer algebra, was Gosper's algorithm [41] for indefinite hypergeometric summation. One of the highlights in the field is Zeilberger's adaption [84] of Gosper's idea to definite hypergeometric summation. This discovery has rendered tables of combinatorial identities obsolete to a large extent. Holonomic sequences (respectively functions), characterized in the univariate case by linear recurrences (respectively differential equations) with polynomial coefficients, constitute a class that is even larger than the hypergeometric sequences (respectively functions). The usefulness of holonomicity for proving combinatorial and special function identities was noted and developed by Stanley, Lipshitz, Zeilberger, Takayama, Chyzak, and Schneider. Now several algorithms are available that can prove (and find) identities involving holonomic objects. Consequently, lots of special function identities from classical tables can be proven by computer algebra. Still, we note in passing that not all pertinent algorithms have as yet been implemented in a form that makes them usable by the uninitiated.

The importance of holonomic functions for automated reasoning entails interest in classification results that allow to say which sequences (respectively functions) are holonomic and which are not. Lots of combinatorial sequences are naturally defined by means of a linear recurrence with polynomial coefficients and are thus holonomic. Moreover, there are several ways to construct new holonomic sequences and functions from known ones, such as addition, multiplication, summation, and integration.

So far the toolbox for proving *non*-holonomicity is fairly small, however. Just like numbers that are not algebraic by construction are usually transcendental, a sequence (respectively function) is usually not holonomic, unless it is holonomic by design. In the first part of this work we present several new methods for proving non-holonomicity. First, number-theoretic results are applied to prove non-holonomicity of certain explicitly defined sequences. We deduce the non-holonomicity of non-integral powers of hypergeometric sequences from a degree property of the number fields that are obtained by extending the rational numbers by the sequence values. Furthermore, we use the transcendence of Euler's number to show that the sequence (n^n) is not holonomic. After these number-theoretic considerations, we present a recent method by Flajolet and Salvy. It proceeds by comparing the asymptotics of a given function with a classical theorem about the asymptotic shape of holonomic functions. This motivates an asymptotic investigation of the generating function of $(\mathrm{e}^{1/n})$, carried out by the saddle point method.

The second part of the thesis deals with a different set of questions. Whereas there are several algorithms known for proving identities of various kinds, not much is known concerning automatic proofs of inequalities. A pity, to be sure, since inequalities pop up almost everywhere in mathematics. The inequalities that we have in mind compare objects that can be defined recursively. Holonomic sequences are an example, but they do not allow to formulate too many interesting inequalities. Polynomial recurrences with unspecified parameters are more appropriate and capture lots of interesting inequalities. But defining a class of inequalities does not necessarily shed light on how to design an algorithm for proving them. We present a proving procedure, developed in joint work with M. Kauers, that accepts as input an inequality (to be proven) from a fairly large class. It tries to construct an inductive proof by replacing the sequences involved in the inequality by real variables. The negation of the induction step is translated into a real variable formula. The defining recurrences of the sequences give rise to polynomial equations in the real variables, which are added to the formula. If the resulting formula is unsatisfiable, then we have established the induction step; all that remains to do is to check sufficiently many initial values. Otherwise, we augment the induction hypothesis by assuming the truth of the inequality for more values than before. Note that unsatisfiability of such real variable formulas can be decided by Cylindrical Algebraic Decomposition. We could not yet determine a useful subclass of the input class on which the procedure provably terminates, but present lots of examples where it works.

The algorithms by Zeilberger and Chyzak mentioned at the beginning are known to terminate on fairly large input classes, viz. proper hypergeometric summands and holonomic functions, respectively. Our inequality proving procedure works often, but not always. This motivates the following more modest question: Is there some small class of recursively defined sequences for which we can decide eventual positivity? A natural candidate – maybe the simplest conceivable – are the C-finite sequences, which satisfy linear homogeneous recurrences with constant coefficients. This subclass of the holonomic sequences is very well studied, most notably from the viewpoint of number theory. Still, it is not known whether the problem of eventual positivity of a C-finite sequence is decidable.

Our contributions to this question consist not of an algorithm, but of theorems that clarify the behaviour in some non-trivial cases. First, we invoke results from Diophantine geometry to prove a result about lattice points in rectangles. This result, together with Kronecker's theorem from Diophantine approximation, is applied in an attempt to prove that C-finite sequences with no positive dominating characteristic root oscillate. Our Diophantine result, while interesting in its own right, is two-dimensional and thus limits the number of dominating roots to four. In a collaboration with J. P. Bell we could modify the approach substantially and prove that sequences with no positive dominating root are indeed neither eventually positive nor eventually negative, with no restriction on the number of dominating roots. We prove even more, viz. that the positivity set and the negativity set have positive density. This investigation also yields a proof that the density of the positivity set of a C-finite sequence always exists, without any restriction on its characteristic roots. Furthermore, we show that each number from the closed unit interval occurs as density of the positivity set of some C-finite sequence. If we restrict attention to sequences without positive dominating roots, then the analogous statement holds for the open unit interval. We also show that the density of the zero set of a C-finite sequence is a rational number. This can be read as a weak version of the Skolem-Mahler-Lech theorem.

The term *combinatorial sequences* in the title demands some explanation. Most of the non-holonomic sequences that we will encounter in Part I cannot be called combinatorial with a clear conscience. Still, the subject of holonomicity certainly has a combinatorial flavour, and *sequences* alone would not be very enlightening, either. The term *recurrence sequences* would be misleading, too, since in the literature it is often reserved for C-finite sequences.

Except for our inequality proving procedure, the new results in this thesis are theorems, not algorithms. They are motivated by questions of symbolic computation, however. Whereas in the proofs some fairly deep results from various areas are applied, most of the problems that we tackle can be formulated and motivated without deep theory. Finally, we mention that the work described herein has lead to six papers [6, 30, 31, 37, 38, 39], four of which are already published.

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Notation

Notation	*Meaning*	*Definition on page*
$\sim$	asymptotically equal to	
$\sharp$	cardinality of a set	
$\langle \cdot, \cdot \rangle$	scalar product	
$\Vert \cdot \Vert_2$	Euclidean norm	
!!	double factorial	43
$(\cdot)_n$	rising factorial (Pochhammer symbol)	8
$[z^n]$	power series coefficient	
$\overline{z}$	complex conjugate of z	
$\equiv 0$	vanishes identically	64
$\gg 0$	sufficiently large	
$\mathbf{1}_A$	characteristic function of the set A	
$\mathbb{C}$	complex numbers	
$\mathbf{d}$	determinant of a lattice	50
$\boldsymbol{\delta}$	density of a set of natural numbers	60
e	Euler's number (base of the natural logarithm)	
E	forward shift	33
γ	Euler-Mascheroni constant	
Γ	Gamma function	
H_n	harmonic numbers	14
$\mathrm{H}_n^{(j)}$	jth order harmonic numbers	15
i	imaginary unit	
$\Im$	imaginary part	
$L_g(\mathbf{u})$	lattice of multiples of $\mathbf{u}$ modulo g	51
λ_k	successive minimum	54
$\boldsymbol{\lambda}$	Lebesgue measure	
lc	leading coefficient	
Li_k	polylogarithm	15
log	natural logarithm	
$\mathbb{N}$	natural numbers $0, 1, 2, \ldots$	
o, O	Landau symbols	
$\mathbb{O}$	set of linear recurrence operators with polynomial coefficients	33
P_n	Legendre polynomials	
$P_n^{(\alpha,\beta)}$	Jacobi polynomials	
$\mathbb{Q}$	rational numbers	
$\mathbb{R}$	real numbers	
$\mathcal{R}_{\lambda_1,\lambda_2}(\mathbf{c})$	rectangle with side lengths $2\lambda_1$, $2\lambda_2$ centered at $\mathbf{c}$	49
$\Re$	real part	
$\mathcal{S}_\lambda(\mathbf{c})$	square with side length 2λ centered at $\mathbf{c}$	50
W	Lambert W function	12
$\mathcal{W}$	set of linear recurrence operators with a certain sign pattern	34
$\mathbb{Z}$	integers	
ζ	Riemann zeta function	

Part I

Non-Holonomicity

Chapter 1

Proving Non-Holonomicity of Sequences and Functions

(a) Develop general methods for determining when a power series is D-finite.

— Richard P. Stanley [78, Miscellaneous problems and examples]

1.1 Definition and Significance of Holonomicity

There are many ways to define concrete real or complex functions. We exemplify some of them by the example of the sine function. The oldest and most elementary definition is the geometric one. Then there is the Taylor series

$$\sin x := \sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

In hypergeometric series notation [2] this reads as

$$\sin x := x\ {}_0F_1\left(\begin{matrix} - \\ 3/2 \end{matrix}\ ;\ -x^2/4\right).$$

The 'explicit' definition

$$\sin x := \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}}$$

uses another previously defined function that has to be defined somehow itself. There is also a definition that involves integration and functional inversion: The sine function is the inverse of the map

$$x \mapsto \int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^2}}.$$

A fifth way is via differential equations. The function $f(x) = \sin x$ is the unique solution of the initial value problem

$$f''(x) + f(x) = 0,$$
$$f(0) = 0, \quad f'(0) = 1.$$

This shows that the sine function is a member of the following class [78].

Definition 1.1.1. *A function $f(z)$ analytic at zero (or a formal power series) is* holonomic *if it satisfies a linear ordinary differential equation*

$$p_0(z)f(z) + p_1(z)\frac{\mathrm{d}f(z)}{\mathrm{d}z} + \cdots + p_d(z)\frac{\mathrm{d}^d f(z)}{\mathrm{d}z^d} = 0 \tag{1.1}$$

with polynomial coefficients $p_k(z)$, not all identically zero.

Holonomic functions are also called *D-finite*. Working with such differential equations as function definitions has some attractive features. First, a great many of the classical special functions are holonomic, e.g., all hypergeometric functions. Furthermore, representability in finite terms is a prerequisite for doing computations. There are symbolic algorithms, for problems like adding two holonomic functions or performing differentiation and indefinite integration, and numerical ones.

What is true for functions, holds for sequences of numbers, too: Doing computations requires picking a subclass with finitely representable members from the uncountable realm of real (or complex) sequences. A useful class is defined analogously to holonomic functions.

Definition 1.1.2. *A* holonomic *sequence is a sequence $(a_n)_{n\geq 0}$ of complex numbers that satisfies a recurrence*

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_d(n)a_{n+d} = 0, \qquad n \geq 0, \tag{1.2}$$

with polynomial coefficients $p_k(n)$, not all identically zero.

The names *P-recursive, P-finite,* and *D-finite* are also in use. Examples of holonomic sequences include all hypergeometric sequences and many other combinatorial sequences, such as the harmonic numbers and the derangement numbers [43]. The analogy between Definitions 1.1.1 and 1.1.2 goes beyond their obvious similarity. It is not difficult to show [78] that a sequence (a_n) is holonomic if and only if its generating function $f(z) = \sum_{n\geq 0} a_n z^n$ is holonomic (the order of the recurrence and the differential equation need not agree). Holonomic sequences and functions enjoy numerous closure properties, some of which are summarized in the following result [78].

Theorem 1.1.3. *(i) The set of holonomic sequences (respectively functions) is a $\mathbb{C}$-algebra.*

(ii) Holonomic functions are closed under algebraic substitution, i.e., $f(g(z))$ is holonomic if $f(z)$ is holonomic and $g(z)$ is algebraic.

There are algorithms for performing these (and other) closure properties. They are implemented in Mallinger's Mathematica package `GeneratingFunctions` [61] and in Salvy and Zimmermann's Maple package `gfun` [76]. A basic example of using these packages is as follows: Suppose we are given two holonomic sequences (a_n) and (b_n) in terms of their recurrences and initial values, and we want to know whether they are equal. Then we can compute a recurrence for their difference $(a_n - b_n)$ and prove or refute equality by checking sufficiently many initial values. Equality of holonomic functions can be checked similarly.

Needless to say, in algorithmics the coefficients of the polynomial recurrence (respectively ODE) coefficients and the initial values of the sequence (respectively function) have to lie in a computable subfield of $\mathbb{C}$. Since we are not concerned with computations, but with non-holonomicity in this chapter, we need not pay attention to this constraint for now.

The original definition of multivariate holonomic functions $f(z_1, \dots, z_r)$, due to Bernstein, is rather technical [86]. Fortunately, it has been shown [49] that holonomicity is equivalent to ∂-finiteness. The latter means that the partial derivatives of a function span a finite-dimensional vector space over the field of multivariate rational functions. Holonomicity of multivariate sequences $a(n_1, \dots, n_s)$ and mixed continuous/discrete functions $f(z_1, \dots, z_r, n_1, \dots, n_s)$ is defined by passing to the generating function w.r.t. the discrete arguments. The closure properties addition, multiplication and algebraic substitution carry over to the multivariate setting. Since summation and integration (both definite and indefinite) also preserve holonomicity [86], many combinatorial and special function identities can be expressed in terms of holonomic objects. This observation has been turned into widely applicable algorithms for identity proving by Zeilberger [84, 86], Takayama [81], and Chyzak [18, 19].

1.2 Known Non-Holonomicity Results

The role of the holonomic sequences in the set of all complex sequences is reminiscent of the role of the algebraic numbers in the set of complex numbers. In this spirit, the analogue of the venerable theory of transcendental numbers is the theory of non-holonomic sequences, which has certainly by no means attained the breadth and impact of the former. Still, we hope to show in the first part of this thesis that the subject of non-holonomicity has attractive relations to several areas of mathematics and promises ample opportunities for future research. On a more down-to-earth scale, proving a sequence to be non-holonomic provides a lower bound concerning its complexity. In all likelihood, there will never be a fully general method for determining (non-)holonomicity. After all, there are too many ways to define sequences (respectively functions), nor is it likely that there are properties (e.g., asymptotic ones) that ensure (non-)holonomicity and can be checked always.

An easy way to show non-holonomicity is to appeal to the following upper bound that restricts the growth rate of holonomic sequences. It is known to hold in the more general setting of coefficients of differentially algebraic[1] power series [60], but we give a short inductive proof for the special case that we are interested in. The idea behind the bound is not difficult to grasp: Due to the polynomial recurrence coefficients, we can bound $|a_{n+1}|$ by $|a_n|$ times a factor of polynomial growth, resulting in the bound $\mathrm{O}(n!^\alpha)$.

Proposition 1.2.1. *Let (a_n) be a holonomic sequence. Then there is a constant α such that $|a_n| \le n!^\alpha$ for all $n \ge 2$.*

Proof. There are rational functions $r_k(n)$ such that

$$a_{n+d} = \sum_{k=0}^{d-1} r_k(n) a_{n+k}, \qquad n \gg 0,$$

and there are positive constants c and β such that for all k

$$|r_k(n)| \le c n^\beta, \qquad n \gg 0.$$

[1] A power series is called *differentially algebraic*, if some of its derivatives are algebraically dependent over the field of rational functions.

Take $n_0 \in \mathbb{N}$ with $n_0 \geq \max(cd, 2)$ and such that the two preceding formulas hold for $n \geq n_0$. Let α be such that $\alpha \geq \beta + 1$ and $|a_{n+d}| \leq (n+d)!^\alpha$ for $n_0 - d \leq n \leq n_0 - 1$. We show by induction that $|a_n| \leq n!^\alpha$ for $n \geq n_0$, that is to say,

$$|a_{n+d}| \leq (n+d)!^\alpha, \qquad n \geq n_0 - d. \tag{1.3}$$

For $n_0 - d \leq n \leq n_0 - 1$ the inequality (1.3) holds by the choice of α. Assume (1.3) holds up to $n-1$ for an arbitrary, but fixed $n \geq n_0$. Then

$$\begin{aligned} |a_{n+d}| &\leq \sum_{k=0}^{d-1} |r_k(n)||a_{n+k}| \\ &\leq dcn^\beta\,(n+d-1)!^\alpha \\ &\leq n^{\beta+1}\,(n+d-1)!^\alpha \\ &\leq (n+d)^\alpha (n+d-1)!^\alpha = (n+d)!^\alpha. \end{aligned}$$

We have established $|a_n| \leq n!^\alpha$ for $n \geq n_0$, and it is clear that α can be enlarged to make the estimate true for all $n \geq 2$. □

Proposition 1.2.1 shows the non-holonomicity of the sequences (2^{n^2}), $(n!^n)$, and (2^{2^n}). On the other hand, $n^n = \mathrm{O}(n!^2)$ does not grow fast enough for this coarse criterion. We will return to this sequence in Sections 1.4 and 1.5.

There are a few deeper and more useful methods available for proving non-holonomicity. A comprehensive overview has been given in a recent article by Flajolet, Salvy and the author [30]. The new method that was introduced there will be explained in Section 1.5 and applied to a fresh example in Section 1.6.

Very recently, M. Klazar [53] and J. Bell [5] have independently devised methods for proving non-holonomicity that are based on the following observation: If a sequence (a_n) is defined via some (real or complex) function $a(z)$ by $a_n := a(n)$, then the recurrence (1.2) asserts that the function

$$z \mapsto \sum_{k=0}^{d} p_k(z) a(z+k) \tag{1.4}$$

vanishes for $z \in \mathbb{N}$. Investigating this function by analytic methods is a promising way of showing non-holonomicity, in which the author is beginning to take part, too. Klazar [53] has successfully applied his method to the sequence $(\log n)$. He shows by arguments from real analysis (notably Rolle's theorem) that the function (1.4) has only finitely many positive real zeros in this case. The first to establish the non-holonomicity of $(\log n)$ were Flajolet and Salvy [30]. The author gave a conditional proof in a previous article [37]; see Section 1.3. As for Klazar's method, the author has observed that Carlson's theorem [2, 13, 74] strongly restricts the possible growth of analytic functions that vanish for $z \in \mathbb{N}$, but are not identically zero. Applying this theorem to the function (1.4) should lead, for instance, to a proof that $(n!^\alpha)$ is not holonomic for non-integral α. This was mentioned in passing and without proof by Lipshitz [59, Example 3.4 (i)]. Bell [5] has shown non-holonomicity of several sequences, for example $(\mathrm{e}^{\sqrt{n}})$, by another approach, which is based on the identity theorem for analytic functions. Note, finally, that both $\log n$ and $\mathrm{e}^{\sqrt{n}}$ pop up as main asymptotic term of some holonomic sequence.

The theory of non-holonomicity is somewhat more advanced for functions than for sequences. For instance, it is known that division, exponentation and composition of holonomic functions 'almost never' lead to holonomic functions, except in the obvious cases (e.g., algebraic substition, cf. Theorem 1.1.3 (ii)). See the above-mentioned survey article [30] for references. So far there seems to be only one result of this kind for sequences: The reciprocal $(1/a_n)$ of a holonomic sequence is holonomic if and only if (a_n) is an interlacing of hypergeometric sequences. This has been shown by van der Put and Singer [82, Chap. 4] by Galois theory of difference equations. On the other hand, we do not know, for instance, if (e^{a_n}) can be holonomic for a holonomic non-linear sequence (a_n).

Returning to reciprocals, the following example, communicated to me by Frédéric Chyzak[2], shows that the reciprocal of a holonomic sequence is not holonomic in general.

Proposition 1.2.2. *Let α, β be complex numbers with $|\alpha| > |\beta| > 0$, and $a_n := \alpha^n + \beta^n$. Then the sequence (a_n) is holonomic, but $(1/a_n)$ is not.*

Proof. The holonomicity of (a_n) follows from Theorem 1.1.3 (i). Now suppose that a recurrence

$$\sum_{k=0}^{d} \sum_{i=0}^{e} \frac{c_{ki} n^i}{\alpha^{n+k} + \beta^{n+k}} = 0, \qquad n \geq 0,$$

with complex numbers c_{ki} holds. Multiplying by α^n and interchanging the order of summation yields

$$\sum_{i=0}^{e} n^i \sum_{k=0}^{d} \frac{c_{ki}}{\alpha^k + \beta^k (\beta/\alpha)^n} = 0, \qquad n \geq 0.$$

The inner sum tends to a constant (depending on i) as $n \to \infty$, and the powers of n constitute an asymptotic basis. Therefore, the inner sum must vanish for large n. Hence, for each i the rational function

$$\sum_{k=0}^{d} \frac{c_{ki}}{\alpha^k + \beta^k X}$$

vanishes for infinitely many distinct values of X, which implies that it is identically zero. Thus, all c_{ki} are zero. □

The multivariate case. So far we have only talked about univariate functions and sequences in this section. In order to show non-holonomicity of multivariate sequences, one can sometimes appeal to closure under taking diagonals: Lipshitz [58] has shown that the univariate sequence $a(n, \dots, n)$ is holonomic for any multivariate holonomic sequence $a(n_1, \dots, n_s)$. For instance, the bivariate sequence (n^k) is not holonomic, since its diagonal sequence (n^n) is not (see Section 1.4).

Another possible tool is Bernstein's elimination lemma [86], which is the bedrock of most of the algorithms that prove identities involving multivariate holonomic functions and sequences. In the discrete bivariate case it asserts that every holonomic sequence $a(n, k)$ satisfies a linear recurrence involving shifts in n and k and coefficients polynomial in n but independent of k. This result has been applied by Wilf and Zeilberger [84, Section 1.5] to show that the bivariate sequence $1/(n^2 + k^2)$ is not holonomic, and by Lipshitz [59, Example 3.4] to do the same for the sequence $1/(n^2 + k)$. In Section 1.3 we will give an entirely different argument that shows that the 'mixed' bivariate continuous/discrete function $1/(z^2 + n)$ is not holonomic. Apart from that, we will not consider multivariate holonomic functions or sequences in the present work.

[2] I could not track down the original source, but F. Chyzak believes to have seen it (at least for some special values of α and β) published somewhere.

1.3 The Number Fields Generated by the Sequence Values

In this section we present a new method [37] for proving non-holonomicity of a given sequence (a_n). It proceeds by studying the chain of field extensions $\mathbb{Q}(\{a_k : 0 \leq k < n\})$ for $n \in \mathbb{N}$.

As a consequence of Theorem 1.1.3 (ii), all algebraic functions are holonomic. One could ask whether this extends to sequences. Clearly, all sequences defined by rational functions in n are holonomic. Essentially, no other algebraic holonomic sequences are known, so we pose the following question: If a holonomic sequence (a_n) satisfies

$$Q(n, a_n) = 0, \qquad n \gg 0,$$

for some nonzero bivariate polynomial $Q(X, Y)$, does it then follow that a_n is an interlacing of rational functions of n? We do not know the answer, but a result in this direction will be given in the present section: The sequence (n^r) is non-holonomic for $r \in \mathbb{Q} \setminus \mathbb{Z}$. More generally, we establish in Theorem 1.3.1 that fractional powers of hypergeometric sequences

$$h_n = \frac{(u_1)_n \cdots (u_p)_n}{(v_1)_n \cdots (v_q)_n}, \qquad n \geq 0, \tag{1.5}$$

where $(c)_n$ denotes the rising factorial

$$(c)_n := \prod_{i=1}^{n} (c + i - 1),$$

are not holonomic under certain assumptions on the parameters u_i, v_i. A generalization into another direction will be obtained in Section 1.5, where we will present a proof by Flajolet and Salvy of the non-holonomicity of (n^r) for any complex non-integral r. The main result of the present section reads as follows.

Theorem 1.3.1. *Let $u_1, \ldots, u_p, v_1, \ldots, v_q$ be pairwise distinct positive integers (possibly $p = 0$ or $q = 0$, but not both). Define the sequence (h_n) by (1.5), and let $r \in \mathbb{Q} \setminus \mathbb{Z}$. Then the sequence (h_n^r) is not holonomic.*

We begin the proof of Theorem 1.3.1 with providing a result that is useful in many non-holonomicity proofs.

Lemma 1.3.2. *Let $\mathbb{K}$ be a subfield of $\mathbb{C}$ and (a_n) be a holonomic sequence with values in $\mathbb{K}$. Then (a_n) satisfies a recurrence of the form (1.2), where the $p_k(n)$ have coefficients in $\mathbb{K}$.*

Proof. Suppose

$$\sum_{k=0}^{d} p_k(n) a_{n+k} = 0 \qquad \text{with} \qquad p_k(n) = \sum_{i=0}^{m_k} c_{ki} n^i, \qquad c_{ki} \in \mathbb{C}, \tag{1.6}$$

and set $m := m_0 + \cdots + m_d + d + 1$. Since $a_n \in \mathbb{K}$, for each n the recurrence (1.6) gives rise to a linear equation $\mathbf{v}_n^T \mathbf{c} = 0$ with $\mathbf{v}_n \in \mathbb{K}^m$ that is satisfied by the coefficient vector

$$\mathbf{c} := (c_{00}, \ldots, c_{0m_0}, \ \ldots \ , c_{d0}, \ldots, c_{dm_d})^T \in \mathbb{C}^m.$$

We may assume that (a_n) is not the zero sequence (otherwise the statement of the lemma is trivial), hence not all $\mathbf{v}_n$ are the zero vector. Let s be maximal such that there are s vectors $\mathbf{v}_{n_1}, \ldots, \mathbf{v}_{n_s}$ that are linearly independent over $\mathbb{C}$. We have $s < m$, since $\mathbf{c} \neq 0$. The linear system

$$\mathbf{v}_{n_1}^T \mathbf{c} = 0$$
$$\vdots$$
$$\mathbf{v}_{n_s}^T \mathbf{c} = 0$$

with coefficients in $\mathbb{K}$ has more unknowns than equations, hence there is a solution $0 \neq \tilde{\mathbf{c}} \in \mathbb{K}^m$. Since any vector $\mathbf{v}_n$ is a $\mathbb{C}$-linear combination of $\mathbf{v}_{n_1}, \ldots, \mathbf{v}_{n_s}$, the vector $\tilde{\mathbf{c}}$ satisfies $\mathbf{v}_n^T \tilde{\mathbf{c}} = 0$ for all n. We obtain the desired recurrence for (a_n) by replacing each c_{ki} in (1.6) with the corresponding entry of $\tilde{\mathbf{c}}$. □

Another proof of Lemma 1.3.2 has been given by Lipshitz [59].

Proof of Theorem 1.3.1. We assume that (h_n^r) is holonomic. Write $r = \frac{\alpha}{\beta}$ with $\beta > 0$ and $\gcd(\alpha, \beta) = 1$, and take integers α', β' such that $\alpha'\alpha + \beta'\beta = 1$.

Case 1. $\alpha' > 0$. The sequence (h_n) is holonomic. Observe that h_n^{-1} is of the form (1.5), too, hence it is also holonomic. By Theorem 1.1.3 (i), we find that

$$(h_n^r)^{\alpha'} h_n^{\beta'} = h_n^{\frac{1-\beta'\beta}{\beta}} h_n^{\beta'} = h_n^{1/\beta}$$

is holonomic.
Case 2. $\alpha' < 0$. In this case

$$(h_n^r)^{-\alpha'} h_n^{-\beta'} = h_n^{\frac{\beta'\beta-1}{\beta}} h_n^{-\beta'} = h_n^{-1/\beta}$$

is holonomic.
Case 3. $\alpha' = 0$. This cannot happen since $\beta \neq \pm 1$.

We assume that we are in case 1. Case 2 can be reduced to case 1 by replacing h_n with h_n^{-1}. For any integer $s \geq 2$ we define

$$\mathbb{K}_s := \mathbb{Q}(2^{1/\beta}, 3^{1/\beta}, \ldots, s^{1/\beta}).$$

Then $\mathbb{K} := \bigcup_{s \geq 2} \mathbb{K}_s$ is a field. Indeed, $\mathbb{K}$ is the intersection of all subfields of $\mathbb{C}$ that contain the set $\{s^{1/\beta} \mid s \in \mathbb{N}\}$. Since $h_n^{1/\beta} \in \mathbb{K}$ for all n, by Lemma 1.3.2 the sequence $(h_n^{1/\beta})$ satisfies a recurrence

$$\sum_{k=0}^{d} p_k(n) h_{n+k}^{1/\beta} = 0, \qquad n \geq 0,$$

where the $p_k(n)$ are polynomials with coefficients in $\mathbb{K}$. There is an integer s_0 such that all these coefficients are in $\mathbb{K}_{s_0}$. For simplicity of notation assume

$$u_1 = \max(u_1, \ldots, u_p, v_1, \ldots, v_q). \tag{1.7}$$

Now choose n_0 larger than the roots of p_d and such that $n_1 := u_1 + n_0 + d - 1$ is larger than s_0 and prime. Then

$$\begin{aligned} h_{n_0+d}^{1/\beta} &= n_1^{1/\beta} \left(\frac{(u_1)_{n_0+d-1}(u_2)_{n_0+d} \cdots (u_p)_{n_0+d}}{(v_1)_{n_0+d} \cdots (v_q)_{n_0+d}} \right)^{1/\beta} \\ &= -p_d(n_0)^{-1} \sum_{k=0}^{d-1} p_k(n_0) h_{n_0+k}^{1/\beta} \end{aligned}$$

implies

$$n_1^{1/\beta} \in \mathbb{K}_{n_1-1}. \tag{1.8}$$

(In the case where the maximum in (1.7) occurs among the denominator parameters v_i it is important to note $h_{n_0+d}^{1/\beta} \neq 0$.) But

$$\mathbb{K}_{n_1-1} = \mathbb{Q}(\rho_1^{1/\beta}, \ldots, \rho_t^{1/\beta}),$$

where $\rho_1, \ldots, \rho_t$ are the primes smaller than n_1, and by Galois theory [36, Section 4.12], the degree of this field over $\mathbb{Q}$ is

$$[\mathbb{K}_{n_1-1} : \mathbb{Q}] = [\mathbb{Q}(\rho_1^{1/\beta}, \ldots, \rho_t^{1/\beta}) : \mathbb{Q}] = \beta^t.$$

Adjoining $n_1^{1/\beta}$ would enlarge the degree to β^{t+1}, hence (1.8) is impossible. This contradiction shows that (h_n^r) is not holonomic. ☐

As an application we show that $f(z, n) = 1/(z^2 + n)$ is not holonomic. We have not given the definition of holonomicity for functions $f(z_1, \ldots, z_r, n_1, \ldots, n_s)$ of several continuous and several discrete arguments, but it suffices to know that definite integration preserves holonomicity [86], as mentioned in Section 1.1, where we have also hinted at the significance of multivariate holonomic functions in automated reasoning. For $n \geq 1$ we have

$$\int_0^\infty \frac{\mathrm{d}z}{z^2 + n} = \frac{1}{\sqrt{n}} \arctan \frac{z}{\sqrt{n}} \bigg|_{z=0}^{\infty} = \frac{\pi}{2\sqrt{n}},$$

thus $1/(z^2 + n)$ is not holonomic by Theorem 1.3.1.

The proof of Theorem 1.3.1 immediately yields the following criterion.

Proposition 1.3.3. *If there are infinitely many n such that*

$$a_n \notin \mathbb{Q}(\{a_k : 0 \leq k < n\}),$$

then the sequence (a_n) is not holonomic.

With this criterion we can give a conditional proof that the sequence $(\log n)$ is not holonomic, assuming the following weak form of Schanuel's conjecture.

Conjecture 1.3.4. *Suppose that $\alpha_1, \ldots, \alpha_s \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$, and that $\mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_s}$ are integers. Then $\alpha_1, \ldots, \alpha_s$ are algebraically independent.*

The standard form of Schanuel's conjecture asserts the following [17]: If $\alpha_1, \ldots, \alpha_s$ are complex numbers linearly independent over $\mathbb{Q}$, then the transcendence degree of the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_s, \mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_s})$ over $\mathbb{Q}$ is at least s. A proof of this well-known conjecture would imply many known results from the theory of transcendental numbers, and solve open problems such as the transcendence of $\mathrm{e} + \pi$.

Proposition 1.3.5. *If Conjecture 1.3.4 holds, then the sequence* $(\log n)$ *is not holonomic.*

Proof. For distinct primes $\rho_1, \ldots, \rho_s$, the numbers $\log \rho_1, \ldots, \log \rho_s$ are linearly independent over $\mathbb{Q}$, since for all $c_1, \ldots, c_s \in \mathbb{Z}$ with

$$\sum_{i=1}^{s} c_i \log \rho_i = \log(\rho_1^{c_1} \ldots \rho_s^{c_s}) = 0$$

we have $\rho_1^{c_1} \ldots \rho_s^{c_s} = 1$, hence $c_i = 0$ for $i = 1, \ldots, s$. By Conjecture 1.3.4, the numbers $\log \rho_1, \ldots, \log \rho_s$ are algebraically independent, and thus the assumption of Proposition 1.3.3 is satisfied. □

Unconditional proofs of the non-holonomicity of $(\log n)$ have been given by Flajolet and Salvy [30], see Section 1.5, and by Klazar [53], see Section 1.2.

1.4 The Sequence (n^n) and the Lambert W Function

We have mentioned in Section 1.2 that the sequence (n^n) does not grow too fast to be ruled out by the general growth restriction (Proposition 1.2.1) for holonomic sequences. We will now show that the non-holonomicity of this sequence follows from the transcendence of Euler's number e. This exhibits a connection between non-holonomicity and transcendence beyond the immediate relation mentioned at the beginning of Section 1.2.

Theorem 1.4.1. *For rational numbers* a, b *with* $b \neq 0$*, the sequence* $((a+n)^{bn})_n$ *is not holonomic.*

Proof. By Theorem 1.1.3 (i) we may assume $b \in \mathbb{Z}$. Now the entries of the sequence are in $\mathbb{Q}$, and if it was holonomic, then by Lemma 1.3.2 there would be polynomials $p_k(n)$ with rational coefficients, $p_d(n)$ not identically zero, such that

$$\sum_{k=0}^{d} p_k(n)(n+a+k)^{b(n+k)} = 0, \qquad n \geq 0.$$

Multiplying both sides with n^{-bn} yields

$$\sum_{k=0}^{d} (n+a+k)^{bk} p_k(n) \left(1 + \frac{a+k}{n}\right)^{bn} = 0.$$

Putting

$$m := \max_{0 \leq k \leq d} (\deg p_k + bk) \qquad \text{and} \qquad M := \{k \mid \deg p_k + bk = m\},$$

we find

$$\sum_{k=0}^{d} n^{bk} p_k(n) \left(1 + \frac{a+k}{n}\right)^{bn} = \mathrm{O}(n^{m-1}) \qquad \text{as} \quad n \to \infty,$$

hence (lc denotes the leading coefficient)

$$\sum_{k \in M} \mathrm{lc}(p_k) \left(1 + \frac{a+k}{n}\right)^{bn} = \mathrm{O}(n^{-1}).$$

Now we take the limit $n \to \infty$. We obtain

$$\sum_{k \in M} \mathrm{lc}(p_k) \mathrm{e}^{b(a+k)} = 0,$$

hence

$$\sum_{k \in M} \mathrm{lc}(p_k) \mathrm{e}^{bk} = 0.$$

This contradicts the transcendence of e^b, a consequence of the Lindemann-Weierstraß theorem [4]. □

By the equivalence of holonomicity of a sequence and its generating function, this implies that the Lambert W function [23], defined implicitly by

$$W(z)\mathrm{e}^{W(z)} = z,$$

is not holonomic, by virtue of the series expansion

$$W(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n.$$

Another proof of this fact will be given in Section 1.5. Before concluding our investigations on non-holonomicity proofs by number theory, a final remark is appropriate. In the present section and the previous one we have applied two number-theoretic results, viz. on the degree of certain algebraic number fields and the transcendence of powers of e, to prove non-holonomicity. Furthermore, we have shown that a special case of Schanuel's conjecture implies the non-holonomicity of $(\log n)$. One could try to turn our arguments around and deduce number-theoretic results from non-holonomicity results. This interesting observation is due to George E. Andrews. For instance, the non-holonomicity of $(\log n)$, which has been shown by Flajolet and Salvy [30] by the method to be presented in Section 1.5, could shed some light on Schanuel's conjecture.

1.5 Asymptotic Structure of Holonomic Functions[3]

A powerful method for proving non-holonomicity is due to Flajolet and Salvy [30]. It appeals to a classical theorem (Theorem 1.5.1 below) that restricts the possible asymptotic behaviour of a holonomic function near a singularity. This result immediately excludes many functions from the holonomic universe. If a function itself is not obviously incompatible to the theorem, then one can try to invoke closure properties in order to arrive at an appropriate function with 'forbidden' asymptotics.

There is a similar structure theorem for difference equations of the form (1.2) (even for more general coefficients than just polynomials) due to Birkhoff and Trjitzinsky [10]. It asserts an asymptotic expansion of solutions of such recurrences in terms of logarithms, powers of n, exponentials, and terms of the form n^{cn}. This result implies, e.g., that the sequence $(\log \log n)$ (or any sequence asymptotic to it) is not holonomic. The drawback of this reasoning is that Birkhoff and Trjitzinsky's work is regarded as almost impenetrable and maybe even faulty. We will give a few further comments on Birkhoff and Trjitzinsky's result in Section 2.1.

[3]The content of this section is due to Philippe Flajolet and Bruno Salvy.

To stay on safe ground, we can start with the asymptotics of some sequence (a_n) in question, deduce asymptotic information on its generating function, and then apply the structure theorem for differential equations, as above. Abelian theorems provide results for the transition from the asymptotics of a sequence to the asymptotics of its generating function. Once again, if the sequence itself does not have 'bad enough' asymptotics, then sometimes we can remedy this by subtracting some holonomic sequence or making use of other holonomic closure properties. The announced structure theorem for solutions of ordinary differential equations reads as follows [83, Theorem 19.1].

Theorem 1.5.1 (Structure theorem for singularities). *Let there be given a differential equation of the form* (1.1)*, a singular point z_0, and a sector S with vertex at z_0. Then, for z in a sufficiently narrow subsector S' of S and for $|z - z_0|$ sufficiently small, there exists a basis of d linearly independent solutions to* (1.1)*, such that any solution Y in that basis admits as $z \to z_0$ in the subsector an asymptotic expansion of the form*

$$Y \sim \exp\left(P(Z^{-1/r})\right) z^\alpha \sum_{j=0}^{\infty} Q_j(\log Z) Z^{js}, \qquad Z := (z - z_0), \tag{1.9}$$

where P is a polynomial, r a natural number, α a complex number, s a positive rational number, and the Q_j are a family of polynomials of uniformly bounded degree. The quantities r, P, α, s, Q_j depend on the particular solution, and the formal asymptotic expansions of (1.9) *are $\mathbb{C}$-linearly independent.*

Examples of functions that do not satisfy (1.9) include iterated logarithms, functions with expansions involving logarithms with unbounded exponents, and many other functions, e.g., $\exp(e^z - 1)$ and $\exp(\sqrt{\log z})$. For the Lambert W function (already encountered in Section 1.4) we have [25, p. 26]

$$W(z) \underset{z\to+\infty}{=} \log z - \log\log z + O(1).$$

Thus, subtracting the holonomic function $\log z$ from $W(z)$ gives a function asymptotic to $-\log\log z$, which is non-holonomic by Theorem 1.5.1. The non-holonomicity of $W(z)$ now follows from closure under addition, which provides a shorter proof than Theorem 1.4.1.

If we are given a sequence instead of a function, then we can appeal to results like the following [9, Corollary 1.7.3].

Theorem 1.5.2 (Basic Abelian theorem). *Let $\phi(x)$ be any of the functions*

$$x^\alpha (\log x)^\beta (\log\log x)^\gamma, \qquad \alpha \geq 0, \quad \beta, \gamma \in \mathbb{C}. \tag{1.10}$$

Let (a_n) be a sequence that satisfies the asymptotic estimate

$$a_n \underset{n\to\infty}{\sim} \phi(n).$$

Then the generating function

$$f(z) := \sum_{n\geq 0} a_n z^n$$

satisfies the asymptotic estimate

$$f(z) \underset{z\to 1^-}{\sim} \Gamma(\alpha+1) \frac{1}{(1-z)} \phi\left(\frac{1}{1-z}\right). \tag{1.11}$$

This estimate remains valid when z tends to 1 in any sector with vertex at 1, symmetric about the horizontal axis, and with opening angle $< \pi$.

As an application, we show that the n-th prime sequence (ρ_n) is not holonomic. From Cipolla's [20] estimate

$$\rho_n = n \log n + n \log \log n + \mathrm{O}(n)$$

we conclude

$$\rho_n/n - \mathrm{H}_n = \log \log n + \mathrm{O}(1).$$

Theorems 1.5.1 and 1.5.2 complete the argument, since multiplication by the rational sequence $(1/n)$ and subtraction of the harmonic numbers $\mathrm{H}_n := \sum_{k=1}^{n} 1/k$ preserve holonomicity.

Other sequences that have been shown to be non-holonomic [30] in this way include $(\log n)$ and (n^r) for $r \in \mathbb{C} \setminus \mathbb{Z}$ (cf. Section 1.3 and the remark around Equation (1.4)). The latter follows from Theorems 1.5.1 and 1.5.2 by the expansion [33]

$$w_n = \frac{(\log n)^{-r}}{\Gamma(1-r)} \left(1 + \mathrm{O}\left(\frac{1}{\log n}\right)\right), \qquad r \in \mathbb{C} \setminus \mathbb{Z},$$

of

$$w_n := \sum_{k=1}^{n} \binom{n}{k} (-1)^k k^r.$$

Note that the Euler transform $\sum_{k=0}^{n} \binom{n}{k} a_k$ of a holonomic sequence (a_n) is holonomic by closure of bivariate holonomic sequences under indefinite sums and diagonals.

This result extends the special case $h_n = n$ of Theorem 1.3, but for the more general situation of nonintegral powers of hypergeometric sequences (1.5) we do not know about an appropriate asymptotic estimate that would allow to extend Theorem 1.3 to exponents $r \in \mathbb{C} \setminus \mathbb{Z}$.

The sequence $(\log n)$, for which we have given a conditional non-holonomicity proof in Section 1.3, can be dealt with by studying

$$v_n := \sum_{k=1}^{n} \binom{n}{k} (-1)^k \log k.$$

If $(\log n)$ was holonomic, then so would be (v_n). But Flajolet and Sedgewick [33] have shown that

$$v_n = \log \log n + \mathrm{O}(1),$$

which we already know to be incompatible with holonomicity. So far we have applied known asymptotic results to prove non-holonomicity. In the following section we deduce a new result on the asymptotics of the generating function of the sequence $(\mathrm{e}^{1/n})$ that will establish its non-holonomicity.

1.6 Saddle Point Analysis of the Generating Function of [4] $(\mathrm{e}^{1/n})$

Continuing the discussion in the previous section, we prove by Flajolet and Salvy's method that the sequence $(\mathrm{e}^{1/n})_{n \geq 1}$ is not holonomic. The Taylor series expansion of $\mathrm{e}^{1/n}$ itself does not lend itself to this end, but the generating function

$$H(z) := \sum_{n \geq 1} \mathrm{e}^{1/n} z^n, \qquad |z| < 1,$$

[4] The material in this section arose from a collaboration with Philippe Flajolet and Bruno Salvy.

admits of an analytic continuation via a Lindelöf integral with appropriate (non-holonomic) asymptotics as $|z| \to \infty$. The latter will be found from the integral representation of $H(z)$ by the saddle point method. As a byproduct of independent interest, we obtain the asymptotics of the alternating sum

$$S_n := \sum_{k=1}^{n} \binom{n}{k} (-1)^k \mathrm{e}^{1/k}, \tag{1.12}$$

which is not quite the innocent modification of the binomial theorem that it may appear. Whereas the non-holonomicity of $(\mathrm{e}^{1/n})$ can be shown by other means, too [5], the example of S_n shows that our main result on the asymptotics of $H(z)$ (viz. Theorem 1.6.2) is a useful tool for the asymptotic treatment of sums involving $\mathrm{e}^{1/n}$.

1.6.1 The Circle of Convergence

Although the sum defining $H(z)$ converges nowhere on the unit circle, we will see below that $z = 1$ is the only singularity of $H(z)$ there. The asymptotic behavior of $H(z)$ near $z = 1$ is given by

$$H(z) = \frac{1}{1-z} - \log(1-z) + C + \mathrm{o}(1) \qquad \text{as} \quad z \to 1^-, \tag{1.13}$$

where

$$C = \sum_{k \geq 2} \frac{\zeta(k)}{k!} - 1 \approx 0.078189.$$

To see this, expand $\mathrm{e}^{1/n}$ into its Taylor series:

$$\begin{aligned} H(z) - \left(\frac{1}{1-z} - \log(1-z) - 1 \right) &= \sum_{n \geq 1} \sum_{k \geq 2} \frac{z^n}{k! n^k} \\ &= \sum_{k \geq 2} \frac{1}{k!} \mathrm{Li}_k(z) \underset{z \to 1^-}{\to} \sum_{k \geq 2} \frac{\zeta(k)}{k!}, \end{aligned}$$

where $\mathrm{Li}_k(z) := \sum_{n \geq 1} z^n / n^k$ denotes the polylogarithm. The second equality follows from absolute convergence, and the interchange of limit and summation can be justified by uniform convergence, which follows from

$$|\mathrm{Li}_k(z)| \leq \mathrm{Li}_k(|z|) \leq \mathrm{Li}_k(1) = \zeta(k), \qquad |z| \leq 1.$$

The asymptotic expansion (1.13) is reflected on the coefficient side as

$$\begin{aligned} \sum_{k=1}^{n} \mathrm{e}^{1/k} &= \sum_{k=1}^{n} \sum_{j \geq 0} \frac{1}{j! k^j} \\ &= n + \mathrm{H}_n + \sum_{j \geq 2} \frac{\mathrm{H}_n^{(j)}}{j!} \\ &= n + \log n + \sum_{k \geq 2} \frac{\zeta(k)}{k!} + \gamma + \mathrm{o}(1), \end{aligned}$$

where $\mathrm{H}_n^{(j)} := \sum_{k=1}^{n} 1/k^j$ are the j-th order harmonic numbers, and γ is the Euler-Mascheroni constant. The sum $\sum \zeta(k)/k! \approx 1.078189$ does not seem to have a closed form.

1.6.2 Analytic Continuation

Lindelöf has found the integral representation [34, 57]

$$\sum_{n\geq 1} a(n)(-z)^n = -\frac{1}{2\mathrm{i}\pi}\int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} a(s)z^s\frac{\pi}{\sin\pi s}\mathrm{d}s$$

for power series whose coefficients $a(n)$ are given by some function $a(s)$ analytic in $\Re(z) > \frac{1}{4}$. The formula holds (possibly in the sense of analytic continuation of the left hand side) for $\Re(z) > 0$ and requires $a(s) = \mathrm{O}(\exp((\pi-\varepsilon)|s|))$ on the part of the coefficient function. For our function $a(s) = \mathrm{e}^{1/s}$ this reads as

$$H(-z) = -\frac{1}{2\mathrm{i}}\int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty}\frac{z^s\mathrm{e}^{1/s}}{\sin\pi s}\mathrm{d}s, \qquad \Re(z) > 0. \tag{1.14}$$

The integral on the right hand side defines an analytic function in the region $\Omega := \mathbb{C}\setminus\,]-\infty,0]$, as seen by standard results. Indeed, $F(z,t) := g(z,\frac{1}{2}+\mathrm{i}t)$ is continuous on $\mathbb{R}\times\Omega$, as is its partial derivative $\partial/\partial z F(z,t)$. Moreover, for fixed t the function $F(z,t)$ is analytic in Ω. Thus [35, Hilfssatz II.3.3], for fixed $N\in\mathbb{N}$ the integral $\int_{-N}^{N}F(z,t)\mathrm{d}t$ defines an analytic function in Ω. It is an easy consequence of

$$|F(z,t)| = (\cosh\pi t)^{-1}\exp\left(\tfrac{2}{4t^2+1}+\tfrac{1}{2}\log|z| - t\arg z\right)$$

and $(\cosh\pi t)^{-1}\sim\mathrm{e}^{-\pi t}$ that

$$\lim_{N\to\infty}\int_N^\infty |F(z,t)|\mathrm{d}t = 0 \qquad \text{locally uniformly in } \Omega.$$

Therefore, the integral in (1.14) is a locally uniform limit of analytic functions, hence analytic [35, Theorem III.1.3]. Since Lindelöf's result is not included in most standard textbooks on complex analysis, we exemplify its proof by our particular example.

Proposition 1.6.1. *Equation* (1.14) *holds for* $|z| < 1$ *and* $z\notin\,]-1,0]$.

Proof. Take an arbitrary z from the specified region. By the residue theorem we have

$$\frac{1}{2\pi\mathrm{i}}\oint\frac{\pi\mathrm{e}^{1/s}z^s}{\sin\pi s}\mathrm{d}s = \left.\frac{\pi\mathrm{e}^{1/s}z^s}{\frac{\mathrm{d}}{\mathrm{d}s}\sin\pi s}\right|_{s=n} = \mathrm{e}^{1/n}(-z)^n, \tag{1.15}$$

where the integration contour is a small circle that surrounds $n\geq 1$ counterclockwise. For $N\in\mathbb{N}$, consider a circle of radius $r_N := N+\frac{1}{2}$ centered at the origin, and let $\mathcal{D}_N$ be the portion of it that satisfies $\Re(s)\geq\frac{1}{2}$.

Let $\mathcal{C}_N$ be the integration path obtained from $\mathcal{D}_N$ by traversing it counterclockwise and joining the two endpoints by a vertical line. Denote the integrand in (1.14) by $g(z,s)$. Summing (1.15) for $n=1,\dots,N$ yields (cf. Figure 1.1)

$$\frac{1}{2\mathrm{i}}\int_{\mathcal{C}_N} g(z,s) = \sum_{n=1}^{N}\mathrm{e}^{1/n}(-z)^n.$$

Thus, it suffices to show that $\int_{\mathcal{D}_N}|g|\mathrm{d}s$ tends to zero as $N\to\infty$ for all z in question. We may safely ignore the factor $\mathrm{e}^{1/s}$, since

$$1\leq|\mathrm{e}^{1/s}|\leq\mathrm{e}^{1/|s|}\leq\mathrm{e}^2, \qquad \Re(s)\geq\tfrac{1}{2}.$$

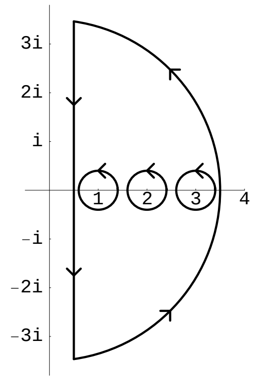

Figure 1.1: Integration over the contour $\mathcal{C}_3$ yields the same result as integration over the small circles surrounding $n = 1, 2, 3$.

For $s = \sigma + i\tau$ we have

$$\left|\frac{z^s}{\sin \pi s}\right| = \frac{\sqrt{2}e^{\sigma \log|z| - \tau \arg z}}{(\cosh 2\pi\tau - \cos 2\pi\sigma)^{1/2}}. \tag{1.16}$$

First we consider the portion of $\mathcal{D}_N$ with $\tau = O(N^{1/3})$. There we have

$$\begin{aligned}\sigma &= \sqrt{r_N^2 - \tau^2} \\ &= r_N \left(1 + O(\tau^2/r_N^2)\right) \\ &= r_N + O(\tau^2/N) \\ &= N + \tfrac{1}{2} + o(1).\end{aligned}$$

Hence for such τ the denominator of the right hand side of (1.16) is larger than some positive constant that does not depend on N, and thus (1.16) is $O(e^{N \log|z|})$ for $\tau = O(N^{1/3})$. Note that $\log|z|$ is negative. If $|\tau|$ is large, then the denominator behaves like $e^{\pi|\tau|}$. Indeed, we have

$$\left|\frac{z^s}{\sin \pi s}\right| = O(\exp(-(\pi + \arg z)N^{1/3})), \qquad \tau \geq N^{1/3},$$

and

$$\left|\frac{z^s}{\sin \pi s}\right| = O(\exp(-(\pi - \arg z)N^{1/3})), \qquad \tau \leq -N^{1/3}.$$

This yields the desired result, since the length of $\mathcal{D}_N$ is $O(N)$. □

1.6.3 Asymptotics near Infinity

We have just proven the first part of our main result:

Theorem 1.6.2. *The function* $H(-z)$ *has an analytic continuation to* $\mathbb{C} \setminus\,]-\infty, -1]$. *In any region* $|\arg z| \leq \pi - \eta$ *with positive* η *and for all* $\varepsilon > 0$ *it satisfies*

$$H(-z) = -\frac{e^{2\sqrt{\log z}}}{2\sqrt{\pi}(\log z)^{1/4}}\left(1 + O((\log|z|)^{-1/4+\varepsilon})\right) \qquad \text{as} \quad |z| \to \infty.$$

From Theorem 1.6.2 we may conclude that $H(z)$ and $(\mathrm{e}^{1/n})$ are not holonomic, since no holonomic function has this asymptotic shape by Theorem 1.5.1. The method of choice for proving Theorem 1.6.2 is saddle point asymptotics. We now give an outline of this method, following de Bruijn [25] and Flajolet and Sedgewick [34]. Let $g(z,s)$ be some analytic function, and $\mathcal{C}_z$ be an integration path that lies in the region of analyticity of $g(z,\cdot)$ for every z of sufficiently large modulus. Integrals of the form

$$F(z) = \int_{\mathcal{C}_z} g(z,s)\mathrm{d}s, \tag{1.17}$$

where both the integrand and the integration path $\mathcal{C}_z$ depend on the complex parameter z, can often be analyzed asymptotically by letting the integration contour cross a saddle point of the integrand. To see why this makes sense, suppose for the moment that we are not after the asymptotics of $F(z)$, but only an upper bound for $|F(z)|$, and let us take a look at the trivial bound

$$|F(z)| \le \mathrm{length}(\mathcal{C}_z) \times \max_{s\in\mathcal{C}_z} |g(z,s)|. \tag{1.18}$$

As long as we stay in the region of analyticity of $g(z,\cdot)$, we have some freedom to move the integration path without altering the value of the integral (1.17). A tight bound will result if we choose a path that makes the maximum in (1.18) small. Here, we assume that the integrand $g(z,s)$ has a rather violent growth, so that moving $\mathcal{C}_z$ a bit will have a large impact on $\max_{s\in\mathcal{C}_z}|g(z,s)|$, but a small effect on the length of $\mathcal{C}_z$. This assumption is a prerequisite for the success of the saddle point method.

We are led to the problem of finding an integration path $\mathcal{C}_z$ that minimizes the maximum in (1.18), at least approximatively. In many cases this will make the bound (1.18) so good that it overshoots the true asymptotic behaviour of $F(z)$ only by a comparatively small factor. For instance [34], for the integral

$$\frac{1}{2\mathrm{i}\pi}\oint \mathrm{e}^s \frac{\mathrm{d}s}{s^{n+1}} = \frac{1}{n!}$$

one can obtain in this way the bound

$$\frac{1}{n!} \le \frac{\mathrm{e}^n}{n^n},$$

which is only off by a factor $\mathrm{O}(n^{1/2})$ from Stirling's formula. This bound can be achieved by choosing a circular integration contour of optimal radius R:

$$\begin{aligned}\frac{1}{n!} &\le \min_R (2\pi R)\frac{1}{2\pi}\max_{|s|=R}\frac{|\mathrm{e}^s|}{|s|^{n+1}} \\ &= \min_R \frac{\mathrm{e}^R}{R^n} \\ &= \frac{\mathrm{e}^n}{n^n}.\end{aligned}$$

Note that since e^z has positive Taylor coefficients, the maximum is attained on the positive real line.

How can we find a 'good' integration path in more complicated situations? Consider, for fixed z, the surface defined by $|g(z,s)|$ as s ranges over some domain Ω, such that $g(s) := g(z,s)$ is analytic in Ω. A *saddle point* of $g(s)$ is a point $s_0 \in \Omega$ with

$$g(s_0) \ne 0 \qquad \text{and} \qquad g'(s_0) = 0. \tag{1.19}$$

Recall that the surface is free of local extrema (apart from minima in the zeros of $g(s)$) by the maximum principle. Thus, the condition (1.19) indeed defines a saddle point and not a local extremum. At a saddle point the shape of the surface is reminiscent

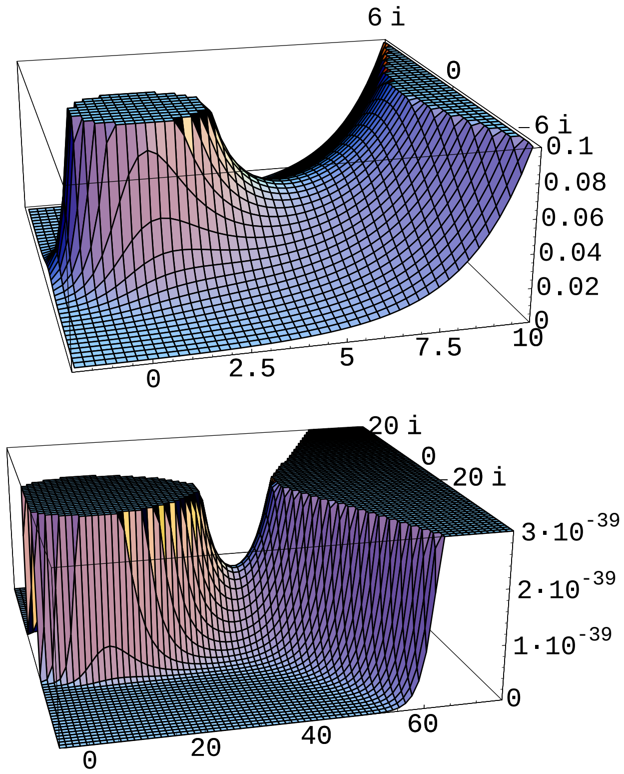

Figure 1.2: The surface $|e^s/s^{n+1}|$ for $n = 4$ and for $n = 34$. The saddle becomes steeper and more pronounced as n increases.

of a saddle (see Figure 1.2). If we assume in addition $g''(s_0) \neq 0$, then it is easy to see from the Taylor series of $g(s)$ around $s = s_0$ that there is a direction of steepest descent from s_0 and a perpendicular direction of steepest ascent. If $g''(s_0) = 0$, then the geometry of the surface around $s = s_0$ is slightly more complicated [34]; we do not go into details. Every mountaineer knows that mountain ranges are most easily traversed at saddles. This idea carries over to our situation. Using the maximum principle the following intuitively clear assertion can be shown: If a path from A to B in Ω has no double points and crosses a saddle point which is a highest point of the path (i.e., $|g(s)|$ is nowhere on the path larger than in $s = s_0$), then this path solves the minimization problem

$$\max_{s \in \mathcal{C}} |g(s)| \to \min,$$

where $\mathcal{C}$ ranges over all paths from A to B that can be continuously deformed to the original path in Ω. This answers the question how to choose a path that minimizes the bound (1.18), at least in theory. In practice, a path that does not cross a saddle point exactly, but only approximately, can be easier to handle and still lead to useful results.

The goal of the saddle point method is not only an upper bound, but an asymptotic estimate for $F(z)$. We assume that $g(s)$ is very large near s_0 in comparison to the rest of the saddle point path. This is again justified by the assumption of violent growth of $g(s)$. By this concentration property, there will be a small part $\mathcal{C}^{(0)}$ (usually depending on z) of the path such that $s_0 \in \mathcal{C}^{(0)}$ and the integral over $\mathcal{C}^{(1)}$, the portion of the path outside of $\mathcal{C}^{(0)}$, is asymptotically smaller than the integral over $\mathcal{C}^{(0)}$. We assume furthermore that $\mathcal{C}^{(0)}$ is so small that we can safely replace the integrand with its quadratic approximation there, which leads to a Gaussian integral that can be done in closed form.

Summing up, a proof of an asymptotic result by the saddle point method comprises the following three parts. Usually, they are technical, but not too difficult. The main difficulty in applying the method consists in the choice of the (often approximate!) saddle point and in the choice of $\mathcal{C}^{(0)}$, which have to ensure the truth of (i) and (iii) below. Note that the saddle point s_0, the integration path $\mathcal{C}$, and the size of the central portion $\mathcal{C}^{(0)}$ usually depend on z. In what follows, it is convenient to write $\mathrm{e}^G := g$.

(i) **Quadratic approximation.** Show that

$$\int_{\mathcal{C}^{(0)}} \exp G(z,s)\mathrm{d}s \underset{|z|\to\infty}{\sim} \mathrm{e}^{G(z,s_0)} \int_{\mathcal{C}^{(0)}} \exp(G''(z,s_0)(s-s_0)^2)\mathrm{d}s. \tag{1.20}$$

(ii) **Central part.** Evaluate the integral on the right hand side of (1.20) asymptotically (leads to a Gaussian integral).

(iii) **Tail estimate.** Show that the tail integral $\int_{\mathcal{C}^{(1)}} g(z,s)\mathrm{d}s$ is asymptotically smaller than the central part.

In (i) the first derivative is not present, because s_0 is assumed to be a saddle point. If s_0 is only an approximation of a saddle point, as will be the case in our example, then the first order term has to be included in the quadratic approximation of G.

After this general prelude we return to our concrete example. Let $g(z,s)$ denote the integrand in (1.14). For every z the function g has a saddle point at $s = (\log z)^{-1/2}$, roughly, since

$$\frac{\partial g(z,s)}{\partial s} = (1 - s^2 \log z + \pi s^2 \cot \pi s) \times \text{nonvanishing factor}.$$

We push the integration path close to this approximate saddle point by replacing 1/2 with $(\log|z|)^{-1/2}$ in (1.14). Observe that this does not alter the value of the integral, as follows readily from the proof of Proposition 1.6.1. We thus obtain

$$-2H(-z) = \int_{-\infty}^{\infty} h(z,t)\mathrm{d}t,$$

where $h(z,t) := g(z,s_0)$ with $s_0 = s_0(z,t) := (\log|z|)^{-1/2} + \mathrm{i}t$. The approximate saddle point that we work with is $s_0(z,0) = (\log|z|)^{-1/2}$. We split the latter integral into

$$\int_{-\infty}^{-\delta} h + \int_{-\delta}^{\delta} h + \int_{\delta}^{\infty} h$$

and call $\int_{-\delta}^{\delta} h$ the *central part.* The integrals $\int_{-\infty}^{-\delta} h$ and $\int_{\delta}^{\infty} h$ are the *tails.* The parameter $\delta = \delta(z) > 0$ has to satisfy the two conflicting requirements noted above: It has to be so small that (1.20) holds, but large enough to make the tails negligible.

We choose $\delta := (\log|z|)^{-\alpha}$ with undetermined α. To avoid some case distinctions that turn out superfluous a posteriori, let us cheat somewhat and assume $\frac{1}{2} < \alpha < 1$ right away. Put $L = L(z) := \log|z|$. We will let $|z| \to \infty$ in the region $|\arg z| \le \pi - \eta$ for some fixed $\eta > 0$. From now on we assume $|z| > 1$, hence L is positive.

1.6.4 Central Part of the Saddle Point Integral

Write $h =: e^f$. We are going to evaluate the central integral $\int_{-\delta}^{\delta}$ that results from replacing f with its quadratic approximation. The expansion of f around $t = 0$ is

$$f(z,t) = c_0(z) + c_1(z)t + c_2(z)t^2 + \dots,$$

where

$$\begin{aligned} c_0(z) &= \log \frac{e^{L^{1/2}} z^{L^{-1/2}}}{\sin(\pi L^{-1/2})} \\ &= \log\left(\exp\left(L^{1/2} + L^{-1/2}\left(L + O\left(1\right)\right)\right)\right) - \log\left(\pi L^{-1/2} + O(L^{-3/2})\right) \\ &= 2L^{1/2} + \tfrac{1}{2}\log L - \log\pi + O(L^{-1/2}), \\ c_1(z) &= -i\pi\cot(\pi L^{-1/2}) - \arg z = O(L^{1/2}), \\ c_2(z) &= -L^{3/2} - \tfrac{\pi^2}{2}\left(\cot(\pi L^{-1/2})\right)^2 - \tfrac{\pi^2}{2} \\ &= -L^{3/2} + O(L). \end{aligned} \tag{1.21}$$

Observe that $c_1(z)$ is not zero, as would be the case if $t = 0$ was a saddle point of g (and thus of f), and not just an approximate one. For $t = O(L^{-\alpha})$ we obtain from (1.21)

$$c_1(z)t + c_2(z)t^2 = -L^{3/2}t^2 + O(L^{1/2-\alpha}). \tag{1.22}$$

Now we can compute the announced integral, provided that $\alpha < \frac{3}{4}$.

$$\begin{aligned} \int_{-\delta}^{\delta} e^{c_0(z)+c_1(z)t+c_2(z)t^2}dt &= e^{c_0(z)} \int_{-L^{-\alpha}}^{L^{-\alpha}} e^{-L^{3/2}t^2}dt\ (1 + O(L^{1/2-\alpha})) \\ &= \tfrac{1}{\pi}e^{2L^{1/2}}L^{1/2} \int_{-\sqrt{2}L^{3/4-\alpha}}^{\sqrt{2}L^{3/4-\alpha}} \tfrac{1}{\sqrt{2}}L^{-3/4}e^{-r^2/2}dr\ (1 + O(L^{1/2-\alpha})) \\ &= \tfrac{1}{\pi\sqrt{2}}e^{2L^{1/2}}L^{-1/4}\int_{-\infty}^{\infty} e^{-r^2/2}dr\ (1 + O(L^{1/2-\alpha})) \\ &= \pi^{-1/2}e^{2L^{1/2}}L^{-1/4}(1 + O(L^{1/2-\alpha})), \end{aligned} \tag{1.23}$$

where the second equality follows from the change of variables $t = rL^{-3/4}/\sqrt{2}$. As for the third equality, note that the constraint $\alpha < \frac{3}{4}$ arises from the need to have integration bounds that tend to infinity as $|z| \to \infty$, so that we arrive at the Gaussian integral. Its tails are negligible, since

$$\int_x^{\infty} e^{-t^2}dt = O(e^{-x^2}) \qquad \text{as} \quad x \to \infty.$$

1.6.5 Quadratic Approximation

We have to show that the relative error that results from replacing f with its second order approximation in the central integral $\int_{-\delta}^{\delta} e^f$ is $o(1)$ as $|z| \to \infty$. We assume $\alpha < \frac{3}{4}$, as dictated by the evaluation of the central part, and $t = O(L^{-\alpha})$. We know from (1.21) and (1.22) that

$$c_0(z) + c_1(z)t + c_2(z)t^2 = 2L^{1/2} - L^{3/2}t^2 + \tfrac{1}{2}\log L - \log\pi + O(L^{1/2-\alpha}). \tag{1.24}$$

Now we investigate the behaviour of f for $t = \mathrm{O}(L^{-\alpha})$. We have

$$\begin{aligned} z^{s_0} &= \mathrm{e}^{s_0 \log z} \\ &= \exp\Big((L^{-1/2} + \mathrm{i}t)(L + \mathrm{O}(1))\Big) \\ &= \mathrm{e}^{L^{1/2}+\mathrm{i}Lt}(1 + \mathrm{O}(L^{-1/2})). \end{aligned}$$

Furthermore, we have the estimate

$$\begin{aligned} \frac{1}{s_0} &= \frac{L^{1/2} - \mathrm{i}Lt}{1 + Lt^2} \\ &= (L^{1/2} - \mathrm{i}Lt)(1 - Lt^2 + \mathrm{O}(L^{2-4\alpha})) \\ &= L^{1/2} - L^{3/2}t^2 - \mathrm{i}Lt + \mathrm{O}(L^{2-3\alpha}), \end{aligned}$$

hence

$$\mathrm{e}^{1/s_0} = \exp(L^{1/2} - L^{3/2}t^2 - \mathrm{i}Lt)(1 + \mathrm{O}(L^{2-3\alpha})).$$

From

$$\sin \pi s_0 = \pi L^{-1/2} + \mathrm{i}\pi t + \mathrm{O}(L^{-3/2})$$

we obtain

$$\log \sin \pi s_0 = -\tfrac{1}{2} \log L + \log \pi + \mathrm{O}(L^{1/2-\alpha}).$$

Summing up, f equals

$$\begin{aligned} f(z,t) &= \log(z^{s_0}\mathrm{e}^{1/s_0}) - \log \sin \pi s_0 \\ &= 2L^{1/2} - L^{3/2}t^2 + \tfrac{1}{2} \log L - \log \pi + \mathrm{O}(L^{2-3\alpha}) \end{aligned}$$

for $t = \mathrm{O}(L^{-\alpha})$. Subtracting the second order approximation (1.24) yields

$$f(z,t) - c_0(z) - c_1(z)t - c_2(z)t^2 = \mathrm{O}(L^{2-3\alpha}). \tag{1.25}$$

Thus, the quadratic approximation is good enough for $\alpha > \frac{2}{3}$. By choosing α close to $\frac{3}{4}$, we can make the exponent of the error term as close to $2 - 3 \times \frac{3}{4} = -\frac{1}{4}$ as we please. As we will see in Section 1.6.6, the error contributed by the tail integrals is smaller, resulting in the overall relative error $\mathrm{O}(L^{-1/4+\varepsilon})$. Note that the error in (1.23) is also smaller than the one in (1.25), since $\alpha < \frac{3}{4}$.

1.6.6 Tail Estimate

It remains to show that the tails $\int_{-\infty}^{\delta} h$ and $\int_{\delta}^{\infty} h$ tend to infinity slower than the central part (1.23). We have (a computer algebra system is helpful here)

$$|h(z,t)| = \left| \frac{\mathrm{e}^{1/s_0} z^{s_0}}{\sin \pi s_0} \right| = \sqrt{2}\, \mathrm{e}^{-t \arg z} \phi(z,t),$$

where

$$\phi(z,t) := \exp\left(\frac{L^{1/2}(Lt^2+2)}{Lt^2+1}\right) \Big(\cosh 2\pi t - \cos(2\pi L^{-1/2})\Big)^{-1/2}. \tag{1.26}$$

For the integral from 1 to ∞ this implies

$$\int_1^\infty |h(z,t)|\mathrm{d}t \le \sqrt{2}\exp\left(\frac{L^{1/2}(L+2)}{L+1}\right)\int_1^\infty \frac{\mathrm{e}^{-t\arg z}}{\sqrt{\cosh 2\pi t - 1}}\mathrm{d}t$$
$$= \mathrm{O}(\exp(\tfrac{3}{2}L^{1/2})), \tag{1.27}$$

where the last integral is O(1), because $|\arg z| \le \pi - \eta$. We are left with bounding the portion of the tail integral from δ to 1. The factor $\mathrm{e}^{-t\arg z}$ is O(1) there. Now note that we have, by Taylor series,

$$\begin{aligned}\cosh 2\pi t - \cos 2\pi L^{-1/2} &\ge 1 + 2\pi^2 t^2 - \cos(2\pi L^{-1/2})\\ &= 2\pi^2(t^2 + L^{-1} + \mathrm{O}(L^{-2}))\\ &= 2\pi^2(t^2 + L^{-1})(1 + \mathrm{O}(L^{-1})),\end{aligned}$$

hence

$$(\cosh 2\pi t - \cos(2\pi L^{-1/2}))^{-1/2} \le \tfrac{1}{\pi\sqrt{2}}(t^2 + L^{-1})^{-1/2}(1 + \mathrm{O}(L^{-1})), \tag{1.28}$$

uniformly for $t \ge L^{-\alpha}$. If we ignore for the moment the exponential in (1.26), then the integral that results from invoking the bound (1.28) can be done in closed form and yields

$$\begin{aligned}\int_{L^{-\alpha}}^1 \frac{\mathrm{d}t}{\sqrt{t^2 + L^{-1}}} &= \log\left(2Lt + 2L\sqrt{\frac{Lt^2+1}{L}}\right)\Bigg|_{t=L^{-\alpha}}^1\\ &= \log\left(2L\left(1 + \left(1 + L^{-1}\right)^{1/2}\right)\right) - \log\left(2L^{1-\alpha} + 2L^{1/2}\left(1 + L^{1-2\alpha}\right)^{1/2}\right)\\ &= \log\left(4L\left(1 + \mathrm{O}(L^{-1})\right)\right) - \log\left(2L^{1/2}\left(1 + \mathrm{O}(L^{1/2-\alpha})\right)\right)\\ &= \tfrac{1}{2}\log L + \mathrm{O}(1).\end{aligned} \tag{1.29}$$

Now we have to estimate the exponential in (1.26). Since the tail integral ought to be smaller than (1.23), we extract $2L^{1/2}$ from the exponent, yielding

$$\frac{L^{1/2}(Lt^2+2)}{Lt^2+1} = 2L^{1/2} - \frac{L^{3/2}t^2}{Lt^2+1}. \tag{1.30}$$

We show that the fraction in the right hand side of (1.30) tends to $-\infty$ as $L \to \infty$. For $L^{-\alpha} \le t \le L^{-1/2}$ we have $Lt^2 + 1 \le 2$, hence

$$-\frac{L^{3/2}t^2}{Lt^2+1} \le -\tfrac{1}{2}L^{3/2}t^2 \le -\tfrac{1}{2}L^{3/2-2\alpha}.$$

In the range $L^{-1/2} < t \le 1$ we have $Lt^2 \ge 1$, which implies $2Lt^2 \ge Lt^2 + 1$, and thus

$$-\frac{L^{3/2}t^2}{Lt^2+1} \le -\tfrac{1}{2}L^{1/2}.$$

Putting both estimates together, we arrive at

$$\exp\left(-\frac{L^{3/2}t^2}{Lt^2+1}\right) = \mathrm{O}\left(\exp\left(-\tfrac{1}{2}L^{3/2-2\alpha}\right)\right), \tag{1.31}$$

uniformly for $L^{-\alpha} \le t \le 1$. From (1.28), (1.29), (1.30), and (1.31) we obtain

$$\int_{L^{-\alpha}}^{1} \phi(z,t)\mathrm{d}t = \mathrm{O}\left(\exp\left(2L^{1/2} - \tfrac{1}{2}L^{3/2-2\alpha}\right)\log L\right).$$

Together with (1.27) this completes the tail estimate

$$\int_{\delta}^{\infty} |h(z,t)|\mathrm{d}t = \mathrm{O}\left(\exp\left(2L^{1/2} - \tfrac{1}{2}L^{3/2-2\alpha}\right)\log L\right).$$

The same bound holds for the integral over $]-\infty,-\delta]$, since $\phi(z,-t) = \phi(z,t)$. This bound is indeed smaller than the absolute error in (1.23), because $\alpha < \frac{3}{4}$. The proof of Theorem 1.6.2 is complete.

1.6.7 An Application

With Theorem 1.6.2 in hand, it is not difficult to crack the asymptotic nut (1.12), where we asked for the asymptotics of

$$S_n = \sum_{k=1}^{n} \binom{n}{k} (-1)^k \mathrm{e}^{1/k},$$

the Euler transform of $(-1)^n\mathrm{e}^{1/n}$. Let

$$S(z) := \sum_{n\ge 1} S_n z^n$$

be the generating function of (S_n).

Proposition 1.6.3. *The function $S(z)$ satisfies*

$$S(z) = \frac{1}{1-z} H\left(-\frac{z}{1-z}\right) \tag{1.32}$$

and is analytic for $|z| < 1$. Its only singularity on the unit circle is at $z = 1$.

Proof. Let $b_n := (-1)^n\mathrm{e}^{1/n}$. Equation (1.32) follows from a standard argument: We have

$$\begin{aligned}
H\left(-\frac{z}{1-z}\right) &= \sum_{k\ge 1} b_k \left(\frac{z}{1-z}\right)^k \\
&= \sum_{k\ge 1} b_k \sum_{n\ge 0} \binom{-k}{n} (-1)^n z^{n+k} \\
&= \sum_{i\ge 1} \left(\sum_{k=1}^{i} b_k \binom{-k}{i-k} (-1)^{i-k}\right) z^i \\
&= \sum_{i\ge 1} \left(\sum_{k=1}^{i} b_k \binom{i-1}{k-1}\right) z^i,
\end{aligned}$$

hence

$$[z^n]\frac{1}{1-z}H\left(-\frac{z}{1-z}\right) = \sum_{i=1}^{n}\sum_{k=1}^{i}\binom{i-1}{k-1}b_k$$
$$= \sum_{k=1}^{n}\sum_{i=k}^{n}\binom{i-1}{k-1}b_k$$
$$= \sum_{k=1}^{n}\binom{n}{k}b_k = S_n,$$

which establishes (1.32). The last but one equality follows from the identity [43, p. 174]

$$\sum_{i=k}^{n}\binom{i-1}{k-1} = \binom{n}{k}.$$

It is clear from (1.32) that $S(z)$ has a singularity at $z = 1$. Mathematica's `Reduce` command shows that $|z| \le 1$ and $z \neq 1$ imply $\Re(z/(1-z)) \ge -\frac{1}{2}$, hence $S(z)$ has no other singularity in the closed unit disc by Theorem 1.6.2. □

By a standard estimate [34], the coefficient sequence of a function that is analytic at the origin and has a singularity of smallest modulus at $z = \alpha$ equals $|\alpha|^{-n}\theta_n$, where θ_n is a subexponential factor, i.e., $\limsup |\theta_n|^{1/n} = 1$. For our example this yields that S_n equals 1^n times a subexponential factor. The latter is determined by the behaviour of $S(z)$ around the singularity and can be obtained by studying the asymptotics of $S(z)$ near $z = 1$. Since for all exponents β

$$\left(\log\frac{z}{1-z}\right)^{\beta} = \left(\log\frac{1}{1-z}\right)^{\beta}\left(1+\left(\log\frac{1}{1-z}\right)^{-1}\log z\right)^{\beta}$$
$$= \left(\log\frac{1}{1-z}\right)^{\beta}\left(1+\mathrm{O}\left(\left(\log\frac{1}{1-z}\right)^{-1}(1-z)\right)\right)^{\beta}$$
$$= \left(\log\frac{1}{1-z}\right)^{\beta}\left(1+\mathrm{O}\left(\left(\log\frac{1}{1-z}\right)^{-1}\right)\right),$$

we have

$$\exp\left(2\left(\log\frac{z}{1-z}\right)^{1/2}\right) = \exp\left(2\left(\log\frac{1}{1-z}\right)^{1/2} + \mathrm{O}\left(\left(\log\frac{1}{1-z}\right)^{-1/2}\right)\right)$$
$$= \exp\left(2\left(\log\frac{1}{1-z}\right)^{1/2}\right)\left(1+\mathrm{O}\left(\left(\log\frac{1}{1-z}\right)^{-1/2}\right)\right)$$

and

$$\left(\log\frac{z}{1-z}\right)^{-1/4} = \left(\log\frac{1}{1-z}\right)^{-1/4}\left(1+\mathrm{O}\left(\left(\log\frac{1}{1-z}\right)^{-1}\right)\right).$$

Theorem 1.6.2 thus implies

$$S(z) \sim -\frac{1}{2\sqrt{\pi}(1-z)}\exp\left(2\left(\log\frac{1}{1-z}\right)^{1/2}\right)\left(\log\frac{1}{1-z}\right)^{-1/4} \tag{1.33}$$

as $z \to 1$ with relative error

$$\mathrm{O}\left(\left(\log \frac{1}{1-z}\right)^{-1/4+\varepsilon}\right).$$

By singularity analysis of slowly varying functions [32], this yields

$$S_n = -\frac{\mathrm{e}^{2\sqrt{\log n}}}{2\sqrt{\pi}(\log n)^{1/4}}\left(1 + \mathrm{O}((\log n)^{-1/4+\varepsilon})\right).$$

In particular, S_n tends to $-\infty$ slower than any polynomial, but faster than any power of $\log n$. We note in passing that (S_n) and $S(z)$ are non-holonomic, as seen from (1.32) and the non-holonomicity of $H(z)$ by closure of holonomic functions under multiplication and algebraic substitution. Alternatively, we could appeal to Theorem 1.5.1 and the asymptotic result (1.33).

Finally, we remark that the saddle point method should still work if $\mathrm{e}^{1/s}$ in the Lindelöf integral (1.14) is replaced by a generic analytic function $a(s)$ with some constraints. The asymptotic behaviour of the integral, generalizing Theorem 1.6.2, will depend on asymptotic properties of $a(s)$ near zero and near infinity [31].

Part II

Positivity of Recursively Defined Sequences

Chapter 2

Proving Positivity of Recursively Defined Sequences by Computer Algebra

Inequalities are deep, while equalities are shallow.

— Doron Zeilberger [87]

2.1 Introduction

As said in the introduction of this thesis, there are several algorithms for proving (and finding) identities involving combinatorial sums and various types of special functions. On the other hand, inequalities have not received much attention from the viewpoint of automated reasoning. The importance of studying inequalities hardly needs to be emphasized. For instance, virtually any proof in analysis contains an estimation of some sort. There are classical [46, 62, 63] and recent [79] textbooks on this broad subject, but, of course, the possibility of proving a given inequality automatically has some obvious advantages in comparison to table lookup.

A possible long term goal of research in algorithmic inequality provers is the Borwein conjecture [1]. It asserts that certain polynomials $A_n(q)$ have non-negative coefficients for all $n \geq 0$, where the $A_n(q)$ satisfy a linear recurrence with coefficients that are polynomials in q^n. As an even more ambitious problem we mention a result by Jensen [73]: The (conjectured) inequality

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(s)\Phi(t)\mathrm{e}^{\mathrm{i}(s+t)x}(s-t)^{2n}\mathrm{d}s\mathrm{d}t \geq 0, \qquad x \in \mathbb{R}, n \in \mathbb{N},$$

where

$$\Phi(t) := 2\sum_{k\geq 1}(2k^4\pi^2\mathrm{e}^{9t/2} - 3k^2\pi\mathrm{e}^{5t/2})\mathrm{e}^{-k^2\pi\mathrm{e}^{2t}},$$

is equivalent to the Riemann Hypothesis.

We now comment on previous applications of computer algebra in proofs of inequalities. For some classical inequalities there is an underlying identity that makes the truth of the inequality obvious. For instance, Lagrange's identity

$$\sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2 - \Big(\sum_{k=1}^{n} x_k y_k\Big)^2 = \sum_{1\le k<i\le n} (x_k y_i - x_i y_k)^2$$

immediately implies the Cauchy-Schwarz inequality

$$\Big(\sum_{k=1}^{n} x_k y_k\Big)^2 \le \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2. \tag{2.1}$$

Askey and Gasper [3] found an identity that establishes the inequality

$$\sum_{k=0}^{n} P_k^{(\alpha,0)}(x) > 0, \qquad \alpha > -1, -1 < x \le 1, \tag{2.2}$$

for a sum of Jacobi polynomials, which was applied in de Branges's [24] proof of the Bieberbach conjecture. Regardless of whether such an identity is algorithmically provable [27, 51], finding a suitable one by human insight is indispensable for this line of attack.

The inequality

$$\sum_{k=n}^{\infty} \frac{1}{k^2\binom{n+k}{k}} < \frac{2}{n\binom{2n}{n}}, \qquad n \ge 1, \tag{2.3}$$

arose in work by Knopp and Schur [77]. Paule's proof [70] of (2.3) contains an application of the extended Gosper algorithm [72]. Also, Paule [69] has applied several computer algebra tools in a proof of another inequality, which implies a conjecture of Knuth. Although computer algebra assisted, both proofs are altogether by no means mechanical. In general, inequalities seem to be much more elusive from the viewpoint of automated reasoning than identities. The main reason why inequalities are difficult to handle is that there is no analogue of what could be called the 'fundamental lemma' of automatic identity proving: *A sequence that satisfies a homogeneous recurrence and has a sufficient (finite!) number of consecutive zeros vanishes everywhere, and similarly for functions and differential equations.*

Even for C-finite sequences, i.e., those that satisfy linear recurrences with constant coefficients, the problem of deciding eventual positivity is not known to be decidable; cf. Section 3.3.7. Therefore, our goal in this chapter cannot be a complete decision algorithm for some class that is larger than the C-finite sequences. Section 2.2 presents a new approach at the positivity of holonomic sequences that works by investigating the signs of the polynomial recurrence coefficients. Alas, we could not find striking examples where the method succeeds. In Section 2.3 we present our main contribution to the subject, which was done in joint work with Manuel Kauers. We will give a method based on induction and Cylindrical Algebraic Decomposition that succeeds in proving many interesting inequalities that involve certain recursively defined sequences. These sequences are members of a class that is considerably larger than the class of holonomic sequences, and the inequalities may even have an unspecified number of parameters, so that inequalities like Cauchy-Schwarz are covered as well. Although not backed by a termination theorem, the procedure works quite well in practice.

Before entering the fray, we comment on the relation between asymptotics and inequalities. One might surmise that proving, say, the positivity of a holonomic sequence, can always be reduced to asymptotics. We have mentioned in Section 1.5 the Birkhoff-Trjitzinsky asymptotic expansion of holonomic sequences. Regardless of any doubts as to the validity of Birkhoff and Trjitzinsky's result, no general method is known to obtain this expansion for a given (in terms of recurrence coefficients and initial values) holonomic sequence. Wimp and Zeilberger [85] have shown how to deduce it for some concrete examples, but it seems necessary to invoke additional knowledge on the sequence, such as monotonicity, which is even stronger than positivity. Furthermore, there are sequences (e.g., (3.23)) where we have not only an asymptotic expansion, but even a closed form, but the eventual sign (or oscillating behaviour) cannot be read off easily.

For some linear homogeneous recurrences, Poincaré's theorem [67] is a useful asymptotic tool. It requires that each recurrence coefficient converge as $n \to \infty$. Using these limits, the characteristic polynomial is defined similarly to the C-finite case. This amounts to neglecting the lower order terms of the recurrence coefficients that have maximal degree, and completely neglecting the recurrence coefficients of smaller degree. The theorem asserts that, if the roots of the characteristic polynomial are simple and of pairwise distinct modulus, then the quotient a_{n+1}/a_n tends to some root for every solution (a_n) that is not the zero sequence. If, in addition, all characteristic roots are positive, then this result implies that there are n_0 and a constant C with

$$\frac{a_{n+1}}{a_n} \geq C > 0, \qquad n \geq n_0.$$

This establishes the induction step for proving $a_n > 0$ for $n \geq n_0$. To apply this approach for proving positivity, we need an effective refinement of Poincaré's theorem that tells us how large n_0 has to be. We did not pursue this idea further, because in non-trivial examples the requirements on the location of the roots are usually not satisfied.

We conclude this introduction with an example of a proof of an inequality by asymptotic reasoning. The example is

$$\prod_{k=0}^{n} \frac{3k+4}{3k+2} > 1 + \frac{2}{3}\sum_{k=1}^{n+1} \frac{1}{k}, \qquad n \geq 1. \tag{2.4}$$

The left hand side can be expressed in terms of the Gamma function. Thus, we obtain by Stirling's formula

$$\begin{aligned}\prod_{k=0}^{n} \frac{3k+4}{3k+2} &= C\frac{\Gamma(n+\frac{7}{3})}{\Gamma(n+\frac{5}{3})}\\ &\sim C\frac{\left(n+\frac{7}{3}\right)^{n+11/6} \mathrm{e}^{-n-7/3}}{\left(n+\frac{5}{3}\right)^{n+7/6} \mathrm{e}^{-n-5/3}}\\ &\sim Cn^{2/3},\end{aligned}$$

where

$$C := \frac{\sqrt{\pi}}{2^{1/3}\Gamma(\frac{7}{6})}.$$

We have used

$$(n+\alpha)^n = n^n \left(1+\tfrac{\alpha}{n}\right)^n \sim \mathrm{e}^{\alpha} n^n$$

for $\alpha = \frac{7}{3}, \frac{5}{3}$. Hence the left hand side of (2.4) is asymptotically larger than the right hand side

$$1 + \tfrac{2}{3}\mathrm{H}_{n+1} \sim \tfrac{2}{3}\log n.$$

The inequality (2.4) is not quite knocked out, however. The standard versions of Stirling's formula and the asymptotics of the harmonic numbers do not tell us from which n_0 on the inequality holds. There are effective refinements of these results, of course, but this is not true for every asymptotic result. The method to be presented in Section 2.3 proves (2.4) automatically.

2.2 Left Multiples of Recurrence Operators

It is easy to decide whether a holonomic sequence (a_n) that satisfies a recurrence

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_d(n)a_{n+d} = 0, \qquad n \geq 0, \tag{2.5}$$

vanishes for all n. To this end it suffices to check d consecutive values (for n's that are larger than all integer roots of $p_d(n)$), since the homogeneous recurrence provides the induction step, once d consecutive zeros have been found. This approach does not extend directly to proving positivity. Indeed, usually we cannot infer eventual positivity of a holonomic sequence from finitely many positive values. Just consider the recurrence

$$a_{n+1} = a_n - 1, \qquad n \geq 0. \tag{2.6}$$

(To see that this inhomogeneous recurrence defines a holonomic sequence, replace n by $n+1$ and subtract, yielding a second order homogeneous recurrence. This works for all linear inhomogeneous recurrences with rational coefficients, after dividing by the inhomogeneous part.) The general solution of (2.6) is $a_n = -n + a_0$ with arbitrary a_0. Thus, by appropriate choice of the initial value a_0, we can build solutions of (2.6) that have an arbitrarily long inital segment of positive entries, but are still not eventually positive.

Occasionally, the $p_k(n)$ have favourable signs. If the recurrence (2.5) is has the sign pattern $++\cdots++-$, by which we mean

$$p_0(n) > 0,\ p_1(n) \geq 0,\ \ldots,\ p_{d-1}(n) \geq 0,\ p_d(n) < 0, \qquad n \gg 0, \tag{2.7}$$

then we can easily prove $a_n \geq 0$ for large n by induction, provided that we find d consecutive non-negative values (with large enough index). Indeed, the recurrence (2.5) immediately yields the induction step.

Example 2.2.1. *With Zeilberger's summation algorithm [71] we can compute a second order recurrence for the indefinite sum*

$$a_n := \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k},$$

the truncated Taylor series of $\log 2$. *The recurrence reads*

$$(n+1)a_n + a_{n+1} - (n+2)a_{n+2} = 0$$

and has the sign pattern (2.7)*. Together with* $a_1 = 1$ *and* $a_2 = \frac{1}{2}$*, this gives an inductive proof of*

$$a_n > 0, \qquad n \geq 1.$$

Of course, few operators that occur in practice will do us the favour of having the nice sign pattern (2.7). For sequences that are defined by a 'non-nice' recurrence, we can sometimes find another recurrence that is satisfied by the sequence and is 'nice'. Upon introducing the *forward shift* $Ea_n := a_{n+1}$, the recurrence (2.5) is equivalent to saying that (a_n) is annihilated by the linear recurrence operator

$$A = A(n, E) := p_0(n)I + p_1(n)E + \cdots + p_d(n)E^d. \tag{2.8}$$

The set $\mathbb{O}$ of linear recurrence operators of the form (2.8), where the $p_k(n)$ have real coefficients, is a non-commutative ring w.r.t. addition and operator composition. Every left multiple of an annihilating operator of (a_n) is an annihilator of (a_n), too. It can be shown by a non-commutative version of the Euclidean algorithm [11, 68] that if the annihilator (2.8) is of minimal order, then all operators in $\mathbb{O}$ annihilating (a_n) are left multiples of this minimal operator.

In some cases, there is a left multiple of A that has the sign pattern (2.7). To find it, we introduce an operator B of order $e \geq 1$ with undetermined coefficients and multiply A with B from the left. The requirement that BA have the sign pattern (2.7) yields a system of polynomial inequalities for the coefficients of B. This system can be simplified by Cylindrical Algebraic Decomposition (see Section 2.3 for more on CAD). If we are lucky, then there is a solution that yields appropriate polynomial coefficients of B. In unlucky cases, there is no solution, or just solutions that are valid for small n only.

Example 2.2.2. *We can show in this way the special case $x = \frac{1}{2}$ of Fejér's inequality. This inequality reads*

$$\sum_{k=0}^{n} P_k(x) > 0, \qquad -1 < x < 1, \tag{2.9}$$

where $P_k(x)$ is the kth Legendre polynomial. With Chyzak's Maple package `Mgfun` *[18] we can compute a recurrence for the sum from the second order recurrence that the Legendre polynomials satisfy. For $x = \frac{1}{2}$ the resulting recurrence is given by the third order operator*

$$A(n, E) = -(2n + 4)I + (4n + 9)E - (4n + 11)E^2 + (2n + 6)E^3.$$

This operator does not have the sign pattern (2.7)*. Having failed for orders* 1 *and* 2*, we try to find a third order operator*

$$B(n, E) = q_0(n)I + q_1(n)E + q_2(n)E^2 + q_3(n)E^3$$

such that BA has the sign pattern (2.7)*. We put $q_0(n) := q_3(n) := 1$ and solve the resulting system of inequalities in the variables n, q_1, q_2 by Mathematica's* `Reduce` *command. Mathematica tells us that one solution is*

$$n \geq 0 \wedge q_1 = \frac{281 + 136n + 16n^2}{149 + 70n + 8n^2} \wedge q_2 = \frac{4 + 15q_1 + 4nq_1}{17 + 4n}.$$

Clearing denominators, we find that BA does the job, where

$$B = (149 + 70n + 8n^2)I + (281 + 136n + 16n^2)E + (283 + 136n + 16n^2)E^2 + (149 + 70n + 8n^2)E^3.$$

The left multiple BA equals

$$BA = 2(2 + n)(149 + 70n + 8n^2)I + (345 + 152n + 16n^2)E + 2(125 + 64n + 8n^2)E^2 + (597 + 280n + 32n^2)E^5 - 2(6 + n)(149 + 70n + 8n^2)E^6.$$

We give another example, this time of no known significance.

Example 2.2.3. *Let (a_n) be defined by $a_0 = 1$, $a_1 = a_2 = 0$, and the annihilating operator*

$$A(n, E) = -I + E - nE^2 + E^3.$$

Then $a_3 = 1$, and the left multiple

$$-(1 + E)A = I + (n - 1)E + nE^3 - E^4,$$

found with Mathematica's `Reduce`, *proves $a_n \geq 0$ for $n \geq 0$.*

If an appropriate left multiple of a given operator A exists, then it is not a big deal to find it by a computer algebra system. As for the existence of such left multiples, we pose the following three questions. To facilitate their formulation, we denote by $\mathcal{W}$ the set of all operators $\pm A \in \mathbb{O}$ such that A has the sign pattern (2.7).

(Q1) Given a recurrence operator $A \in \mathbb{O}$, is there always $B \in \mathbb{O}$ such that $BA \in \mathcal{W}$?

(Q2) Does there exist for arbitrary $d \geq 1$ an operator $A \in \mathbb{O}$ of order d such that A annihilates some eventually positive sequence and A itself is not in $\mathcal{W}$, but has a left multiple in $\mathcal{W}$?

(Q3) Given A, is there a useful criterion that guarantees the existence of a left multiple in $\mathcal{W}$?

We will see shortly that the answer to question (Q1) is negative, as was to be expected. Note that even the eventual positivity of C-finite sequences is not known to be decidable; more on this in Section 3.3.7. This makes the far more modest question (Q2) interesting. It asks whether for each d there is some operator of order d to which our method can be applied successfully. This is true, as shown by the following proposition. Unfortunately, we know nothing regarding the third and most interesting question.

Proposition 2.2.4. *For every $d \geq 3$ there is an operator $A_d \in \mathbb{O} \setminus \mathcal{W}$ of order d, such that A_d has a left multiple in $\mathcal{W}$ and there is a positive sequence (a_n) annihilated by A_d.*

Proof. For even d we put

$$A_d(E) := 1 + \sum_{k=1}^{d} (-1)^{k+1} E^k.$$

Then the required left multiple is

$$\begin{aligned}(1 + E)A_d &= 1 + \sum_{k=1}^{d} (-1)^{k+1} E^k + E + \sum_{k=1}^{d} (-1)^{k+1} E^{k+1} \\ &= 1 + \sum_{k=0}^{d-1} (-1)^k E^{k+1} + 2E + \sum_{k=0}^{d} (-1)^{k+1} E^{k+1} \\ &= 1 + 2E - E^{d+1} \in \mathcal{W}.\end{aligned}$$

If (a_n) is some sequence annihilated by A_d, then we have

$$a_d = a_0 + \sum_{k=1}^{d-1} (-1)^{k+1} a_k.$$

Therefore, we can make (a_n) positive by choosing positive initial values $a_0, \ldots, a_{d-1}$ such that $a_0 + \sum_{k=1}^{d-1}(-1)^{k+1}a_k$ is positive.

If d is odd, then we define

$$A_d(E) := \sum_{k=0}^{d}(-1)^k E^k.$$

Again the multiplier $(1+E)$ does the job:

$$\begin{aligned}(1+E)A_d &= \sum_{k=0}^{d}(-1)^k E^k + \sum_{k=0}^{d}(-1)^k E^{k+1} \\ &= 1 + \sum_{k=1}^{d}(-1)^k E^k + \sum_{k=1}^{d+1}(-1)^{k-1} E^k \\ &= 1 - E^{d+1} \in \mathcal{W},\end{aligned}$$

and if (a_n) is some sequence annihilated by A_d, then we have

$$a_d = \sum_{k=0}^{d-1}(-1)^k a_k.$$

Hence (a_n) is positive if $a_0, \ldots, a_{d-1}$ are positive numbers such that $\sum_{k=0}^{d-1}(-1)^k a_k$ is positive. □

The following negative result shows that our method, i.e., exhibiting a left multiple in $\mathcal{W}$, never succeeds for second order recurrences with constant coefficients, thus answering question (Q1) above.

Theorem 2.2.5. *Let $A \notin \mathcal{W}$ be a linear recurrence operator of order one or two with constant real coefficients. Then*

$$\text{there is } B \in \mathbb{O} \text{ with } BA \in \mathcal{W} \tag{2.10}$$

if and only if

every real sequence annihilated by A except the zero sequence has infinitely many positive and infinitely many negative values. (2.11)

Proof. If the order of A is one, then we may assume $A = \alpha + E$ with $\alpha > 0$. The general solution of $Aa = 0$ is $a_n = (-\alpha)^n a_0$, hence (2.11) holds. Furthermore, we have

$$(\alpha - E)A = \alpha^2 - E^2 \in \mathcal{W}.$$

If A is of order two, then we may assume $A = \gamma + \beta E + E^2$, where $\beta, \gamma \in \mathbb{R}$ are not both negative and $\gamma \neq 0$. From now on (a_n) always denotes a solution of $Aa = 0$ with initial values a_0, a_1. If $\beta^2 \neq 4\gamma$, then it can be expressed as

$$a_n = \frac{1}{\delta}\left(\left(\frac{a_0}{2}(\delta - \beta) - a_1\right)\left(\frac{-\beta-\delta}{2}\right)^n + \left(\frac{a_0}{2}(\beta + \delta) + a_1\right)\left(\frac{\delta-\beta}{2}\right)^n\right), \tag{2.12}$$

where $\delta := \sqrt{\beta^2 - 4\gamma}$.

Case 1. $\beta \geq 0$, $\gamma > 0$. If $\beta > 0$, then we put $\alpha := \max(\beta, \gamma/\beta)$ and find

$$(\alpha - E)A = \alpha\gamma + (\alpha\beta - \gamma)E + (\alpha - \beta)E^2 - E^3 \in \mathcal{W}.$$

If $\beta = 0$, then we have

$$(\gamma - E^2)A = \gamma - E^4 \in \mathcal{W}.$$

Assertion (2.11) holds, because if $a_n > 0$ for some n, then either $a_{n+1} < 0$ or $a_{n+1} \geq 0$, and the latter implies $a_{n+2} < 0$ by our assumptions on β and γ. Analogously, if $a_n < 0$, then one of the values a_{n+1}, a_{n+2} must be positive.

Case 2. $\beta \geq 0$, $\gamma < 0$. To show that (2.11) does not hold in this case, let $a_0 > 0$ be arbitrary and set $a_1 := \frac{1}{2}a_0(\delta - \beta) > 0$, where δ is as in (2.12). By (2.12),

$$a_n = a_0 \left(\frac{\delta - \beta}{2} \right)^n > 0, \qquad n \geq 0.$$

Now we show that (2.10) does not hold either. An operator $B = \sum_{k=0}^{m} q_k(n)E^k$ with $BA \in \mathcal{W}$ would have to satisfy (for n large)

$$\gamma q_0 > 0, \tag{2.13}$$
$$\gamma q_1 + \beta q_0 \geq 0, \tag{2.14}$$
$$\gamma q_k + \beta q_{k-1} + q_{k-2} \geq 0, \qquad 2 \leq k \leq m, \tag{2.15}$$
$$\beta q_m + q_{m-1} \geq 0, \tag{2.16}$$
$$q_m < 0. \tag{2.17}$$

We assume that (2.13), (2.14), and (2.15) hold and show $q_k \leq 0$, $0 \leq k \leq m$, which shows that (2.16) and (2.17) cannot both be true. The base cases $q_0 \leq 0$ and $q_1 \leq 0$ follow from (2.13) and (2.14), respectively. (2.15) provides the induction step.

Case 3. $\beta < 0$, $\gamma > 0$.

Subcase a. $\beta^2 \geq 4\gamma$. Now δ is a real number and $\delta - \beta > 0$. If $\beta^2 > 4\gamma$, then

$$a_n \sim \frac{(\delta + \beta)a_0 + 2a_1}{2\delta} \left(\frac{\delta - \beta}{2} \right)^n \qquad \text{as} \quad n \to \infty,$$

unless $(\delta + \beta)a_0 + 2a_1 = 0$. Hence (2.11) is violated if we choose a_0 arbitrarily and a_1 such that

$$a_1 > -\frac{\delta + \beta}{2} a_0.$$

If $\beta^2 = 4\gamma$, then the recurrence $Aa = 0$ has a double characteristic root, and the solution is

$$a_n = -\frac{1}{\beta} \left(-\frac{\beta}{2} \right)^n (\beta (n-1) a_0 + 2na_1),$$

which is positive for $n \geq 0$ if we take $a_0 > 0$ and $a_1 \geq -\beta a_0/2$, since then

$$a_1 \geq -\frac{\beta a_0}{2} > -\frac{\beta(n-1)a_0}{2n}, \qquad n \geq 1.$$

Next we prove that (2.10) does not hold. We assume that $B = \sum_{k=0}^{m} q_k(n)E^k$ satisfies (2.13), (2.14), and (2.15) and show that (2.17) does not hold. We define the sequence (ϕ_j) by

$$\begin{aligned} &\phi_0 := 1,\ \phi_1 := -\beta/\gamma \\ &\phi_j := (-\beta\phi_{j-1} - \phi_{j-2})/\gamma, \qquad j \geq 2, \end{aligned}$$

and claim (again for n large)

$$\phi_j \geq 0, \qquad j \geq 0, \tag{2.18}$$

$$q_m \geq \phi_j q_{m-j} - \phi_{j-1} q_{m-j-1}/\gamma, \qquad 1 \leq j \leq m-1. \tag{2.19}$$

Then it follows (by (2.19) with $j = m-1$, (2.18), (2.14), the definition of ϕ_j, and (2.13))

$$\begin{aligned} q_m &\geq \phi_{m-1}q_1 - \phi_{m-2}q_0/\gamma \\ &\geq -\beta\phi_{m-1}q_0/\gamma - \phi_{m-2}q_0/\gamma \\ &= \phi_m q_0 \geq 0, \end{aligned}$$

violating (2.17).

Proof of (2.18): If $\beta^2 = 4\gamma$, then

$$\phi_j = 2^j(-\beta)^{-j}(j+1) > 0, \qquad j \geq 0.$$

If $\beta^2 > 4\gamma$, then

$$\phi_j = \frac{1}{2\delta(2\gamma)^j}\left((-1)^j(\beta+\delta)^{j+1} + (\delta-\beta)^{j+1}\right), \qquad j \geq 0.$$

This is positive, since $\delta - \beta > 0$ and

$$\delta - \beta > -\beta - \delta = |\beta + \delta|.$$

Proof of (2.19): We use induction on j. For $j = 1$ (2.19) follows from (2.15). Now suppose (2.19) holds for some j with $1 \leq j \leq m-2$. Then, by (2.18), (2.15) and the definition of ϕ_j,

$$\begin{aligned} q_m &\geq \phi_j q_{m-j} - \phi_{j-1}q_{m-j-1}/\gamma \\ &\geq \phi_j\left(-\beta q_{m-j-1}/\gamma - q_{m-j-2}/\gamma\right) - \phi_{j-1}q_{m-j-1}/\gamma \\ &= \left(-\beta\phi_j/\gamma - \phi_{j-1}/\gamma\right) q_{m-j-1} - \phi_j q_{m-j-2}/\gamma \\ &= \phi_{j+1}q_{m-j-1} - \phi_j q_{m-j-2}/\gamma. \end{aligned}$$

Subcase b. $\beta^2 < 4\gamma$. The recurrence $Aa = 0$ has a pair of conjugate complex characteristic roots. Hence (2.11) follows from a result of Burke and Webb [12], cf. the remark after Theorem 3.3.1. We proceed to prove (2.10). Define the sequence (ϕ_j) as in subcase a. The sequence (ϕ_j) is annihilated by the operator $\frac{1}{\gamma} + \frac{\beta}{\gamma}E + E^2$, and $\beta^2 < 4\gamma$ implies $(\beta/\gamma)^2 < 4/\gamma$. Hence we may apply Burke and Webb's result again and find some m with $\phi_m < 0$. Let m be minimal with this property. Then $m \geq 2$ and conditions (2.13) to (2.17) are satisfied for $q_k = \phi_k$, hence $BA \in \mathcal{W}$ for $B := \sum_{k=0}^{m} \phi_k E^k$. □

2.3 A Proving Procedure Based on Cylindrical Algebraic Decomposition[1]

2.3.1 The Proving Procedure

It is clear that Cylindrical Algebraic Decomposition [16, 21] can prove inequalities. After all, CAD is a method that allows to decide whether a system

$$\begin{aligned} P_1(X_1, \ldots, X_m) &= 0 \\ &\vdots \\ P_k(X_1, \ldots, X_m) &= 0 \\ Q_1(X_1, \ldots, X_m) &> 0 \\ &\vdots \\ Q_l(X_1, \ldots, X_m) &> 0 \end{aligned} \tag{2.20}$$

of polynomial equations and inequalities in several variables $X_1, \ldots, X_m$ has a solution over the real numbers. CAD decomposes $\mathbb{R}^m$ into so-called cells with the property that the signs of the polynomials P_i and Q_i are invariant in each cell. The satisfiability of the system (2.20) can then be decided by inspecting the values of the polynomials at sample points, one from each cell. We do not describe how CAD works [16, 21], but use it as a black box procedure for deciding whether a system of the form (2.20) is unsatisfiable over the reals. The implementation we use is the one behind Mathematica's `CylindricalDecomposition` command, due to Strzeboński [80]. As an example of special purpose software for CAD we mention QEPCAD [22] by Collins and Hong. We are not interested in proving inequalities like

$$XY \leq \tfrac{1}{2}(X^2 + Y^2), \qquad X, Y \in \mathbb{R},$$

that are in the obvious scope of CAD, but in inequalities that involve recursively defined objects. In the present section, we introduce a proving procedure that can be applied to many inequalities that involve some discrete parameter, usually called n. The procedure invokes CAD to perform induction steps. As an introductory toy example, we present a preposterously complicated inductive proof of the inequality

$$2^n > n, \qquad n \geq 0. \tag{2.21}$$

To perform the induction step from n to $n+1$, we have to show that

$$(2^n > n) \wedge (2^{n+1} \leq n+1) \quad \text{is false for all } n \geq 0. \tag{2.22}$$

Replacing the constituents of this formula by real variables A, A', B, and B' yields

$$(A > B) \wedge (A' \leq B'). \tag{2.23}$$

If (2.23) was false for all real A, A', B, and B', then (2.22) would certainly hold. But (2.23) is of course satisfiable over the reals. Therefore, we have to put in additional knowledge, say, the recurrences

$$a_{n+1} = 2a_n \qquad \text{and} \qquad b_{n+1} = b_n + 1$$

[1]The material in this section arose from a collaboration with Manuel Kauers.

that the sequences $(a_n) := (2^n)$ and $(b_n) := (n)$ satisfy. This translates into $A' = 2A$ and $B' = B + 1$. The augmented formula

$$(A > B) \wedge (A' \leq B') \wedge (A' = 2A) \wedge (B' = B + 1)$$

is, by Mathematica's `CylindricalDecomposition`, equivalent to

$$(A < 1) \wedge (2A - 1 \leq B < A) \wedge (A' = 2A) \wedge (B' = B + 1). \tag{2.24}$$

This is still satisfiable (otherwise `CylindricalDecomposition` would return `False`), but we have made important progress. Assuming the validity of the easier inequality $2^n \geq 1$ for $n \geq 0$, we enter $A \geq 1$ as additional knowledge. Then the resulting formula is obviously unsatisfiable, as seen from the first clause in (2.24). After checking the initial value $n = 0$, the inequality (2.21) is established. If we are in a puristic mood, we can prove the inequality $2^n \geq 1$, which we applied in the above proof, by the same method. The pertinent real variable formula is

$$(A \geq 1) \wedge (A' < 1) \wedge (A' = 2A),$$

which is indeed unsatisfiable.

This simple example shows the main ideas of our method for proving a given inequality: First, reduce the induction step of the desired proof to a formula θ over the reals that has to be refuted. To perform this reduction, introduce real variables for the sequences that occur in the inequality and their shifts. The recursive definitions of the sequences are entered as equations that the real variables have to satisfy. Second, check by Cylindrical Algebraic Decomposition whether θ is unsatisfiable. If it is, then checking initial values proves the inequality. If, on the other hand, the formula θ is satisfiable, then we can often enter additional knowledge (usually the positivity of things that are obviously positive, like n or 2^n) and arrive at an unsatisfiable formula. These additional facts can often be proven by our method themselves. Other useful facts can be identities that the involved sequences satisfy; cf. the examples in Section 2.3.2. In some cases it also helps to extend the induction hypothesis, i.e., to take some $r \geq 2$ and translate

$$(a_n > b_n) \wedge \cdots \wedge (a_{n+r-1} > b_{n+r-1}) \wedge (a_{n+r} \leq b_{n+r}) \tag{2.25}$$

into a real variable formula. Clearly, once (2.25) is established, r consecutive initial values have to be checked.

It is time to clarify to what kind of inequalities our method can be applied. Suppose that we want to prove positivity (or non-negativity) of a sequence (a_n). To be *admissible*, the sequence should be defined in a rationally recursive way, i.e.,

$$a_{n+s} = R(a_n, a_{n+1}, \ldots, a_{n+s-1}), \qquad n \geq 0, \tag{2.26}$$

with some rational function R. For instance, the sequence $a_n = 2^{2^n}$ (non-holonomic by Proposition 1.2.1) satisfies the polynomial recurrence $a_{n+1} = a_n^2$. The full richness of the class is only seen after specifying the coefficient domain of the rational function R. Its coefficients may be defined in a rationally recursive way themselves, and so on. A precise definition can be given by structural induction [50, 52]. To get a feeling for this, we show that the sequence defined by

$$d_n := \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \qquad n \geq 0,$$

is admissible. The numbers $n!d_n$ are known in combinatorics as *derangement numbers* [43]. They count the number of permutations of n objects without fixed points. Admissibility of (d_n) follows from the tower of recurrences

$$\begin{aligned}
a_{n+1} &= a_n + 1 && (\, a_n = n \,), \\
b_{n+1} &= (a_n + 1) b_n && (\, b_n = n! \,), \\
c_{n+1} &= -c_n && (\, c_n = (-1)^n \,), \\
d_{n+1} &= d_n - \frac{c_n}{(a_n + 1) b_n} && (\, d_n = \sum_{k=0}^{n} (-1)^k / k! \,).
\end{aligned} \tag{2.27}$$

The class of admissible sequences contains many sequences of practical significance. Clearly, all holonomic sequences are admissible. We have already mentioned the admissible quadratically recursive sequence (2^{2^n}). Concerning closure properties, the class sports closure under addition, multiplication, indefinite sums and products, and continued fractions [50]. The latter means that the sequence defined by

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}, \qquad n \geq 0,$$

is admissible for an admissible sequence (a_n). The definition of admissibility is not yet complete, since we have not fixed the ground field so far. In the example (2.27) the basic recurrence (for $a_n = n$) had coefficients in $\mathbb{Q}$, and the others had coefficients in $\mathbb{Q}$ extended by the previously defined sequences. But we may also allow real parameters in the recurrences. This captures a lot of interesting inequalities, such as the Bernoulli inequality

$$(x + 1)^n \geq 1 + nx, \qquad n \geq 0, x \geq -1. \tag{2.28}$$

Upon agreeing on some finite extension of $\mathbb{Q}$ as our ground field, the definition of admissibility is finally complete.

This definition is not quite the same as in Kauers's original article [50], where he presented a zero decision algorithm for admissible sequences. There, the rational function R in the recurrence has to be either a polynomial or the reciprocal of a polynomial. It can be shown [52] that both definitions are equivalent.

The method we are presenting here is an adaption of Kauers's algorithm. Where the zero decision algorithm uses Gröbner bases computations to perform induction steps, we use CAD. It has to be remarked that Kauers's algorithm always terminates, whereas our inequality proving procedure might not.

Why not prove Bernoulli's inequality (2.28) right away? The recurrences we need are, clearly,

$$\begin{aligned}
a_{n+1} &= a_n + 1 && (\, a_n = n \,), \\
b_{n+1} &= (x + 1) b_n && (\, b_n = (x + 1)^n \,).
\end{aligned}$$

Thus the formula we have to refute is

$$\begin{aligned}
&(B \geq 1 + xA) \wedge (B' < 1 + xA') \wedge (A' = A + 1) \\
&\qquad \wedge (B' = (x + 1)B) \wedge (x \geq -1) \wedge (A \geq 0).
\end{aligned}$$

This is indeed false for all real A, A', B, B', and x. No additional facts have to be entered into the knowledge base.

Incorporating the strategy outlined around Equation (2.25), our proving procedure takes the form displayed below. Our focus is on the construction of the induction step, and we assume that the initial values can be checked algorithmically. This is questionable if there are parameters involved. If they enter polynomially, then the initial values pose no problem, since they can be checked by CAD. Otherwise, the initial values may require human insight. Another thing that might require human help is the choice of the knowledge base of additional facts. Of course, the first try will always be to prove the desired inequality without additional facts. If we need additional knowledge, then it can sometimes itself be proven by our procedure, or by Kauers's zero decision algorithm. Note, however, that we do not know how to determine automatically which facts should be put into the knowledge base.

INPUT:
– finitely many admissible sequences $(a_n), (b_n), \dots$, defined by recurrences and initial values
– an inequality ϕ_n that is polynomial in these sequences and their shifts, to be proven for all $n \geq 0$
– a formula ψ_n (the knowledge base) that is a boolean combination of equations and inequalities that are polynomial in the sequences and their shifts, known to hold for all $n \geq 0$

OUTPUT:
– true if ϕ_n holds for all $n \geq 0$, false otherwise

1. $r := 0$
2. repeat

   If ϕ_r is false, return false.

   $r := r + 1$

   Translate $\phi_n \wedge \cdots \wedge \phi_{n+r-1} \wedge \neg\phi_{n+r} \wedge \psi_n$ into a formula θ in real variables, by using the defining recurrences of the sequences, and replacing the sequences and their shifts with real variables $A^{(0)}, A^{(1)}, \dots, B^{(0)}, B^{(1)}, \dots, \dots$.

   If θ is unsatisfiable, return true.

We note that the procedure achieves quantifier elimination: If successful, it computes an r such that the truth of the formula

$$\forall n \geq 0 : \phi_n$$

is equivalent to the truth of

$$\phi_0 \wedge \cdots \wedge \phi_r.$$

Our procedure can easily be adapted to prove inequalities that hold for $n \geq n_0$ instead of $n \geq 0$.

2.3.2 Examples and Variations

We have yet to argue that our procedure succeeds in other interesting examples than just the Bernoulli inequality. Determining a useful class of inequalities on which the procedure terminates is a goal of future research. Presently, we confine ourselves to exhibiting a range of inequalities that can be proven by our method. We do not bore the reader with details on the computations, but refer to a Mathematica notebook [40] that contains all the following examples.

- Turán's inequality [2]

$$P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0, \qquad -1 < x < 1, n > 1,$$

  for the Legendre Polynomials $P_n(x)$.

- Levin's inequality [63, 3.2.13]

$$1 \leq \frac{nx^{n+1}+1}{x^n(n+1)} \leq \tfrac{1}{2}n(1-x)^2x^{-n}+1, \qquad 0 < x \leq 1, n > 0.$$

- If a_n is defined by

$$a_{n+1} = 1 + \frac{n}{a_n}, \qquad n \geq 1, \qquad a_1 = 1,$$

  then

$$\sqrt{n - \tfrac{3}{4}} \leq a_n - \tfrac{1}{2} \leq \sqrt{n + \tfrac{1}{4}}, \qquad n \geq 1$$

  (Nanjundiah [65]).

The following inequalities from Mitrinović's book 'Elementary Inequalities' [62] can be done as well. The labels are from the book. We specify which additional knowledge we had to enter to make the proof go through, if any.

- 3.57

$$(n+a)! + n! > (n+a-1)! + (n+1)!, \qquad n \geq 1, a \geq 1;$$

  additional knowledge $n! > 0, (n+a)! > 0$.

- 4.3

$$\mathrm{H}_{2n} > \mathrm{H}_n + \tfrac{1}{2}, \qquad n > 1.$$

- 4.8

$$\sum_{k=1}^{n} a^{2k} \leq n(a^{2n+1}+1), \qquad n \geq 1, a \geq 1;$$

  additional knowledge $a^{2n} \geq 0, \sum_{k=1}^{n} a^{2k} \geq 0$.

- 4.22

$$\prod_{k=1}^{n-1}(a^k+1) < \frac{1-a}{a^n-2a+1}, \qquad 0 < a < \tfrac{1}{2}, n \geq 1;$$

  additional knowledge $a \geq a^n > 0$.

- 4.29

$$\frac{1}{2} - \frac{1}{n+1} < \sum_{k=2}^{n} \frac{1}{k^2} < 1 - \frac{1}{n}, \qquad n \geq 2.$$

- 4.30

$$\frac{1}{a+1} - \frac{1}{a+n+1} < \sum_{k=1}^{n} \frac{1}{(a+k)^2} < \frac{1}{a} - \frac{1}{a+n}, \qquad a > 0, n \geq 1.$$

- 4.41

$$(2n)! \sum_{k=2}^{2n} \frac{(-1)^k}{k!} \geq (2n-1)!!^2, \qquad n > 1;$$

additional knowledge $(2n)! > 1, (2n-1)!! > 1$. The *double factorial* is defined by $(2n-1)!! := (2n-1)(2n-3) \cdot \ldots \cdot 3 \cdot 1$.

- 7.36

$$1 - x^{2n} \geq 2nx^n(1-x), \qquad n \geq 0, 0 \leq x \leq 1.$$

Many inequalities of practical importance, such as the Cauchy-Schwarz inequality

$$\Big(\sum_{k=1}^{n} x_k y_k\Big)^2 \leq \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2,$$

contain an unspecified number of parameters. Our procedure is applicable to these without additional work: Just represent the parameters by a 'free' sequence without defining recurrence. Constraints like 'the x_k increase' or 'the x_k are bounded' can be easily incorporated, of course. This simple observation goes a long way: besides Cauchy-Schwarz, the method proves, e.g., Weierstraß's inequalities [63]

$$\begin{aligned}
&\prod_{k=1}^{n}(a_k+1) > 1 + \sum_{k=1}^{n} a_k, && a_k > 0, n \geq 1,\\
&\prod_{k=1}^{n}(1-a_k) > 1 - \sum_{k=1}^{n} a_k, && 0 < a_k < 1, \sum_{k=1}^{n} a_k < 1, n \geq 1,\\
&\prod_{k=1}^{n}(a_k+1) < \frac{1}{1-\sum_{k=1}^{n} a_k}, && 0 < a_k < 1, \sum_{k=1}^{n} a_k < 1, n \geq 1,\\
&\prod_{k=1}^{n}(1-a_k) < \frac{1}{1+\sum_{k=1}^{n} a_k}, && 1 > \prod_{k=1}^{n}(1-a_k) > 0.
\end{aligned}$$

As additional knowledge we entered $\sum_{k=1}^{n} a_k > 0$ in the first three cases, and $0 < \prod_{k=1}^{n}(1-a_k) < 1$ in the fourth. Another example is Beesack's inequality

$$\sum_{k=1}^{n} x_k^a \Big(\sum_{i=1}^{k} x_i\Big)^b \leq \Big(\sum_{k=1}^{n} x_k\Big)^{a+b}, \qquad a \geq 1, a+b \geq 1, x_1, \ldots, x_n > 0, n \geq 1.$$

We can do it for specific integral values of a and b, using only the positivity of the sums involved as additional knowledge. The inequality [63, p. 112]

$$\sum_{k=1}^{n}(-1)^{k-1}a_k^2 \geq \Big(\sum_{k=1}^{n}(-1)^{k-1}a_k\Big)^2, \qquad n \geq 1,$$

valid for any positive decreasing sequence (a_n), can even be shown without entering additional facts. More examples, again taken from Mitrinović's book [62], follow.

- 4.23
$$(n+1)\prod_{k=1}^{n}(a_k+1) \geq 2^n\Big(\sum_{k=1}^{n}a_k+1\Big), \qquad a_k > 0, n \geq 1;$$
additional knowledge $2^{-n}\prod_{k=1}^{n}(a_k+1) > 1, \sum_{k=1}^{n}a_k > 1$.
- 7.44
$$(n-1)\Big(\sum_{k=1}^{n}a_k\Big)^2 \geq 2n\sum_{k=1}^{n}a_k\sum_{i=1}^{k-1}a_i, \qquad n \geq 1.$$
- 8.3
$$\sum_{k=1}^{n}\frac{1}{a_k}\sum_{k=1}^{n}a_k \geq n^2, \qquad a_k > 0, n > 0;$$
additional knowledge $\sum_{k=1}^{n}a_k \geq 0$.
- 8.4
$$\Big(\sum_{k=1}^{n}a_kb_k\Big)^2 \leq \Big(\sum_{k=1}^{n}ka_k^2\Big)\sum_{k=1}^{n}\frac{1}{k}b_k^2, \qquad n \geq 1;$$
additional knowledge $\sum_{k=1}^{n}ka_k^2 \geq 0, \sum_{k=1}^{n}b_k^2/k \geq 0$.
- 8.8
$$\Big(\sum_{k=1}^{n}\frac{1}{k}a_k\Big)^2 \leq \Big(\sum_{k=1}^{n}k^3a_k^2\Big)\sum_{k=1}^{n}\frac{1}{k^5}, \qquad n \geq 1;$$
additional knowledge $\sum_{k=1}^{n}k^3a_k^2 \geq 0, \sum_{k=1}^{n}1/k^5 \geq 0$.
- 8.25
$$\Big(\sum_{k=1}^{n}a_i\Big)^2 \leq n\sum_{k=1}^{n}a_k^2, \qquad n \geq 0.$$

Another straightforward variation accomodates algebraic sequences. For instance, a suitable defining relation for the sequence $(\sqrt{n})$ (non-holonomic by Theorem 1.3.1) is $a_n^2 = n \wedge a_n > 0$. In this way, the inequality

$$\Big(\sum_{k=1}^{n}\sqrt{k}\Big)^2 \leq \Big(\sum_{k=1}^{n}\sqrt[3]{k}\Big)^3, \qquad n \geq 0,$$

can be proven automatically. We conclude this section with some more examples from Mitrinović's book [62] that we could prove automatically, without entering additional facts.

- 3.27
$$\frac{1}{2\sqrt{n}} < 4^{-n}\binom{2n}{n} < \frac{1}{\sqrt{3n+1}}, \qquad n \geq 2.$$
- 3.28
$$\prod_{k=1}^{n} \frac{4k-1}{4k+1} < \sqrt{\frac{3}{4n+3}}, \qquad n \geq 1.$$
- 3.29
$$\frac{1}{2\sqrt{n}} < \prod_{k=1}^{n} \frac{2k-1}{2k} < \frac{1}{\sqrt{2n+1}}, \qquad n \geq 2.$$
- 4.1
$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > 2(\sqrt{n+1}-1), \qquad n \geq 1.$$
- 11.1
$$\sqrt{n+\sqrt{(n-1)+\sqrt{\cdots+\sqrt{2+\sqrt{1}}}}} < \sqrt{n}+1, \qquad n \geq 1.$$

2.3.3 Sign Patterns of C-Finite Sequences

The procedure of Section 2.3.1 can be modified slightly in order to analyze the sign patterns of oscillating sequences. Consider, for instance, the C-finite sequence (a_n) defined by

$$\begin{aligned} &a_0 = 2+\sqrt{2}, \quad a_1 = 2+\sqrt{10}, \quad a_2 = -2+5\sqrt{2}, \\ &a_n = (4+\sqrt{5})a_{n-1} - (5+4\sqrt{5})a_{n-2} + 5\sqrt{5}a_{n-3}, \qquad n \geq 3. \end{aligned} \tag{2.29}$$

The initial values and recurrence coefficients are chosen such that a_n has the closed form

$$a_n = \sqrt{2}\; 5^{n/2}(1-2\sin(n\xi - \tfrac{\pi}{4})), \qquad n \geq 0,$$

with $\xi := \arctan \frac{1}{2}$. The sequence (a_n) clearly has infinitely many positive and infinitely many negative values by Kronecker's theorem (Theorem 3.3.5). Our goal is to obtain finer information on the sign of a_n. As additional knowledge, we use the identity

$$\begin{aligned} &25a_n^2 - \tfrac{10}{11}(14+13\sqrt{5})a_n a_{n+1} - \tfrac{20}{11}(2-6\sqrt{5})a_n a_{n+2} \\ &\quad + (6+4\sqrt{5})a_{n+1}^2 + a_{n+2}^2 - \tfrac{2}{11}(14-13\sqrt{5})a_{n+1}a_{n+2} = 0, \end{aligned} \tag{2.30}$$

which was found by an ansatz with undetermined coefficients and verified by Kauers's zero decision algorithm [50].

In order to study the sign pattern of (a_n), we proceed similarly as in Section 2.3.1 to prove that a certain sequence of sign changes determines the sign of the next value. Indeed, if we let A, A', and A'' correspond to a_n, a_{n+1}, and a_{n+2}, respectively, and the formula ψ denotes the conjunction of the formulas arising from the recurrence (2.29) and the identity (2.30), then the formula

$$\psi \wedge (A \geq 0) \wedge (A' < 0) \wedge (A'' \geq 0)$$

is unsatisfiable. Hence, if $a_n \geq 0$ and $a_{n+1} < 0$, then also $a_{n+2} < 0$. Let us write this as $(+|0) - \Rightarrow -$. In the same way, we can show that

$$\begin{aligned}
&(+|0) - - \Rightarrow -,\\
&(+|0) - - - \Rightarrow -,\\
&+++++(+|0) - - - - * \Rightarrow +,\\
&(+|0) - - - - * + \Rightarrow +,\\
&(+|0) - - - - * + + \Rightarrow +,\\
&(+|0) - - - - * + + + \Rightarrow +,\\
&(+|0) - - - - * + + + + \Rightarrow +,\\
&(+|0) - - - - * + + + + + \Rightarrow +,\\
&(+|0) - - - - * + + + + + + \Rightarrow +,\\
&(+|0) - - - - * + + + + + + + \Rightarrow +,\\
&(+|0) - - - - (-|0) + + + + + + + + \Rightarrow +,\\
&(+|0) - - - - * + + + + + + + + + * \Rightarrow -,
\end{aligned}$$

where $*$ may be $+$, 0, or $-$. Putting all these rules together, we find that the sign pattern of (a_n) may be written in the form

$$+^9[+ \mid 0] -^4 [- \mid 0],$$

i.e., there are alternating runs of nine positive and four negative values, but each run may be prolonged by one, or there might be a zero between two runs. (Following standard notation, we write $[a]$ for 'a or nothing,' $a \mid b$ for 'a or b', and a^n for n repetitions of a.)

In fact, the first values of the sequence have the signs

$$+^3 -^5 +^9 -^4 +^9 -^5 +^9 -^4 +^{10} -^4 +^9 -^5 +^9 -^4 +^9 \cdots$$

which is in accordance with the predicted pattern.

What can be said about the optional prolongations $[+ \mid 0]$ and $[- \mid 0]$ in the pattern? When do they occur? By a more careful application of the same technique, we obtained the following refined description of the sign pattern:

$$(0 \mid +) +^9 P^5 -^5 +^9 P^4 -^4 \left[- +^9 P^4 \left[P(-^5 +^9 P^4)^2\right] -^4\right]$$

where P stands for the subpattern $-^4 +^9 -^5 +^9$. This gives much more detail compared to the first pattern, even though it still doesn't describe the pattern entirely. In this pattern, there are only two uncertain signs per 674 values left. The computations we did for the sign pattern analysis can also be found in the cited Mathematica notebook [40].

Chapter 3

C-Finite Sequences

It seems almost magical that,
in many applications, linear recurrence sequences[1]
show up from several quite unrelated directions.

— G. Everest, A. van der Poorten, I. Shparlinski, T. Ward [29]

3.1 Introduction

In the previous chapter we have presented a procedure that proves a lot of interesting inequalities automatically, but we know nothing else about the class of inequalities on which it terminates. In the present chapter we restrict attention to a small subclass of its input class, and ask whether positivity is decidable for its members. Namely, we investigate if a sequence (a_n) defined by a linear recurrence with constant coefficients

$$a_{n+d} = s_1 a_{n+d-1} + \cdots + s_{d-1} a_{n+1} + s_d a_n, \qquad n \geq 0, \tag{3.1}$$

is positive (for large n). Following Zeilberger [86] we will call sequences that satisfy such a recurrence *C-finite,* although the less suggestive term *recurrence sequence* dominates in the literature. Since we are concerned with questions of positivity, we always tacitly assume that our C-finite sequences have real recurrence coefficients and real initial values. Recall [29] that a C-finite sequence (a_n) can be written in terms of the roots $\alpha_1, \ldots, \alpha_s$ of the *characteristic polynomial*

$$z^d - s_1 z^{d-1} - \cdots - s_{d-1} z - s_d$$

of the recurrence (3.1) as a generalized power sum

$$a_n = Q_1(n)\alpha_1^n + \ldots Q_s(n)\alpha_s^n, \tag{3.2}$$

where the $Q_k(n)$ are polynomials in n with complex coefficients. We refer to the α_k that occur in (3.2) with nonzero coefficient as *characteristic roots of* (a_n). The characteristic roots of maximal modulus will be called *dominating characteristic roots of* (a_n). This well-known explicit representation may seem to render the determination of the eventual sign an easy matter. Indeed, usually it can be read off this representation immediately, but there are some nasty C-finite sequences for which the question of eventual positivity

[1] Called *C-finite sequences* in the present work.

is non-trivial. In fact it is not known [8] whether eventual positivity of C-finite sequences is a decidable problem (for rational recurrence coefficients and rational initial values, say).

We have seen in Section 2.3.3 that the inequality proving procedure from Section 2.3.1 can be used to analyze the sign pattern of oscillating C-finite sequences. When employed to prove positivity, however, the procedure does not seem to terminate for any C-finite sequence whose sign is non-trivial. In fact we will not obtain a decision algorithm in this chapter, but provide theorems that show that certain C-finite sequences are neither eventually positive nor eventually negative. Some results that are interesting in their own right will be obtained in passing. The first one stems from the area of lattice points in specified regions, a subfield of Diophantine geometry. Secondly, in Section 3.3 we will show that the density of the positivity set of a C-finite sequence always exists and determine its possible values.

3.2 A Result from Diophantine Geometry

3.2.1 The Connection to C-Finite Sequences

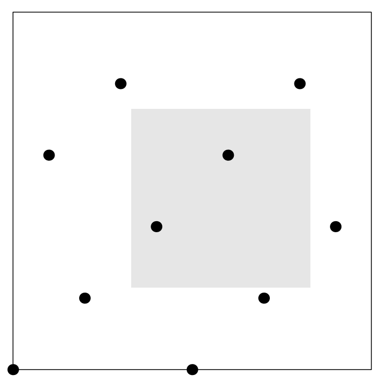

Figure 3.1: The set N (the dots) in the unit square and a square (shaded) with side length $\frac{1}{2}$.

Figure 3.1 shows a discrete set N in the plane, defined by

$$N := \{(7n/10, n/5) \bmod 1 : n \in \mathbb{N}\}.$$

We pose the following question: If we choose an arbitrary square with sides of length $\frac{1}{2}$ parallel to the coordinate axes, will its interior contain a point of N wherever we put the square? Here we take the square modulo one, so that it reappears at the left side of the unit square if we push it across the right side. Figure 3.1 suggests that the answer is yes. The reader will ask what on earth this has to do with C-finite sequences. Consider, for instance, the order four recurrence

$$2b_{n+4} = -(1+\sqrt{5})b_{n+2} - 2b_n.$$

Its characteristic polynomial is

$$(z - \mathrm{e}^{7\mathrm{i}\pi/5})(z - \mathrm{e}^{-7\mathrm{i}\pi/5})(z - \mathrm{e}^{2\mathrm{i}\pi/5})(z - \mathrm{e}^{-2\mathrm{i}\pi/5}),$$

and the solution is given by

$$b_n = c_1\alpha_1^n + \overline{c}_1\overline{\alpha}_1^n + c_2\alpha_2^n + \overline{c}_2\overline{\alpha}_2^n, \qquad n \geq 0, \tag{3.3}$$

where $\alpha_1 := \mathrm{e}^{7\mathrm{i}\pi/5}$, $\alpha_2 := \mathrm{e}^{2\mathrm{i}\pi/5}$, and the complex coefficients c_1, c_2 depend on the real initial values $b_0, \ldots, b_3$. We may ask ourselves what the sign of b_n is for large n. The sequence (b_n) seems to have positive and negative values for arbitrarily large n, but it is not obvious how to prove this. Replacing (α_k, c_k) by $(\overline{\alpha}_k, \overline{c}_k)$ and vice versa if necessary, we may assume $\Im(c_k) \geq 0$. Putting $\theta_k := (\arg \alpha_k)/2\pi$, we then obtain by standard formulas

$$\begin{aligned} b_n &= 2\sum_{k=1}^{2} \Re\left(c_k \exp\left(2\pi \mathrm{i} n\theta_k\right)\right) \\ &= 2\sum_{k=1}^{2} \left(\Re(c_k)\cos 2\pi n\theta_k - \Im(c_k)\sin 2\pi n\theta_k\right) \\ &= \sum_{k=1}^{2} w_k \sin(2\pi n\theta_k + \varphi_k), \end{aligned} \tag{3.4}$$

where the coefficients are nonzero real numbers

$$w_k := \begin{cases} -2|c_k|, & c_k \in \mathbb{C}\setminus\mathbb{R} \\ 2c_k, & c_k \in \mathbb{R} \end{cases},$$

and the φ_k are given by

$$\varphi_k := \begin{cases} -\arctan(\Re(c_k)/\Im(c_k)), & c_k \in \mathbb{C}\setminus\mathbb{R} \\ \frac{1}{2}\pi, & c_k \in \mathbb{R} \end{cases}.$$

We turn our attention to the signs of $\sin(2\pi n\theta_k + \varphi_k)$. If we can prove that for every pair (S_1, S_2) of $+1$'s and -1's there are infinitely many n such that the sign of $\sin(2\pi n\theta_k + \varphi_k)$ equals S_k for $k = 1, 2$, then we will have shown that (b_n) oscillates, whatever the initial values (and thus the w_k) are. Take, for instance, the pair $(S_1, S_2) = (1, -1)$. We are looking for infinitely many n such that

$$(2\pi n\theta_1 + \varphi_1) \bmod 2\pi \in\ ]0, \pi[\qquad \text{and} \qquad (2\pi n\theta_2 + \varphi_2) \bmod 2\pi \in\ ]\pi, 2\pi[, \tag{3.5}$$

and similarly for the other three sign combinations (S_1, S_2). Rescaling to the unit interval and inserting our concrete values $\theta_1 = \frac{7}{10}$ and $\theta_2 = \frac{1}{5}$, we find that (3.5) is equivalent to

$$(7n/10 + \varphi_1/2\pi) \bmod 1 \in\ ]0, \tfrac{1}{2}[\qquad \text{and} \qquad (n/5 + \varphi_2/2\pi) \bmod 1 \in\ ]\tfrac{1}{2}, 1[.$$

Summarizing, the presence of a point of N in any open square of the type described above is a sufficient condition for the oscillating behaviour of (b_n). One point is enough, because the purely periodic sequence (b_n) attains each of its values infinitely often.

3.2.2 Statement of the Diophantine Result and First Part of its Proof

We define the open rectangle parallel to the axes with side lengths $2\lambda_1, 2\lambda_2 \in \mathbb{R}$ centered at $\mathbf{c} = (c_1, c_2) \in \mathbb{R}^2$ as

$$\mathcal{R}_{\lambda_1,\lambda_2}(\mathbf{c}) := \left\{\mathbf{x} \in \mathbb{R}^2 : |x_1 - c_1| < \lambda_1, |x_2 - c_2| < \lambda_2\right\}.$$

For an open square parallel to the axes we write

$$\mathcal{S}_\lambda(\mathbf{c}) := \mathcal{R}_{\lambda,\lambda}(\mathbf{c}), \qquad \lambda \in \mathbb{R}, \mathbf{c} \in \mathbb{R}^2.$$

The main goal of this section and the next one is to prove the following theorem.

Theorem 3.2.1. *Let $u_1, u_2, v_1, v_2 \in \mathbb{N}$, $2 \le v_2 \le v_1$, $1 \le u_k < v_k$, $\gcd(u_k, v_k) = 1$ for $k = 1, 2$, and $\frac{u_1}{v_1} \not\equiv \pm\frac{u_2}{v_2} \pmod 1$. Then there is $\mathbf{c} \in [0,1]^2$ such that for all $n \in \mathbb{N}$*

$$n(\tfrac{u_1}{v_1}, \tfrac{u_2}{v_2}) \bmod 1 \notin \mathcal{S}_{1/4}(\mathbf{c}) \bmod 1$$

provided that

$$(v_1, v_2) \in \{(5,5), (6,3), (8,4)\} \cup \{(v_1, 2) : 2 \le v_1 \in \mathbb{N}\}, \tag{3.6}$$

and there is no such $\mathbf{c}$ if (3.6) does not hold.

Proof of the right to left implication of Theorem 3.2.1. If $v_2 = 2$, then we necessarily have $u_2 = 1$, and we may take $c_2 = \frac{1}{4}$ and $c_1 \in \mathbb{R}$ arbitrary. (See Figure 3.2 for an example.) If $(v_1, v_2) = (5, 5)$, then it is easy to see that for all $\mathbf{u}$ in question the set of integer multiples modulo one is one of the two sets

$$\{n(\tfrac{1}{5}, \tfrac{2}{5}) \bmod 1 : n \in \mathbb{N}\} \qquad \text{and} \qquad \{n(-\tfrac{1}{5}, \tfrac{2}{5}) \bmod 1 : n \in \mathbb{N}\},$$

obtained from $\mathbf{u} = (1, 2)$ and $\mathbf{u} = (-1, 2)$, respectively. Similarly, for $(v_1, v_2) = (6, 3)$ it suffices to consider $\mathbf{u} = (\pm 1, 2)$. This is also true for $(v_1, v_2) = (8, 4)$, if we take $\mathbf{u} = (\pm 3, 1)$ instead of $(\pm 1, 2)$. The number of $\mathbf{u}$'s to check can be reduced further by taking advantage of some obvious symmetries. By the subsequent lemma, the alternative with negative first entry can be discarded in each of the three cases. Figure 3.2 illustrates that in the remaining cases we may take $\mathbf{c} = (\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{12}, \frac{1}{3})$ and $(\frac{1}{2}, \frac{1}{2})$, respectively. ◻

Lemma 3.2.2. *Define the maps s and τ on $\mathbb{R}^2$ by*

$$s(x_1, x_2) = ((1 - x_1) \bmod 1, x_2) \qquad \text{and} \qquad \tau(x_1, x_2) = (x_2, x_1).$$

Then for all real numbers θ_1, θ_2

$$s((\theta_1, \theta_2) \bmod 1) = s(\theta_1, \theta_2) \bmod 1,$$
$$\tau((\theta_1, \theta_2) \bmod 1) = \tau(\theta_1, \theta_2) \bmod 1.$$

Proof. Obvious. ◻

The more interesting part of Theorem 3.2.1 for our purpose is the converse implication. Its proof is the content of the remainder of this section and of the following one. We start out by recalling the basics of the theory of (point) lattices [15]. A *lattice* is a discrete subgroup $\Lambda \subset \mathbb{R}^m$. Equivalently, a lattice is a group

$$\Lambda = \mathbb{Z}\mathbf{v}_1 + \cdots + \mathbb{Z}\mathbf{v}_m$$

generated by some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^m$. The *determinant* of Λ is defined as

$$\mathbf{d}(\Lambda) := |\det(\mathbf{v}_1, \ldots, \mathbf{v}_m)|.$$

It is independent of the choice of generators and equals the volume of the parallelepiped spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$. The determinant is positive if $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent. A lattice point $\mathbf{x} \in \Lambda$ is called *primitive* if there is no integer $t > 1$ with $t^{-1}\mathbf{x} \in \Lambda$.

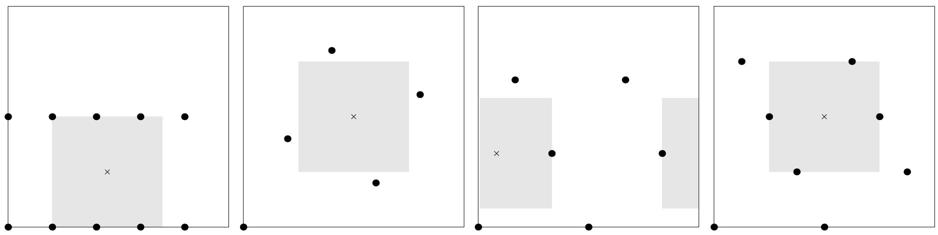

Figure 3.2: The unit square with the set $\{n(\frac{u_1}{v_1}, \frac{u_2}{v_2}) \bmod 1 : n \in \mathbb{N}\}$ for $(\frac{u_1}{v_1}, \frac{u_2}{v_2}) = (\frac{1}{5}, \frac{1}{2})$, $(\frac{1}{5}, \frac{2}{5})$, $(\frac{1}{6}, \frac{2}{3})$ and $(\frac{3}{8}, \frac{1}{4})$, respectively.

Definition 3.2.3. *Let g be a positive integer and u_1, u_2 be integers relatively prime to g. Then we define the* lattice of multiples of $\mathbf{u} = (u_1, u_2)$ modulo g *as*

$$L_g(\mathbf{u}) = L_g(u_1, u_2) := \left\{\mathbf{z} \in \mathbb{Z}^2 : n\mathbf{u} \equiv \mathbf{z} \pmod g \quad \text{for some } n \in \mathbb{N}\right\}.$$

Alternatively [75], $L_g(u_1, u_2)$ can be defined as the lattice generated by the vectors $(0, g), (g, 0)$ and (u_1, u_2). The lattices $L_g(u_1, u_2)$ will provide a convenient representation of the sets of integer multiples of rational numbers modulo one, which we encountered in Theorem 3.2.1. For this purpose we require a version of the well-known Chinese remainder theorem for moduli that are not necessarily pairwise relatively prime.

Theorem 3.2.4 (Generalized Chinese remainder theorem). *Let $v_1, \ldots, v_m$ be positive integers and $z_1, \ldots, z_m$ be integers. Then there is an integer z with*

$$0 \le z < \operatorname{lcm}(v_1, \ldots, v_m) \qquad \textit{and} \qquad z \equiv z_i \mod v_i, \quad 1 \le i \le m,$$

provided that

$$z_i \equiv z_j \mod \gcd(v_i, v_j), \qquad 1 \le i, j \le m.$$

Proof. See Knuth [54, Exercise 4.3.2.3]. □

Lemma 3.2.5. *Let u_1, u_2 be integers and v_1, v_2 be positive integers with $\gcd(u_k, v_k) = 1$ for $k = 1, 2$ and $g := \gcd(v_1, v_2)$. Then*

(i) $\left\{n(\frac{u_1}{v_1}, \frac{u_2}{v_2}) \bmod 1 : n \in \mathbb{N}\right\} = \left\{(\frac{z_1}{v_1}, \frac{z_2}{v_2}) : \mathbf{z} \in L_g(u_1, u_2),\ 0 \le z_k < v_k\right\}$

(ii) $L_g(u_1, u_2) = \left\{\mathbf{z} \in \mathbb{Z}^2 : u_1 z_2 \equiv u_2 z_1 \pmod g\right\}$

Proof. We have

$$\begin{aligned}\left\{n(\tfrac{u_1}{v_1}, \tfrac{u_2}{v_2}) \bmod 1 : n \in \mathbb{N}\right\} &= \left\{(\tfrac{nu_1 \bmod v_1}{v_1}, \tfrac{nu_2 \bmod v_2}{v_2}) : n \in \mathbb{N}\right\} \\ &= \left\{(\tfrac{z_1}{v_1}, \tfrac{z_2}{v_2}) : n\mathbf{u} \equiv \mathbf{z} \pmod{\mathbf{v}},\ 0 \le z_k < v_k,\ k = 1, 2, \quad \text{for some } n \in \mathbb{N}\right\} \\ &= \left\{(\tfrac{z_1}{v_1}, \tfrac{z_2}{v_2}) : \mathbf{z} \in L_g(u_1, u_2),\ 0 \le z_k < v_k\right\}.\end{aligned}$$

The latter equality and assertion (ii) follow from Theorem 3.2.4. □

Example 3.2.6. *In the example* (3.3) *of Section 3.2.1 we have $\theta_1 = \frac{7}{10}$ and $\theta_2 = \frac{1}{5}$. The corresponding lattice $L_5(7, 1) = L_5(2, 1)$ is displayed in Figure 3.3.*

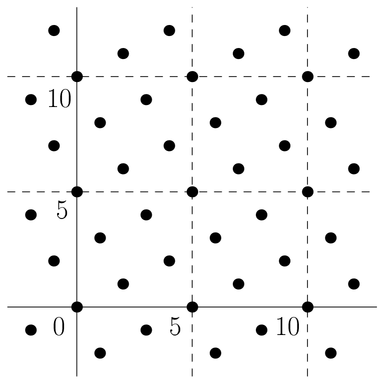

Figure 3.3: The lattice $L_5(2,1)$.

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be as in the assumptions of Theorem 3.2.1, but such that $\mathbf{v}$ is not in the set (3.6), and put $g := \gcd(v_1, v_2)$. In the light of Lemma 3.2.5, it is an immediate consequence of the periodicity property

$$L_g(u_1, u_2) = L_g(u_1, u_2) + g\mathbb{Z}^2 \tag{3.7}$$

that searching a point $n(\frac{u_1}{v_1}, \frac{u_2}{v_2}) \bmod 1$ in a 'modded' square $\mathcal{S}_{1/4}(\mathbf{c}) \bmod 1$ amounts to looking for a point of the lattice $L_g(u_1, u_2)$ in the rectangle $\mathcal{R}_{v_1/4, v_2/4}(v_1c_1, v_2c_2)$ with side lengths $v_1/2$, $v_2/2$. We let c_k absorb v_k and write again $\mathbf{c} = (c_1, c_2)$ for the arbitrary center (v_1c_1, v_2c_2).

Questions about lattice points in specified regions belong to the 'geometry of numbers', a subject founded by Minkowski. Its starting point was Minkowski's theorem [47]: Any convex region in the two-dimensional plane, symmetrical about the origin and of area greater than 4, contains a non-zero point with integral coordinates. The theorem extends to m-dimensional lattices Λ in $\mathbb{R}^m$ [15, 44], where the number 4 in the statement has to be replaced by $2^m\mathbf{d}(\Lambda)$. Since then, various other conditions on sets in real space have been derived that ensure the presence of a lattice point in the set [28]. We will have occasion to apply one such result (Lemma 3.2.13), which appeals to the quotient of area and perimeter of the set, in our proof of Theorem 3.2.1.

Example 3.2.7. *If we want to show that the sequence* (3.3) *oscillates, then we are lead to the problem of finding a point of* $L_5(2,1)$ *in any rectangle* $\mathcal{R}_{5/2,5/4}(\mathbf{c})$, $\mathbf{c} \in \mathbb{R}^2$.

If the numbers v_1/g and v_2/g are large, then it is easy to find a point of $L_g(u_1, u_2)$ in the rectangle, whereas $v_1 = v_2 = g$ is the most difficult case. This is so because if we fix u_1, u_2 and g and enlarge v_1/g and v_2/g, then the lattice $L_g(u_1, u_2)$ remains invariant, while the rectangle becomes bigger.

At first glance, the problem seems to be easily reducible to the case of equal denominators $v_1 = v_2 = g$. In Example 3.2.7, if we could show that any square $\mathcal{S}_{5/4}(\mathbf{c})$ contains a point of $L_5(2,1)$, then it would follow at once that every rectangle $\mathcal{R}_{5/2,5/4}(\mathbf{c})$ contains a point of $L_5(2,1)$. But we have already seen (Theorem 3.2.1) that there are squares $\mathcal{S}_{5/4}(\mathbf{c})$ without points of $L_5(2,1)$. In general, the catch is that even if (u_1, u_2, v_1, v_2) satisfy the requirements of Theorem 3.2.1 and (v_1, v_2) is not in the set (3.6), then it may still happen that $(u_1 \bmod g, u_2 \bmod g, g, g)$ violate the requirements of Theorem 3.2.1 or that (g, g) is in (3.6). Therefore we choose a different approach for the case $v_1 \neq v_2$.

For relatively prime v_1 and v_2 the lattice $L_g(u_1, u_2)$ equals $\mathbb{Z}^2$. All rectangles $\mathcal{R}_{v_1/4,v_2/4}(\mathbf{c})$ with $\mathbf{c} \in \mathbb{R}^2$ have side lengths greater than one and therefore contain a point of $\mathbb{Z}^2$. If $g = 2$, then u_1 and u_2 must both be odd, hence

$$L_g(u_1, u_2) = \left\{\mathbf{z} \in \mathbb{Z}^2 : z_1 \equiv z_1 \pmod 2\right\}.$$

Since $v_1 > 4$ in this case, it is easy to see that this lattice contains a point of any rectangle $\mathcal{R}_{v_1/4,v_2/4}(\mathbf{c})$.

From now on we assume $g \geq 3$. The following proposition deals with the case $(v_1, v_2) = (2g, g)$. Recall that $(v_1, v_2) = (4, 2)$, $(6, 3)$, and $(8, 4)$ need not be considered, because they are in the set (3.6).

Proposition 3.2.8. *Let u_1, u_2, v_1, v_2 be as in Theorem 3.2.1. Suppose $g \geq 5$, $v_1 = 2g$, and $v_1 = g$. Then for all $\mathbf{c} \in \mathbb{R}^2$*

$$L_g(u_1, u_2) \cap \mathcal{R}_{v_1/4,v_2/4}(\mathbf{c}) \neq \emptyset.$$

Proof. Observe that by the periodicity property (3.7) of $L_g(u_1, u_2)$ it suffices to find a point of the lattice in the set

$$\mathcal{R}_{v_1/4,v_2/4}(\mathbf{c}) + g\mathbb{Z}^2. \tag{3.8}$$

Let $\mathbf{p} = (p_1, p_2)$ be the lower left corner of $\mathcal{R}_{v_1/4,v_2/4}(\mathbf{c})$. We assume w.l.o.g. $0 \leq p_1, p_2 < g$ and define $I :=]p_2, p_2 + \frac{1}{2}g[$. Then (3.8) contains the set

$$([0, g[\setminus \{p_1\}) \times I = ([0, g[\times I) \setminus (\{p_1\} \times I)\,. \tag{3.9}$$

The interval I contains at least two integers, since its length is $\frac{1}{2}g > 2$. Since u_2 is invertible modulo g, there are at least two points of $L_g(u_1, u_2)$ in $[0, g[\times I$ by part (ii) of Lemma 3.2.5, and at least one of them lies in (3.9). ☐

Now we consider values of v_1 that are at least $3g$, which completes the proof of the case $v_1 \neq v_2$ of Theorem 3.2.1.

Proposition 3.2.9. *Let u_1, u_2, v_1, v_2 be as in Theorem 3.2.1. Suppose $g \geq 3$ and $v_1 \geq 3g$. Then for all $\mathbf{c} \in \mathbb{R}^2$*

$$L_g(u_1, u_2) \cap \mathcal{R}_{v_1/4,v_2/4}(\mathbf{c}) \neq \emptyset.$$

Proof. It suffices to consider $v_1 = 3g$ and $v_2 = g$. Proceeding analogously to the proof of Proposition 3.2.8, we arrive at the set $[0, g[\times I$ instead of (3.9). The result follows from part (ii) of Lemma 3.2.5 and $\frac{1}{2}g > 1$. ☐

3.2.3 The Proof in the Case of Equal Denominators

In order to finish the proof of Theorem 3.2.1 we will establish the following proposition.

Proposition 3.2.10. *Let u_1, u_2, v_1, v_2 be as in Theorem 3.2.1. Suppose $v_1 = v_2 = g \neq 5$. Then for all $\mathbf{c} \in \mathbb{R}^2$*

$$L_g(u_1, u_2) \cap \mathcal{S}_{g/4}(\mathbf{c}) \neq \emptyset.$$

If $L_g(u_1, u_2)$ contains one or two sufficiently short vectors, its points are dense enough so that the square $\mathcal{S}_{g/4}(\mathbf{c})$ is populated by at least one lattice point. This is the basic idea of our proof of Proposition 3.2.10. Although there are algorithms [56, 75] tailored to $L_g(u_1, u_2)$ for computing a reduced lattice basis, we do not know of any specialized a priori bounds for the norm of the basis elements. Therefore, we appeal to the standard bound.

Definition 3.2.11. *Let $\mathcal{K}$ be a subset of $\mathbb{R}^m$ and $\Lambda \subset \mathbb{R}^m$ be a lattice. Then the* successive minima *of $\mathcal{K}$ w.r.t. Λ are defined for $1 \le k \le m$ by*

$$\lambda_k(\mathcal{K}, \Lambda) := \inf \{\lambda > 0 : \lambda\mathcal{K} \text{ contains } k \text{ linearly independent points of } \Lambda\}.$$

No confusion should arise with the Lebesgue measure $\boldsymbol{\lambda}$. The geometric meaning of λ_k is as follows: Suppose that $\mathcal{K}$ is a sufficiently small ball around the origin, such that $\mathcal{K}$ does not contain a non-zero point of Λ. For a large enough parameter $\lambda > 0$, the boundary of the blown up set $\lambda\mathcal{K}$ will hit another lattice point. The smallest such λ is the first successive minimum $\lambda_1(\mathcal{K}, \Lambda)$. The smallest λ that makes the boundary of $\lambda\mathcal{K}$ hit yet another lattice point is $\lambda_2(\mathcal{K}, \Lambda)$, and so on. The following theorem is one of the fundamental results in the geometry of numbers. The term *body* denotes a set $\mathcal{K} \subset \mathbb{R}^m$ with non-empty interior such that $\mathcal{K}$ is contained in the closure of its interior. This technical definition should not scare us, since in our application of Theorem 3.2.12 the set $\mathcal{K}$ will be the humble closed unit ball.

Theorem 3.2.12 (Minkowski's second theorem). *If Λ is an m-dimensional lattice in $\mathbb{R}^m$ and $\mathcal{K} \subset \mathbb{R}^m$ is a bounded zero-symmetric convex body with volume $\boldsymbol{\lambda}(\mathcal{K})$, then*

$$\lambda_1(\mathcal{K}, \Lambda) \cdots \lambda_m(\mathcal{K}, \Lambda)\boldsymbol{\lambda}(\mathcal{K}) \le 2^m \mathbf{d}(\Lambda).$$

Proof. See Gruber and Lekkerkerker's monograph [44, Theorem 2.16.3]. □

From this theorem we will deduce that $L_g(u_1, u_2)$ must contain either two 'short' linearly independent vectors with norm $\mathrm{O}(g)$ or one 'very short' nonzero vector with norm $\mathrm{O}(1)$. If the first case occurs, then we will apply the following result of Bender [7].

Lemma 3.2.13. *Let $\{\mathbf{w}_1, \mathbf{w}_2\}$ be a basis of a lattice $\Lambda \subset \mathbb{R}^2$, and let $0 < \vartheta < \pi$ be the angle between $\mathbf{w}_1$ and $\mathbf{w}_2$. Suppose further that $\mathcal{C} \subset \mathbb{R}^2$ is a bounded convex set such that the quotient of its area and perimeter is greater than*

$$\tfrac{1}{2} \max\left(\|\mathbf{w}_1\|_2, \|\mathbf{w}_2\|_2 \sin\vartheta\right).$$

Then $\mathcal{C}$ contains a point of Λ.

For the second case, where we will find one vector of 'very small' norm in $L_g(u_1, u_2)$, we could not locate an applicable result in the literature that would ensure a lattice point in the square, so we provide one now.

Lemma 3.2.14. *Let $\Lambda \subset \mathbb{R}^2$ be a lattice and $\mathbf{r} = (r_1, r_2)$ be a primitive point of Λ with $0 < r_2 \le r_1$. Let further $\mathcal{Q}$ be an open square with sides parallel to the axes and side length $A > 0$. If $\mathcal{Q}$ contains no point of Λ, then*

$$A \le \max\left(r_1, \tfrac{\mathbf{d}(\Lambda) + 2r_1r_2}{r_1 + r_2}\right).$$

Proof. There is a family $\mathfrak{L}$ of parallel equidistant lines with slope $s := r_2/r_1$ such that $\Lambda \subset \bigcup \mathfrak{L}$ and the perpendicular distance between two adjacent lines of $\mathfrak{L}$ is $\mathbf{d}(\Lambda)/\|\mathbf{r}\|_2$ [15, Lemma III.5]. Then the vertical distance between two adjacent lines is $D := \mathbf{d}(\Lambda)/r_1$. We claim

$$\min_{\mathbf{c}\in\mathbb{R}^2} \max_{\ell\in\mathfrak{L}} \left(\text{horizontal length of } \ell \cap \mathcal{S}_{A/2}(\mathbf{c})\right)$$
$$= \begin{cases} A, & D \le A(1-s) \\ \frac{A(1+s)-D}{2s}, & A(1-s) \le D \le A(1+s) \\ 0, & D \ge A(1+s) \end{cases}. \quad (3.10)$$

If $D \leq A(1-s)$, then for each square $\mathcal{S} = \mathcal{S}_{A/2}(\mathbf{c})$ there is a line in $\mathfrak{L}$ that goes through the left and the right edge of the square (see Figure 3.4). This settles the first case in the right hand side of (3.10).

If D is larger than $A(1+s)$, then there is a square that is not intersected by any line from $\mathfrak{L}$ (see again Figure 3.4).

We are left with the intermediate case $A(1-s) \leq D \leq A(1+s)$. To achieve the minimum in (3.10), we must certainly place $\mathcal{S}$ such that there is no line from $\mathfrak{L}$ in the parallelogram $\mathcal{P}(\mathcal{S})$ of Figure 3.4. But then there is always a line $\ell \in \mathfrak{L}$ that intersects $\mathcal{S} \setminus \mathcal{P}(\mathcal{S})$, say in the upper triangle of $\mathcal{S} \setminus \mathcal{P}(\mathcal{S})$. If no line intersects the lower triangle of $\mathcal{S} \setminus \mathcal{P}(\mathcal{S})$, we can make the maximum in (3.10) smaller by pushing $\mathcal{S}$ downwards. The smallest possible value of the maximum is achieved as soon as the intersections of $\mathcal{S}$ with ℓ and the line from $\mathfrak{L}$ just below ℓ have equal length. It is easy to see that these intersections both have horizontal length $(A(1+s)-D)/2s$.

Now that (3.10) is established, let $\mathcal{Q}$ be an open square with sides parallel to the axes and side length

$$A > \max\left(r_1, \tfrac{\mathbf{d}(\Lambda)+2r_1r_2}{r_1+r_2}\right). \tag{3.11}$$

Our goal is to show $\mathcal{Q} \cap \Lambda \neq \emptyset$. If the first case in the right hand side of (3.10) occurs, we are well off: Since $A > r_1$, the line segment in $\mathcal{Q} \cap \bigcup \mathfrak{L}$ of horizontal length A must contain a point of Λ. The third case in (3.10) cannot happen, since it would imply $\mathbf{d}(\Lambda) \geq A(r_1+r_2)$, contradicting (3.11). As for the second case,

$$A > \frac{\mathbf{d}(\Lambda) + 2r_1r_2}{r_1 + r_2}$$

implies

$$r_1 < \frac{A(r_1+r_2) - \mathbf{d}(\Lambda)}{2r_2} = \frac{A(1+s) - D}{2s},$$

hence $\mathcal{Q} \cap \Lambda \neq \emptyset$. □

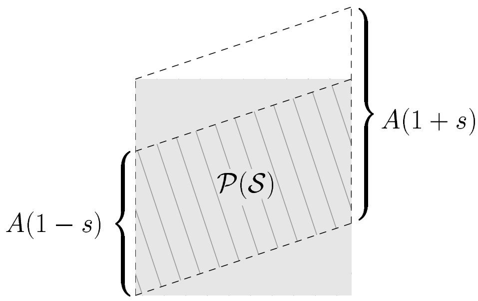

Figure 3.4: The square $\mathcal{S}$ (shaded) and the parallelogram $\mathcal{P}(\mathcal{S})$ (hatched), which lies between two lines of slope s that go through the upper right and the lower left corner of $\mathcal{S}$, respectively.

Proof of Proposition 3.2.10. We begin this proof by settling the cases where g is at most 9. The only numbers to consider are $g = 7, 8, 9$, since for smaller $g \neq 5$ there are no u_1, u_2 that satisfy the requirements of Theorem 3.2.1 (and hence of Proposition 3.2.10).

First let $g = 7$. If we have proven the desired result for a pair (u_1, u_2), then we need not consider the five pairs

$$(u_2, u_1), (g - u_1, u_2), (u_1, g - u_2), (g - u_2, u_1) \quad \text{and} \quad (u_2, g - u_1)$$

any more by Lemma 3.2.2. It is readily seen that under our restrictions on u_1, u_2 all lattices $L_7(u_1, u_2)$ are equal to $L_7(1, 3)$ modulo these symmetries. Similarly, for $g = 8$ and $g = 9$ it suffices to consider $L_8(3, 1)$ and $L_9(2, 1)$, respectively. In all three cases it is easy to verify the desired result. From now on we assume $g \geq 10$. Put $\Lambda := L_g(u_1, u_2)$, and let

$$\mathcal{B} := \left\{ \mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 \leq 1 \right\}$$

be the closed unit disc. It is not difficult to see [56, Section 2] that the determinant of Λ is $\mathbf{d}(\Lambda) = g$. Then Theorem 3.2.12 shows

$$\lambda_1(\mathcal{B}, \Lambda)\lambda_2(\mathcal{B}, \Lambda)\pi \leq 4g.$$

First suppose $\lambda_2(\mathcal{B}, \Lambda) < g/4$. The quotient of the area of $\mathcal{S}_{g/4}(\mathbf{c})$ and its perimeter is $\frac{g^2}{4}/2g = g/8$, hence we can apply Lemma 3.2.13.

If, on the other hand, $\lambda_2(\mathcal{B}, \Lambda) \geq g/4$, then we have $\lambda_1(\mathcal{B}, \Lambda) \leq 16/\pi$, which provides us with a nonzero point $\mathbf{r} \in \Lambda$ with $\|\mathbf{r}\|_2 \leq 16/\pi$. W.l.o.g. assume that $\mathbf{r}$ is primitive and satisfies $0 < r_2 \leq r_1$. According to Lemma 3.2.14, it suffices to show

$$\frac{g}{2} > \frac{g + 2r_1r_2}{r_1 + r_2},$$

i.e.,

$$4r_1r_2 < g(r_1 + r_2 - 2). \tag{3.12}$$

This inequality is satisfied for $g \geq 10$ and

$$\mathbf{r} \in \{(2, 1), (3, 1), (4, 1), (2, 2), (3, 2), (4, 2), (3, 3), (4, 3)\},$$

which are all values of $\mathbf{r}$ in question. Observe that $u_1 \not\equiv u_2 \pmod g$ implies $(1, 1) \notin \Lambda$. ☐

This completes the proof of Theorem 3.2.1. We remark that the successive minima approach from the preceding proof can be applied to the case of distinct denominators v_1, v_2, too. However, the number of special cases that have to be checked separately is much larger than for equal denominators.

3.3 The Positivity Set of a C-Finite Sequence

3.3.1 Sequences with no Positive Dominating Root I

The topic of this section is the application of Theorem 3.2.1 to C-finite sequences, as hinted at in Section 3.2.1. Since Theorem 3.2.1 is limited to two dimensions, we can only accomodate recurrences with two pairs of conjugated complex roots. The generalization of Theorem 3.3.1 to an arbitrary number of characteristic roots will be the topic of Section 3.3.3.

Theorem 3.3.1. *Let (a_n) be a C-finite sequence, not identically zero and with at most four dominating characteristic roots, none of which is real positive. Then there are infinitely many n with $a_n > 0$ and infinitely many n with $a_n < 0$.*

So far this result has only been verified for one dominating characteristic root (trivial) and, by Burke and Webb [12], for one pair of conjugate complex roots. We cannot follow an argument from Nagasaka and Shiue [64], viz. that this special case should immediately imply the truth of the result for any number of dominating roots.

Let (a_n) be as in Theorem 3.3.1. We order the characteristic roots $\alpha_1, \dots, \alpha_s$ of (a_n) such that $\alpha_1, \dots, \alpha_t$ contain all real dominating characteristic roots, precisely one element of every pair of conjugate non-real dominating characteristic roots and no other characteristic roots. Note that this implies $t = 1$ or $t = 2$.

Moreover, let $\alpha_1, \dots, \alpha_t$ be ordered such that, with the notation of (3.2),

$$D := \deg Q_1 = \dots = \deg Q_l > \deg Q_{l+1} \geq \dots \geq \deg Q_t$$

for some $1 \leq l \leq t \leq 2$. Then we obtain [43]

$$n^{-D} a_n = \sum_{k=1}^{l} (c_k \alpha_k^n + \overline{c}_k \overline{\alpha}_k^n) + \mathrm{O}(n^{-1} |\alpha_1|^n), \tag{3.13}$$

where c_k is the leading coefficient of $Q_k(n)$. This formula shows that Theorem 3.3.1 can be deduced from the above-mentioned result of Burke and Webb ($l = 1$) and the following theorem ($l = 2$). Observe that we may safely assume $|\alpha_1| = |\alpha_2| = 1$, since we can divide a_n by the positive factor $|\alpha_1|^n$.

Theorem 3.3.2. *Let $\alpha_1, \alpha_2 \in \mathbb{C} \setminus [0, \infty[$ with $|\alpha_1| = |\alpha_2| = 1$ and $\alpha_1 \neq \alpha_2 \neq \overline{\alpha}_1$. Let further c_1, c_2 be nonzero complex numbers and*

$$b_n := c_1 \alpha_1^n + \overline{c}_1 \overline{\alpha}_1^n + c_2 \alpha_2^n + \overline{c}_2 \overline{\alpha}_2^n, \qquad n \geq 0. \tag{3.14}$$

Then there is $\delta > 0$ such that $b_n > \delta$ for infinitely many n and $b_n < -\delta$ for infinitely many n.

Note that if δ was replaced by zero, then it might happen that, e.g., all negative values b_n are so small in absolute value that the remainder term of a_n, which comes from the characteristic roots of smaller modulus, takes over and makes the corresponding values a_n positive. This uniformity condition was missed by Burke and Webb [12]. They only argue that $c_1 \alpha_1^n + \overline{c}_1 \overline{\alpha}_1^n$ has infinitely many positive and infinitely many negative values, which is not sufficient, but their proof can be easily repaired.

Theorem 3.3.3. *Let $\theta_1, \theta_2 \in]0, 1[\setminus \{\frac{1}{2}\}$ such that $\theta_1 \not\equiv \pm\theta_2 \pmod 1$ and, if both θ_1 and θ_2 are rational, then the pair of their denominators (written with the larger denominator first) is none of $(5, 5)$, $(6, 3)$, $(8, 4)$. Then for all $\mathbf{c} \in \mathbb{R}^2$ there is $\varepsilon > 0$ such that there are infinitely many n with*

$$n(\theta_1, \theta_2) \bmod 1 \ \in \ \mathcal{S}_{1/4-\varepsilon}(\mathbf{c}) \bmod 1.$$

Since the sine function is continuous, applying this theorem with

$$(\theta_1, \theta_2) = ((\arg \alpha_1)/2\pi, \ (\arg \alpha_2)/2\pi)$$

and $c_k = \frac{1}{4} - \varphi_k/2\pi$ to make $\sin(2\pi n \theta_k + \varphi_k)$ positive and $c_k = \frac{3}{4} - \varphi_k/2\pi$ for a negative sign proves Theorem 3.3.2 (cf. Section 3.2.1), unless one of the α_k is a negative real number (which implies $\theta_k = \frac{1}{2}$) or θ_1, θ_2 are rational numbers with denominators in

$\{(5,5),(6,3),(8,4)\}$. In these special cases of Theorem 3.3.2 our approach with (lattice) points in squares does not work, since then the signs of the pairs

$$\big(\sin(2\pi n\theta_1+\varphi_1),\ \sin(2\pi n\theta_2+\varphi_2)\big), \qquad n\geq 0,$$

do not assume all four combinations of ± 1. Still, all sequences

$$w_1\sin(2\pi n\theta_1+\varphi_1)+w_2\sin(2\pi n\theta_2+\varphi_2) \tag{3.15}$$

built from these θ_1, θ_2 oscillate (for $w_1w_2\neq 0$). The author has shown this [38] by unnecessarily involved arguments that are not of independent interest. We do not go into details, since Theorem 3.3.1 will be superseded by Theorem 3.3.11 anyways. Still, we give a complete proof of Theorem 3.3.3 here. Let us distinguish the following three cases:

(1) $\theta_1,\theta_2,1$ are linearly independent over $\mathbb{Q}$.

(2) θ_1,θ_2 are not both rational, but satisfy a linear relation $r_1\theta_1+r_2\theta_2=z$ with $r_1,r_2,z\in\mathbb{Z}$.

(3) θ_1 and θ_2 are both rational.

Case 3 of Theorem 3.3.3, the most difficult case, follows immediately from Theorem 3.2.1. Note that the purely periodic sequence

$$n(\tfrac{u_1}{v_1},\tfrac{u_2}{v_2}) \bmod 1 = (\tfrac{nu_1 \bmod v_1}{v_1},\tfrac{nu_2 \bmod v_2}{v_2}), \qquad n\geq 0,$$

assumes each of its finitely many values infinitely often. The ε has disappeared because the set of all $n(\tfrac{u_1}{v_1},\tfrac{u_2}{v_2}) \bmod 1$ is finite and $\mathcal{S}_{1/4}(\mathbf{c})$ is open.

We remark that in order to prove Theorem 3.3.1 for one pair of conjugate complex dominating roots, it suffices to show that for every real number $\theta\neq\frac{1}{2}$ with $0<\theta<1$ and every real number c there is $\varepsilon>0$ such that for infinitely many n

$$n\theta \bmod 1 \ \in\ \big]c-\tfrac{1}{4}+\varepsilon, c+\tfrac{1}{4}-\varepsilon\big[\bmod 1.$$

This is essentially what was done (without ε, cf. the remark after Theorem 3.3.2) by Burke and Webb [12].

We now turn to the proof of Theorem 3.3.3 in the cases 2 and 3. The closure of the set of integer multiples of a vector $\boldsymbol{\theta}=(\theta_1,\theta_2)$ modulo one is described by a classical result from Diophantine approximation.

Theorem 3.3.4 (Kronecker's theorem in two dimensions). *Let θ_1, θ_2 be real numbers.*

(i) If $\theta_1,\theta_2,1$ are linearly independent over the rationals, then the points $n\boldsymbol{\theta} \bmod 1$, $n\in\mathbb{N}$, lie dense in the unit square.

(ii) If θ_1,θ_2 are not both rational, but satisfy a relation $r_1\theta_1+r_2\theta_2=z$ with $r_1,r_2,z\in\mathbb{Z}$ and $\gcd(r_1,r_2,z)=1$, then the points $n\boldsymbol{\theta} \bmod 1$, $n\in\mathbb{N}$, lie dense on the portions of the lines

$$\ell_t := \left\{\mathbf{x}\in\mathbb{R}^2 : r_1x_1+r_2x_2=t\right\}, \qquad t\in\mathbb{Z},$$

which lie within the unit square.

Proof. See, e.g., Niven [66, Theorems 3.4 and 3.6]. Figure 3.5 depicts an example that illustrates part (ii) of the theorem. □

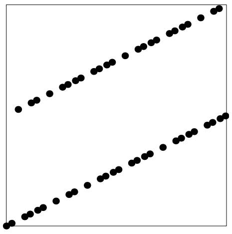

Figure 3.5: The unit square with $(n\theta_1, n\theta_2) \bmod 1$ for $\theta_1 = 2\sqrt{2}$, $\theta_1 - 2\theta_2 = 2$ and $n = 0, \dots, 50$.

Part (i) of Theorem 3.3.4 settles case 1 of Theorem 3.3.3. We proceed to case 2. Let $\mathbf{c} \in \mathbb{R}^2$ be arbitrary but fixed and ℓ_t, r_1, r_2 be as in part (ii) of Theorem 3.3.4. Since

$$\bigcup_{t\in\mathbb{Z}} \ell_t + \mathbb{Z}^2 = \bigcup_{t\in\mathbb{Z}} \ell_t,$$

it suffices to find infinitely many n such that $n\boldsymbol{\theta} \bmod 1$ is in the set

$$\mathcal{S}_{1/4-\varepsilon}(\mathbf{c}) \cap \bigcup_{t\in\mathbb{Z}} \ell_t,$$

where $\varepsilon > 0$ is yet to be chosen. First suppose that θ_1 and θ_2 are irrational. Then the parallel lines ℓ_t are neither horizontal nor vertical, since $r_1 r_2 \neq 0$. Two adjacent lines ℓ_t, ℓ_{t+1} have horizontal distance $1/|r_1|$ and vertical distance $1/|r_2|$. Since $\theta_1 \not\equiv \pm\theta_2 \pmod 1$, one of these quantities must be smaller than or equal to $\frac{1}{2}$. Thus

$$\mathcal{S}_{1/4}(\mathbf{c}) \cap \bigcup_{t\in\mathbb{Z}} \ell_t \neq \emptyset.$$

In fact this set is not only non-empty but contains a line segment. Clearly, we can find $\varepsilon > 0$ such that the set $\mathcal{S}_{1/4-\varepsilon}(\mathbf{c}) \cap \bigcup_t \ell_t$ still contains a line segment of length greater than zero. Filling this line segment densely with points $n\boldsymbol{\theta} \bmod 1$ requires infinitely many n.

Now let θ_1 be rational and θ_2 be irrational, and let $v_1 \in \mathbb{N}$ be the denominator of θ_1. This implies $r_2 = 0$. Then the lines ℓ_t are vertical, and the horizontal distance between ℓ_t and ℓ_{t+1} is $1/v_1 \leq \frac{1}{3}$, since $v_1 > 2$ by the assumptions of Theorem 3.3.3. Case 2 of Theorem 3.3.3 is proven, thus the proof of Theorem 3.3.3 is finished. Modulo the remark after (3.15), this completes the proof of Theorem 3.3.1.

Theorem 3.3.1 is unsatisfactory, since it does not come as a surprise that it holds for any number of dominating roots. In order to extend our proof to l dominating characteristic roots, we would have to show that the tuples

$$\big(\sin(2\pi n\theta_1 + \varphi_1), \dots, \sin(2\pi n\theta_l + \varphi_l))\big), \qquad n \geq 0, \tag{3.16}$$

attain each of the 2^l possible sign combinations (with only ± 1 and no zeros) infinitely often. This approach suffers from two problems. First, this sufficient condition does not hold for all $(\theta_1, \dots, \theta_l) \in \,]0,1[^l$; it follows from Theorem 3.2.1 that for $l = 2$ there are θ_1, θ_2 that do not produce all four sign combinations, although all sequences

$$w_1 \sin(2\pi n\theta_1 + \varphi_1) + w_2 \sin(2\pi n\theta_2 + \varphi_2), \qquad n \geq 0,$$

built from these θ_1, θ_2 oscillate (for $w_1 w_2 \neq 0$). Second, determining the vectors $(\theta_1, \dots, \theta_l)$ for which the above sufficient condition holds poses Diophantine geometry problems that are somewhat involved for $l = 2$ already and do not seem to become easier for larger l. We would have to show that infinitely many $n(\theta_1, \dots, \theta_l) \bmod 1$ lie in any given l-dimensional hypercube (modulo one) with side length $\frac{1}{2} - \varepsilon$. Theorem 3.3.4 generalizes in the following way [14, Theorem III.5.IV]:

Theorem 3.3.5 (Kronecker's theorem). *The points* $n\boldsymbol{\theta} \bmod 1$ *lie dense in the set of all* $\mathbf{x} \in [0,1]^l$ *that satisfy* $\langle \mathbf{u}, \mathbf{x} \rangle \in \mathbb{Z}$ *for all integer vectors* $\mathbf{u}$ *with* $\langle \mathbf{u}, \boldsymbol{\theta} \rangle \in \mathbb{Z}$. *In particular, if* $\theta_1, \dots, \theta_l, 1$ *are linearly independent over* $\mathbb{Q}$*, then the points* $n\boldsymbol{\theta} \bmod 1$ *lie dense in the unit hypercube* $[0,1]^l$.

Again the case of rational $\theta_1, \dots, \theta_l$ with equal denominators $b_1 = \cdots = b_l = g > 0$ will be the crux of the proof. This case seems to become more and more difficult for fixed denominator g as m increases, since the set

$$\left\{ n \left(\tfrac{u_1}{g}, \dots, \tfrac{u_l}{g} \right) \bmod 1 : n \in \mathbb{N} \right\} \tag{3.17}$$

has g elements for all l, whereas the volume of the hypercube is $(\frac{1}{2} - \varepsilon)^l$. Minkowski's second theorem (Theorem 3.2.12) is certainly a valuable tool. Hadwiger [45] has extended Bender's two-dimensional result (Lemma 3.2.13) that we used in the proof of Proposition 3.2.10 to arbitrary dimension m. A significant extension of Lemma 3.2.14 is still needed. Anyways there are exceptional $\boldsymbol{\theta}$ (e.g., those that have $\frac{1}{2}$ as a component) for which the hypercube might contain no point of (3.17); they require a separate argument to prove oscillation of the corresponding sequences. In the following two sections we will generalize Theorem 3.3.1 to l dominating characteristic roots by another approach that relieves us at an early stage of the troublesome integer relations between the θ_k.

3.3.2 The Density of the Positivity Set[2]

Our goal is the announced generalization of Theorem 3.3.1. We will pass from $\theta_1, \dots, \theta_l$ to a module basis of $\mathbb{Z} + \sum \mathbb{Z}\theta_k$ with the property that one of its elements is rational and the other elements do not satisfy integer relations. By splitting the sequence into subsequences we will then reduce the problem to two extreme cases: Either the θ_k do not satisfy integer relations, or they are all rational. The latter case, which caused us some troubles in the proof of Theorem 3.3.1, loses its daunting character if we consider the original problem ((a_n) oscillates) instead of the sufficient condition involving the signs of the tuples (3.16).

In fact we will prove more than the oscillating behaviour, viz. that the positivity set and the negativity set both have positive density. The density of a set A of natural numbers is defined as

$$\boldsymbol{\delta}(A) := \lim_{x \to \infty} x^{-1} \sharp\{n \leq x : n \in A\},$$

provided that the limit exists. In the present section we will show that the density the positivity set

$$\{n \in \mathbb{N} : a_n > 0\}$$

of a C-finite sequence (a_n) always exists, without any assumption on the characteristic roots. The proof will be adapted in the next section in order to obtain the announced generalization of Theorem 3.3.1 to l dominating characteristic roots.

[2]The material in Sections 3.3.2 to 3.3.5 arose from a collaboration with Jason P. Bell.

Theorem 3.3.6. *Let (a_n) be a C-finite sequence. Then the density of the set $\{n \in \mathbb{N} : a_n > 0\}$ exists.*

Since $(-a_n)$ is C-finite, too, the analogous result for the negativity set is equivalent to Theorem 3.3.6. The goal of this section is to prove Theorem 3.3.6. We use the notation of Equation (3.2). Dividing a_n by $n^D|\alpha_1|^n$, where α_1 is a dominating root of (a_n), and D is the maximal degree of the Q_k with $|\alpha_k| = |\alpha_1|$, we obtain (cf. (3.4) and (3.13))

$$n^{-D}|\alpha_1|^{-n}a_n = \sum_{i=1}^{l} w_i \sin(2\pi n\theta_i + \varphi_i) + v - r_n,$$

where $r_n = \mathrm{O}(1/n)$ is a C-finite sequence, $\theta_1, \dots, \theta_l$ are in $]0, 1[$, and $w_i, \varphi_i, v \in \mathbb{R}$. From now on we will assume w.l.o.g. $D = 0$ and $|\alpha_1| = 1$. As a first step we get rid of any integer relations that the θ_i's might satisfy.

Lemma 3.3.7. *Let $\theta_1, \dots, \theta_l$ be real numbers. Then there is a basis $\{\tau_1, \dots, \tau_{m+1}\}$ of the $\mathbb{Z}$-module*

$$M := \mathbb{Z} + \mathbb{Z}\theta_1 + \dots \mathbb{Z}\theta_l$$

such that $1/\tau_{m+1}$ is a positive integer and $1, \tau_1, \dots, \tau_m$ are linearly independent over $\mathbb{Q}$.

Proof. M is finitely generated and torsion free, hence it is free [55, Theorem III.7.3]. Let $\{\gamma_1, \dots, \gamma_{m+1}\}$ be a basis. Since $1 \in M$, there are integers $e_1, \dots, e_{m+1}$ such that

$$e_1\gamma_1 + \dots + e_{m+1}\gamma_{m+1} = 1.$$

We complete $(e_1/g, \dots, e_{m+1}/g)$, where $g := \gcd(e_1, \dots, e_{m+1})$, to a unimodular integer matrix C with last row $(e_1/g, \dots, e_{m+1}/g)$ [55, §XXI.3]. Then

$$(\tau_1, \dots, \tau_{m+1})^T := C(\gamma_1, \dots, \gamma_{m+1})^T$$

yields a basis of M with $\tau_{m+1} = 1/g$. Now suppose

$$u_1\tau_1 + \dots + u_m\tau_m = u$$

for integers $u_1, \dots, u_m, u$. Since u has also the representation

$$ug\tau_{m+1} = u,$$

it follows $u_1 = \dots = u_m = u = 0$. □

Take $\tau_1, \dots, \tau_{m+1}$ as in Lemma 3.3.7, with $\tau_{m+1} = 1/g$. Roughly speaking, we have put all integer relations among the θ_i into the rational basis element τ_{m+1}. There are integers b_{ij} with

$$\theta_i = \sum_{j=1}^{m+1} b_{ij}\tau_j.$$

Now we split the sequence (a_n) into the subsequences $(a_{gn+k})_{n\ge0}$ for $0 \le k < g$. We have

$$a_{gn+k} = G_n - s_n,$$

where $s_n := r_{gn+k}$ and G_n is the dominant part. Both s_n and G_n depend on k. Defining the integer matrix

$$B := (gb_{ij})_{\substack{1\le i\le l\\ 1\le j\le m}} \in \mathbb{Z}^{l\times m}$$

and the real vector $\mathbf{c} = (c_1, \ldots, c_l)$ with

$$c_i := 2\pi k \sum_{j=1}^{m+1} b_{ij}\tau_j + \varphi_i, \qquad 1 \le i \le l,$$

it can be written as

$$\begin{aligned} G_n &= \sum_{i=1}^{l} w_i \sin\left(2\pi\,(gn+k)\sum_{j=1}^{m+1} b_{ij}\tau_j + \varphi_i\right) + v \\ &= \sum_{i=1}^{l} w_i \sin\left(2\pi n \sum_{j=1}^{m} g b_{ij}\tau_j + 2\pi k \sum_{j=1}^{m+1} b_{ij}\tau_j + \varphi_i\right) + v \\ &= \mathbf{w}^T \sin(2\pi n B\boldsymbol{\tau} + \mathbf{c}) + v, \end{aligned}$$

where sin is applied component-wise. We show that the density of $\{n \in \mathbb{N} : a_{gn+k} > 0\}$ exists for each k. Since (s_n) is a C-finite sequence with fewer characteristic roots than (a_n), we may assume inductively that $\boldsymbol{\delta}(\{n \in \mathbb{N} : s_n < 0\})$ exists. Thus, if (G_n) is the zero sequence, then we are done. Now let k be such that (G_n) is not identically zero. It is plain that $G_n = H(n\boldsymbol{\tau})$, where

$$H(\mathbf{t}) := \mathbf{w}^T \sin(2\pi B\mathbf{t} + \mathbf{c}) + v, \qquad \mathbf{t} \in [0,1]^m.$$

The following theorem shows that the function H can be used to evaluate the density of the positivity set of G_n, which equals, as we will see below, that of the set $\{n \in \mathbb{N} : a_{gn+k} > 0\}$.

Theorem 3.3.8 (Kronecker-Weyl). *Let $\tau_1, \ldots, \tau_m$ be real numbers such that the numbers $1, \tau_1, \ldots, \tau_m$ are linearly independent over $\mathbb{Q}$. Then for every Jordan measurable set $A \subseteq [0,1]^m$ we have*

$$\boldsymbol{\delta}(\{n \in \mathbb{N} : n\boldsymbol{\tau} \bmod 1 \in A\}) = \boldsymbol{\lambda}(A).$$

Proof. We refer to Cassels [14, Theorems IV.I and IV.II]. ☐

The notions from measure theory that we use can be found in any introductory textbook on this subject. Theorem 3.3.8 extends part (i) of Theorem 3.3.4 to m real numbers and makes the stronger assertion that the points $n\boldsymbol{\tau} \bmod 1$ are not only dense, but uniformly distributed in the unit hypercube. By Theorem 3.3.8 and the 1-periodicity of H, the density of the positivity set of (G_n) exists and equals

$$\begin{aligned} \boldsymbol{\delta}(\{n \in \mathbb{N} : G_n > 0\}) &= \boldsymbol{\delta}(\{n \in \mathbb{N} : H(n\boldsymbol{\tau}) > 0\}) \\ &= \boldsymbol{\lambda}(\{t \in [0,1]^m : H(\mathbf{t}) > 0\}). \end{aligned}$$

We define

$$L_\varepsilon := \{n \in \mathbb{N} : G_n \ge \varepsilon\} \qquad \text{and} \qquad S_\varepsilon := \{n \in \mathbb{N} : |G_n| < \varepsilon\}. \tag{3.18}$$

The corresponding sets for the function H are defined as

$$\widetilde{L}_\varepsilon := \{\mathbf{t} \in [0,1]^m : H(\mathbf{t}) \ge \varepsilon\} \qquad \text{and} \qquad \widetilde{S}_\varepsilon := \{\mathbf{t} \in [0,1]^m : |H(\mathbf{t})| < \varepsilon\}.$$

Since for all $\varepsilon \geq 0$

$$L_\varepsilon = \{n \in \mathbb{N} : n\boldsymbol{\tau} \bmod 1 \in \widetilde{L}_\varepsilon\},$$

we have $\boldsymbol{\delta}(L_\varepsilon) = \boldsymbol{\lambda}(\widetilde{L}_\varepsilon)$ for all $\varepsilon \geq 0$ by Theorem 3.3.8. Similarly,

$$\boldsymbol{\delta}(S_\varepsilon) = \boldsymbol{\lambda}(\widetilde{S}_\varepsilon), \qquad \varepsilon > 0. \tag{3.19}$$

Note that the boundary of the bounded set $\widetilde{S}_\varepsilon$ (respectively $\widetilde{L}_\varepsilon$) is a Lebesgue null set (as seen by applying the following lemma with $F(\mathbf{t}) = H(\mathbf{t}) - \varepsilon$), hence $\widetilde{S}_\varepsilon$ and $\widetilde{L}_\varepsilon$ are Jordan measurable, and Theorem 3.3.8 is indeed applicable.

Lemma 3.3.9. *Let $F : \mathbb{R}^m \to \mathbb{R}$ be a real analytic function. Then the zero set of F has Lebesgue measure zero, unless F vanishes identically.*

Lemma 3.3.9 seems to be known [48], but we could not find a complete proof in the literature. We give a proof at the end of this section. Since (G_n) is not the zero sequence, the function H does not vanish identically on $[0,1]^m$. By the Lebesgue dominated convergence theorem and Lemma 3.3.9 we thus find

$$\lim_{\varepsilon \to 0} \boldsymbol{\lambda}(\widetilde{S}_\varepsilon) = 0 \qquad \text{and} \qquad \lim_{\varepsilon \to 0} \boldsymbol{\lambda}(\widetilde{L}_\varepsilon) = \boldsymbol{\lambda}(\widetilde{L}_0).$$

This yields $\boldsymbol{\delta}(\{n \in \mathbb{N} : G_n > s_n\}) = \boldsymbol{\lambda}(\widetilde{L}_0)$ by the following lemma, which completes the proof of Theorem 3.3.6.

Lemma 3.3.10. *Let G_n and s_n be real sequences with $s_n = \mathrm{o}(1)$, and let L_ε, S_ε be as in (3.18). Suppose that $\boldsymbol{\delta}(L_\varepsilon)$ and $\boldsymbol{\delta}(S_\varepsilon)$ exist for all $\varepsilon \geq 0$, and that*

$$\lim_{\varepsilon \to 0} \boldsymbol{\delta}(L_\varepsilon) = \boldsymbol{\delta}(L_0) \qquad \textit{and} \qquad \lim_{\varepsilon \to 0} \boldsymbol{\delta}(S_\varepsilon) = 0.$$

Then

$$\boldsymbol{\delta}(\{n \in \mathbb{N} : G_n > s_n\}) = \boldsymbol{\delta}(L_0).$$

Proof. For any set $A \subseteq \mathbb{N}$ we write $A(x) := \{n \leq x : n \in A\}$. Define

$$P := \{n \in \mathbb{N} : G_n > s_n\}.$$

Let $\varepsilon > 0$ be arbitrary. Take n_0 such that $|s_n| < \varepsilon$ for $n > n_0$. It follows

$$\begin{aligned} \sharp P(x) = \sharp\{n \leq n_0 : G_n > s_n\} &+ \sharp\{n_0 < n \leq x : G_n \geq \varepsilon\} \\ &+ \sharp\{n_0 < n \leq x : s_n < G_n < \varepsilon\}, \end{aligned}$$

hence

$$|\sharp P(x) - \sharp L_\varepsilon(x)| \leq \sharp S_\varepsilon(x) + \mathrm{o}(x)$$

as $x \to \infty$. Thus we have

$$\begin{aligned} |x^{-1}\sharp P(x) - \boldsymbol{\delta}(L_0)| &\leq |x^{-1}\sharp P(x) - x^{-1}\sharp L_\varepsilon(x)| + |x^{-1}\sharp L_\varepsilon(x) - \boldsymbol{\delta}(L_0)| \\ &\leq x^{-1}\sharp S_\varepsilon(x) + |x^{-1}\sharp L_\varepsilon(x) - \boldsymbol{\delta}(L_0)| + \mathrm{o}(1). \end{aligned}$$

The right hand side tends to

$$\boldsymbol{\delta}(S_\varepsilon) + |\boldsymbol{\delta}(L_\varepsilon) - \boldsymbol{\delta}(L_0)|$$

as $x \to \infty$. By assumption, this can be made arbitrarily small, which implies $\boldsymbol{\delta}(P) = \boldsymbol{\delta}(L_0)$. ☐

Proof of Lemma 3.3.9. For $m = 1$ this is clear, since then the zero set is countable. Now assume that we have established the result for $1, \dots, m-1$. Put

$$V := \{(t_2, \dots, t_m) \in \mathbb{R}^{m-1} : F(\cdot, t_2, \dots, t_m) \text{ vanishes identically } \}.$$

Take a real number s such that $F(s, \cdot, \dots, \cdot)$ is not identically zero. Clearly, we have $F(s, t_2, \dots, t_m) = 0$ for all $(t_2, \dots, t_m) \in V$. By the induction hypothesis, this implies $\boldsymbol{\lambda}(V) = 0$. Note that V is closed, hence measurable. Since F is real analytic in the first argument, we have

$$\int_{\mathbb{R}} \mathbf{1}_Z(t_1, \dots, t_m) \mathrm{d}\boldsymbol{\lambda}(t_1) = 0$$

for all $(t_2, \dots, t_m) \notin V$, where $\mathbf{1}_Z$ is the characteristic function of the zero set

$$Z := \{(t_1, \dots, t_m) \in \mathbb{R}^m : F(t_1, \dots, t_m) = 0\}.$$

Since V has measure zero, this implies

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathbf{1}_Z(t_1, \dots, t_m) \mathrm{d}\boldsymbol{\lambda}(t_1) \cdots \mathrm{d}\boldsymbol{\lambda}(t_m) = 0.$$

This argument works for any order of integration, hence we obtain $\int_{\mathbb{R}^m} \mathbf{1}_Z = 0$ by Tonelli's theorem. □

3.3.3 Sequences with no Positive Dominating Root II

We return to the problem of showing that C-finite sequences with no real positive dominating root always have infinitely many positive and infinitely many negative values. The ideas from the previous section yield the desired generalization of Theorem 3.3.1.

Theorem 3.3.11. *Let (a_n) be a C-finite sequence, not identically zero and with no positive dominating characteristic root. Then the sets $\{n \in \mathbb{N} : a_n > 0\}$ and $\{n \in \mathbb{N} : a_n < 0\}$ have positive density.*

We begin by settling the special cases where the θ_i are all irrational or all rational, and then put them together.

Lemma 3.3.12. *Let $\theta_1, \dots, \theta_l$ be irrational numbers, and let w_i, φ_i be real numbers such that the sequence*

$$u_n := \sum_{i=1}^{l} w_i \sin(2\pi n \theta_i + \varphi_i)$$

is not identically zero. Let further (r_n) be a C-finite sequence with $r_n = \mathrm{o}(1)$. Then the set $\{n \in \mathbb{N} : u_n > r_n\}$ has positive density.

Proof. Proceeding as in the proof of Theorem 3.3.6, we can write

$$G_n := u_{gn+k} = \mathbf{w}^T \sin(2\pi n B \boldsymbol{\tau} + \mathbf{c}),$$

where B is an integer matrix no row of which is zero, g a positive integer, $\mathbf{c}$ a real vector, and $1, \tau_1, \dots, \tau_m$ are linearly independent over $\mathbb{Q}$. If k is such that $G_n = u_{gn+k}$ vanishes for all n, which we abbreviate by $G_n \equiv 0$, then the density of $\{n \in \mathbb{N} : G_n > s_n\}$, where $s_n := r_{gn+k}$, exists by Theorem 3.3.6, but may be zero. Now choose a k_0 such that the corresponding sequence $G_n = u_{gn+k_0}$ is not the zero sequence. We have $G_n = H(n\boldsymbol{\tau})$, where

$$H(\mathbf{t}) := \mathbf{w}^T \sin(2\pi B\mathbf{t} + \mathbf{c}).$$

Moreover, with the notation of the proof of Theorem 3.3.6, we have

$$\boldsymbol{\delta}(\{n \in \mathbb{N} : G_n > s_n\}) = \boldsymbol{\lambda}(\widetilde{L}_0).$$

The function H is not identically zero on $[0,1]^m$. But

$$\int_0^1 \cdots \int_0^1 H(t_1, \ldots, t_m)\mathrm{d}t_1 \cdots \mathrm{d}t_m = 0, \tag{3.20}$$

because no row of B is the zero vector. Hence H has a positive value on $[0,1]^m$, and since it is continuous, we have $\boldsymbol{\lambda}(\widetilde{L}_0) > 0$. □

Observe that the integral in (3.20) need not vanish if B has a zero row, which can only happen if the θ_i corresponding to this row is a rational number. This is the reason why we consider rational θ_i's separately.

Lemma 3.3.13. *Let $\theta_1, \ldots, \theta_l$ be rational numbers in $]0,1[$, and let w_i, φ_i be real numbers such that the purely periodic sequence*

$$u_n = \sum_{i=1}^{l} w_i \sin(2\pi n\theta_i + \varphi_i)$$

is not identically zero. Then u_n has a positive and a negative value (and thus infinitely many of each).

Proof. By the identity

$$\sum_{k=0}^{q-1} \cos \tfrac{2\pi kp}{q} + \mathrm{i} \sum_{k=0}^{q-1} \sin \tfrac{2\pi kp}{q} = \sum_{k=0}^{q-1} \mathrm{e}^{2\pi \mathrm{i} kp/q} = 0,$$

valid for integers $0 < p < q$, and the addition formula of the sine function we obtain

$$\begin{aligned} \sum_{k=0}^{q-1} u_k &= \sum_{k=0}^{q-1} \sum_{i=1}^{l} w_i \sin(2\pi k\theta_i + \varphi_i) \\ &= \sum_{i=1}^{l} w_i \sum_{k=0}^{q-1} (\cos\varphi_i \sin 2\pi k\theta_i + \sin\varphi_i \cos 2\pi k\theta_i) = 0, \end{aligned}$$

where q is a common denominator of $\theta_1, \ldots, \theta_l$. Since not all of $u_0, \ldots, u_{q-1}$ are zero, there must be positive and negative elements among them. □

Proof of Theorem 3.3.11. It suffices to consider the positivity set. We may write

$$a_n = u_n + v_n - r_n,$$

where $r_n = \mathrm{o}(1)$ is a C-finite sequence,

$$u_n = \sum_{i=1}^{l} w_i \sin(2\pi n\theta_i + \varphi_i),$$

$$v_n = \sum_{i=l+1}^{e} w_i \sin(2\pi n\theta_i + \varphi_i),$$

$\theta_1, \ldots, \theta_l$ are irrational, $\theta_{l+1}, \ldots, \theta_e$ are rational numbers in $]0,1[$ with common denominator $q > 0$, and $u_n + v_n \not\equiv 0$. If $v_n \equiv 0$, then the result follows from Lemma 3.3.12. Now suppose $v_n \not\equiv 0$. Then for each k the density of the set $\{n \in \mathbb{N} : a_{qn+k} > 0\}$ exists by Theorem 3.3.6. By Lemma 3.3.13 there is k_0 such that $v_{qn+k_0} = v > 0$. It suffices to show that the set $\{n \in \mathbb{N} : a_{qn+k_0} > 0\}$ has positive density. This is clear if $u_{qn+k_0} \equiv 0$. Otherwise, notice that

$$\{n \in \mathbb{N} : a_{qn+k_0} > 0\} \supseteq \{n \in \mathbb{N} : u_{qn+k_0} > r_{qn+k_0}\},$$

and the latter set has positive density by Lemma 3.3.12. □

The proof of Theorem 3.3.11 is complete.

3.3.4 The Possible Values of the Density

In Section 3.3.2 we established that the density of the positivity set of a C-finite sequence always exists. Now a natural question is what values it can assume. In its basic form, the question is readily answered:

Example 3.3.14. *Let w be a real number and define*

$$a_n := \sin(2\pi n\sqrt{2}) - w.$$

Then, by Theorem 3.3.8,

$$\begin{aligned} \boldsymbol{\delta}(\{n \in \mathbb{N} : a_n > 0\}) &= \boldsymbol{\lambda}(\{t \in [0,1] : \sin(2\pi t) > w\}) \\ &= \begin{cases} 1, & w \leq -1 \\ \frac{1}{2} - \frac{1}{\pi}\arcsin w, & -1 \leq w \leq 1 \\ 0, & w \geq 1 \end{cases}. \end{aligned}$$

Since the range of arcsin *is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, for every $\kappa \in [0,1]$ this yields a C-finite sequence (a_n) such that*

$$\boldsymbol{\delta}(\{n \in \mathbb{N} : a_n > 0\}) = \kappa.$$

More generally, we could try to simultaneously prescribe the densities of the positivity set and the zero set. The latter is restricted by the following well-known theorem [29], one of the most charming and celebrated results about C-finite sequences.

Theorem 3.3.15 (Skolem-Mahler-Lech). *The zero set of a C-finite sequence is the union of a finite set and finitely many arithmetic progressions.*

Thus, the density of the zero set of a C-finite sequence is always a rational number. (In Section 3.3.5 we will prove this without using Theorem 3.3.15.)

Proposition 3.3.16. *Let κ be a real number and r be a rational number with $0 \leq \kappa, r \leq 1$ and $\kappa + r \leq 1$. Then there is a C-finite sequence (a_n) such that*

$$\boldsymbol{\delta}(\{n \in \mathbb{N} : a_n > 0\}) = \kappa \qquad \text{and} \qquad \boldsymbol{\delta}(\{n \in \mathbb{N} : a_n = 0\}) = r.$$

Proof. Suppose that $r = p/q$ for positive integers p and q. As seen in Example 3.3.14, there is a C-finite sequence (g_n) such that the density of the zero set of (g_n) is zero and the density of its positivity set is $\kappa/(1-r)$ (The case $r=1$ is trivial). The interlacing sequence

$$a_{bn+k} := \begin{cases} 0, & 0 \le k < p \\ g_n, & p \le k < q \end{cases}$$

is a C-finite sequence [29, Section 4.1]. Clearly, the density of its zero set is r, and the density of its positivity set is

$$\boldsymbol{\delta}(\{n \in \mathbb{N} : a_n > 0\}) = \frac{q-p}{q} \times \frac{\kappa}{1-r} = \kappa,$$

as required. □

If we restrict attention to sequences without dominating real positive roots, then Theorem 3.3.11 tells us that the density of the positivity set can be neither zero nor one. Computer experiments with arbitrary numerical values for the parameters usually yield approximations of the density that are close to $\frac{1}{2}$. Still, all values from the open unit interval occur.

Theorem 3.3.17. *Let $\kappa \in]0,1[$. Then there is a C-finite sequence (a_n) with no positive dominating characteristic root and $\boldsymbol{\delta}(\{n \in \mathbb{N} : a_n > 0\}) = \kappa$.*

Proof. Let $\varepsilon > 0$ be arbitrary. We define a function H on $[0, \frac{1}{2}]$ by

$$H(t) := \begin{cases} \frac{(\varepsilon-1)^2}{\varepsilon}\left(1 - \frac{2t}{\varepsilon}\right), & 0 \le t \le \frac{\varepsilon}{2} \\ \varepsilon - 2t, & \frac{\varepsilon}{2} \le t \le \frac{1}{2} \end{cases}$$

and extend it to an even, 1-periodic function H on $\mathbb{R}$ (see Figure 3.6). It is continuous and satisfies

$$\int_0^1 H(t)\mathrm{d}t = 0 \qquad \text{and} \qquad \boldsymbol{\lambda}(\{t \in [0,1] : H(t) > 0\}) = \varepsilon.$$

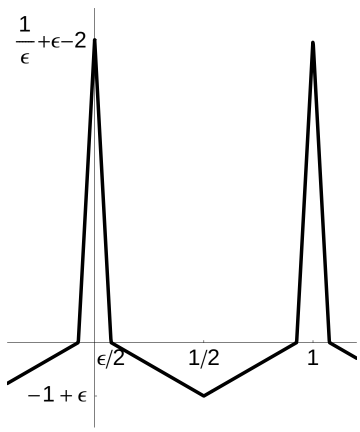

Figure 3.6: The function H

Expanding H into a Fourier series, we find that there are real w_j such that H is the pointwise limit of

$$H_m(t) := \sum_{j=1}^{m} w_j \sin(2\pi j t)$$

as $m \to \infty$. The Lebesgue dominated convergence theorem yields

$$\begin{aligned}
\lim_{m\to\infty} \boldsymbol{\lambda}(\{t \in [0,1] : H_m(t) > 0\}) &= \lim_{m\to\infty} \int_0^1 \mathbf{1}_{\{H_m>0\}}(t)\mathrm{d}t \\
&= \int_0^1 \lim_{m\to\infty} \mathbf{1}_{\{H_m>0\}}(t)\mathrm{d}t \\
&= \int_0^1 \mathbf{1}_{\{H>0\}}(t)\mathrm{d}t \\
&= \boldsymbol{\lambda}(\{H > 0\}) = \varepsilon.
\end{aligned}$$

In the third equality we have used that the zero set of H is a null set. Now take l such that

$$\boldsymbol{\lambda}(\{t \in [0,1] : H_l(t) > 0\}) \le 2\varepsilon.$$

The function

$$\phi(W_1, \dots, W_l) := \boldsymbol{\lambda}\Big(\Big\{t \in [0,1] : \sum_{j=1}^{l} W_j \sin(2\pi j t) > 0\Big\}\Big)$$

is continuous on $\mathbb{R}^l \setminus \{\mathbf{0}\}$. To see this, observe that ϕ is continuous at all points $(W_1, \dots, W_l)$ for which $\sum_{j=1}^{l} W_j \sin(2\pi j t)$ is not identically zero (this follows from the Lebesgue dominated convergence theorem and Lemma 3.3.9) and appeal to the uniqueness of the Fourier expansion. Since $\phi(1, 0, \dots, 0) = \frac{1}{2}$ and $\phi(w_1, \dots, w_l) \le 2\varepsilon$, the function ϕ assumes every value from $[2\varepsilon, \frac{1}{2}]$ by the intermediate value theorem.

Hence the positivity sets of the sequences

$$a_n := \sum_{j=1}^{l} W_j \sin(2\pi j n \sqrt{2})$$

assume all densities from $[2\varepsilon, \frac{1}{2}]$ for appropriate choices of $(W_1, \dots, W_l)$ by Theorem 3.3.8. Repeating the whole argument with $-H$ instead of H yields the desired result for $\kappa \in [\frac{1}{2}, 1 - 2\varepsilon]$. Since ε was arbitrary, the theorem is proven. □

3.3.5 A Weak Version of the Skolem-Mahler-Lech Theorem

Without using the Skolem-Mahler-Lech theorem (Theorem 3.3.15), it follows from Theorem 3.3.6 and the partition

$$\mathbb{N} = \{n : a_n = 0\} \cup \{n : a_n > 0\} \cup \{n : a_n < 0\}$$

that the density of the zero set of a C-finite sequence (a_n) exists. We can show a bit more with our approach. Recall, however, that we only deal with real sequences, whereas the Skolem-Mahler-Lech theorem holds for any field of characteristic zero.

Proposition 3.3.18. *The density of the zero set of a (real) C-finite sequence (a_n) is a rational number.*

Proof. Let k be a natural number, and let g, G_n, and s_n be as in the proof of Theorem 3.3.6. If k is such that $G_n \equiv 0$, then the density of the zero set of a_{gn+k} is rational, since we may assume inductively that the density of $\{n : s_n = 0\}$ is rational.

Now suppose $G_n \not\equiv 0$. The zero set of a_{gn+k} can be partitioned as

$$\{n \in \mathbb{N} : G_n = s_n\} = \{n : G_n = s_n, |G_n| < \varepsilon\} \cup \{n : G_n = s_n, |G_n| \geq \varepsilon\},$$

where $\varepsilon \geq 0$ is arbitrary. The latter set is finite, and the first one is contained in S_ε, defined in (3.18). Hence

$$\boldsymbol{\delta}(\{n \in \mathbb{N} : G_n = s_n\}) \leq \boldsymbol{\delta}(S_\varepsilon)$$

for all $\varepsilon \geq 0$. But we know that $\lim_{\varepsilon \to 0} \boldsymbol{\delta}(S_\varepsilon) = 0$ from the proof of Theorem 3.3.6, which yields

$$\boldsymbol{\delta}(\{n \in \mathbb{N} : G_n = s_n\}) = 0.$$

Thus, the zero sets of all subsequences $(a_{gn+k})_{n\geq 0}$, $0 \leq k < g$, have rational density, which proves the desired result. □

3.3.6 A Positive Real Characteristic Root

We continue our investigations on the sign of C-finite sequences with some results on sequences with positive dominating roots. Consider the sequence defined by

$$a_n := \sum_{k=1}^{l} w_k \sin(2\pi n\theta_k + \varphi_k) + 1 + \mathrm{o}(1), \qquad n \geq 0, \tag{3.21}$$

where $\theta_1, \ldots, \theta_l$, $w_1, \ldots, w_l$ are nonzero real numbers, and $\varphi_1, \ldots, \varphi_l$ are real numbers. Here and throughout this section we assume that the coefficient of the real positive root is positive (and thus w.l.o.g. equals one). Analogous considerations apply for a negative coefficient. The behaviour of (a_n) depends on how 1 compares to

$$\begin{aligned} S &:= \sup_{n\geq 0} \left(- \sum_{k=1}^{l} w_k \sin(2\pi n\theta_k + \varphi_k) \right) \\ &= - \inf_{n\geq 0} \sum_{k=1}^{l} w_k \sin(2\pi n\theta_k + \varphi_k) \;\in\;]-W, W], \end{aligned}$$

where $W := \sum_{k=1}^{l} |w_k|$. The sequence (a_n) is positive for large n if $S < 1$ (in particular, if $W < 1$), and it oscillates if $S > 1$. If $S = 1$, the behaviour of (a_n) depends on how well $\sum_{k=1}^{l} w_k \sin(2\pi n\theta_k + \varphi_k)$ approximates -1 and on the o(1) term.

The preceding discussion gives a handy criterion only for $W < 1$, which was already noted by Burke and Webb [12]. For $W \geq 1$ we confine ourselves to arbitrary, but fixed parameters φ_k and w_k and show how (a_n) behaves for almost all values of the θ_k. For $W > 1$ it is an easy consequence of Kronecker's theorem (Theorem 3.3.5) that (a_n) oscillates for almost all choices of the θ_k. If $W = 1$, then it is not immediately obvious what happens generically. In order to produce infinitely many negative values in this case, the o(1) term has to be negative for all n from some infinite set, and the sum $\sum_{k=1}^{l} w_k \sin(2\pi n\theta_k + \varphi_k)$ has to approximate -1 well enough as n runs through this set. The latter observation suggests appealing to the following result from Diophantine approximation.

Lemma 3.3.19. *Let $\boldsymbol{\alpha} \in \mathbb{R}^l$ and let (ψ_n) be a sequence of positive real numbers such that $\sum_{n\geq 0} \psi_n^l$ converges. Then the system of inequalities*

$$(n\theta_k - \alpha_k) \bmod 1 < \psi_n, \qquad 1 \leq k \leq l,$$

has infinitely many solutions $n \in \mathbb{N}$ for almost no $\boldsymbol{\theta} \in \mathbb{R}^l$.

Proof. See Cassels [14, Lemma VII.2.1]. □

In order to apply the following theorem we require the dominating characteristic roots to be simple. This assumption makes the remainder term r_n go to zero exponentially. Parts (i) and (iii) hold for multiple roots as well, since they only require $r_n = \mathrm{o}(1)$. Our proof of part (ii), however, breaks down for $l = 1$ in case of a multiple root, because then we can ensure only $r_n = \mathrm{O}(n^{-1})$, and this leads to a divergent series in Lemma 3.3.19.

Theorem 3.3.20. *Let $w_1, \ldots, w_l$ be nonzero real numbers with $W := \sum_{k=1}^{l} |w_k|$, $\varphi_1, \ldots, \varphi_l$ be real numbers, and (r_n) be a real sequence with $r_n = \mathrm{O}(\omega^n)$ for some $0 < \omega < 1$.*

(i) If $W < 1$, then for all $\boldsymbol{\theta} \in \mathbb{R}^l$ the sequence (a_n) defined by

$$a_n := \sum_{k=1}^{l} w_k \sin(2\pi n\theta_k + \varphi_k) + 1 + r_n$$

is positive for large n.

(ii) If $W = 1$, then for almost all $\boldsymbol{\theta} \in \mathbb{R}^l$ the sequence (a_n) is positive for large n.

(iii) If $W > 1$, then (a_n) oscillates for almost all $\boldsymbol{\theta} \in \mathbb{R}^l$.

Proof. (i) is clear. (iii) follows from Theorem 3.3.5, because $\theta_1, \ldots, \theta_l$ are linearly independent over the rationals for almost all $\boldsymbol{\theta}$. We proceed to prove (ii). Suppose $a_n \le 0$ for all n in an infinite set $I \subseteq \mathbb{N}$. To make a_n non-positive, $\sin(2\pi n\theta_k + \varphi_k)$ has to be very close to -1 for the k's with $w_k > 0$ and very close to 1 if $w_k < 0$. To be precise, we must have

$$\lim_{\substack{n\to\infty \\ n\in I}} f(n) = 0$$

for

$$f(n) := (f_1(n), \ldots, f_l(n))$$

with

$$f_k(n) := \begin{cases} (2\pi n\theta_k + \varphi_k - \frac{1}{2}\pi) \bmod 2\pi, & w_k < 0 \\ (2\pi n\theta_k + \varphi_k - \frac{3}{2}\pi) \bmod 2\pi, & w_k > 0 \end{cases}.$$

By Taylor expansion, we obtain

$$\begin{aligned} \sum_{k=1}^{l} w_k \sin(2\pi n\theta_k + \varphi_k) + 1 &= -\sum_{k=1}^{l} |w_k| + \frac{1}{2}\sum_{k=1}^{l} |w_k| f_k(n)^2 + 1 + \mathrm{O}\left(\sum_{k=1}^{l} f_k(n)^4\right) \\ &= \frac{1}{2}\sum_{k=1}^{l} |w_k| f_k(n)^2 + \mathrm{O}\left(\sum_{k=1}^{l} f_k(n)^4\right) \qquad \text{as } n \to \infty \quad \text{in } I. \end{aligned}$$

Removing finitely many elements from I if necessary, we thus have

$$\sum_{k=1}^{l} w_k \sin(2\pi n\theta_k + \varphi_k) + 1 > \frac{w}{3}\sum_{k=1}^{l} f_k(n)^2, \qquad n \in I,$$

where $w := \min_{1\le k\le l} |w_k| > 0$. Since $a_n \le 0$ for $n \in I$, this implies

$$\sum_{k=1}^{l} f_k(n)^2 < -\frac{3r_n}{w} = \mathrm{O}(\omega^n), \qquad n \in I,$$

hence for $1 \le k \le l$

$$f_k(n) = \mathrm{O}(\omega^{n/2}) \qquad \text{as } n \to \infty \text{ in } I.$$

According to Lemma 3.3.19 this holds for almost no $\boldsymbol{\theta}$. □

Finer questions may be asked about the sets of measure zero alluded to in Theorem 3.3.20. As for part (ii) of the theorem, we note that there are $\varphi_1, \dots, \varphi_l$ and (r_n) such that there are infinitely many $\boldsymbol{\theta}$ such that (a_n) oscillates for all nonzero $w_1, \dots, w_l$ with $W = \sum |w_k| = 1$. To see this, define

$$\varphi_k := \begin{cases} \frac{1}{2}\pi, & w_k < 0 \\ \frac{3}{2}\pi, & w_k > 0 \end{cases},$$

let $\boldsymbol{\theta} \in \mathbb{Q}^l$ be arbitrary, and $r_n := (-\omega)^{n+1}$ for some $0 < \omega < 1$. Then $a_n \ge (-\omega)^{n+1} = \omega^{n+1} > 0$ for odd n, and

$$\begin{aligned} a_n &= \sum_{w_k<0} w_k \sin \tfrac{1}{2}\pi + \sum_{w_k>0} w_k \sin \tfrac{3}{2}\pi + 1 + (-\omega)^{n+1} \\ &= -W + 1 - \omega \\ &= -\omega < 0 \end{aligned}$$

if n is two times a common multiple of the denominators of $\theta_1, \dots, \theta_l$. The preceding example is a special case of the following proposition, which completely describes the behaviour of (a_n) under the assumptions of part (ii) of Theorem 3.3.20 and the additional constraint that all θ_k be rational.

Proposition 3.3.21. *Let $\theta_k = u_k/v_k$ be rational numbers for $1 \le k \le m$, let $\varphi_1, \dots, \varphi_l$ be real numbers, let $w_1, \dots, w_l$ be nonzero real numbers with $\sum_{k=1}^{l} |w_k| = 1$, and define*

$$a_n := \sum_{k=1}^{l} w_k \sin(2\pi n\theta_k + \varphi_k) + 1 + \mathrm{o}(1), \qquad n \ge 0.$$

(i) If there is a k such that φ_k/π is irrational, then (a_n) is positive for large n.

(ii) Suppose that $\varphi_k/2\pi$ is a rational number c_k/d_k for $1 \le k \le m$. If for all $1 \le k, l \le m$

$$v_k(A_k d_k - 4c_k) \equiv v_l(A_l d_l - 4c_l) \pmod{4 \gcd(d_k v_k, d_l v_l)} \qquad (3.22)$$

with

$$A_k := \begin{cases} 1, & w_k < 0 \\ 3, & w_k > 0 \end{cases},$$

then there are infinitely many n with $b_n = 0$, where

$$b_n := \sum_{k=1}^{l} w_k \sin(2\pi n\theta_k + \varphi_k) + 1 \ge 0,$$

and the behaviour of (a_n) *depends in an obvious way on the sign of the* $o(1)$ *term for these* n*. If there are* k*,* l *such that* (3.22) *does not hold, then* (a_n) *is positive for large* n*.*

Proof. The purely periodic sequence (b_n) satisfies $b_n \geq 0$ for all $n \geq 0$. If none of its finitely many values are zero, then (a_n) is positive for large n. We have $b_n = 0$ if and only if $\sin(2\pi n\theta_k + \varphi_k)$ equals 1 for the k's with $w_k < 0$ and -1 for the k's with $w_k > 0$, i.e.

$$2\pi n\theta_k + \varphi_k \equiv \tfrac{1}{2}A_k\pi \pmod{2\pi}, \qquad 1 \leq k \leq l,$$

which is equivalent to

$$n\tfrac{u_k}{v_k} + \tfrac{\varphi_k}{2\pi} \equiv \tfrac{1}{4}A_k \pmod{1}, \qquad 1 \leq k \leq l.$$

Clearly, this cannot hold if one of the φ_k/π is irrational. Under the assumption of part (ii), we are lead to the system of congruences

$$4d_k u_k n \equiv v_k(A_k d_k - 4c_k) \pmod{4d_k v_k}, \qquad 1 \leq k \leq l.$$

Now the result follows from Theorem 3.2.4. □

3.3.7 Algorithmic Aspects

There is no algorithm known that decides, given a C-finite sequence (a_n), whether $a_n > 0$ for all n, nor has the problem been shown to be undecidable [8]. When we are talking about algorithmics, it is natural to assume that the recurrence coefficients and the initial values are rational numbers. In this case Gourdon and Salvy [42] have proposed an efficient method for ordering the characteristic roots w.r.t. to their modulus. Thus, the dominating characteristic roots can be identified algorithmically. If none of them is real positive, then we know that the sequence oscillates by Theorem 3.3.11. On the other hand, sequences where a positive dominating root is accompanied by complex dominating roots seem to pose difficult Diophantine problems. For instance, we do not know if the sequence

$$a_n := \sin(2\pi\theta n) + 1 + \left(-\tfrac{1}{2}\right)^n \tag{3.23}$$

is positive for $\theta = \sqrt{2}$, say. It is positive for $n \leq 10^5$. We only know that the set of θ's for which the corresponding sequence (a_n) (defined by (3.23)) is eventually positive has measure one, by virtue of Theorem 3.3.20. Finally, we remark that the problem of deciding positivity of the power series coefficients of rational functions in two non-commuting variables has been shown to be undecidable [26]. This is a generalization of our problem, since C-finite sequences are the power series coefficients of univariate commutative rational functions.

Another algorithmic question is the following: Given some oscillating C-finite sequence, what can we say about the location of its positive and negative values? If all dominating characteristic roots have arguments that are commensurable to π, then this is easy, since then the sequence is purely periodic. Otherwise, the sequence of signs seems to have a somewhat random behaviour. In Section 2.3.3 we have shown how to obtain some information on the sequence of signs in such cases.

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