A Geometrical Decision Algorithm Based on the Gröbner Bases Algorithm *)

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Abstract

Gröbner bases have been used in various ways for dealing with the problem of geometry theorem proving as posed by Wu. Kutzler and Stifter have proposed a procedure centered around the computation of a basis for the module of syzygies of the geometrical hypotheses. We elaborate this approach and extend it to a complete decision procedure. Also, in geometry theorem proving the problem of constructing subsidiary (or degeneracy) conditions arises. Such subsidiary conditions usually are not uniquely determined and obviously one wants to keep them as simple as possible. This problem, however, has not received enough attention in the geometry theorem proving literature. Our algorithm is able to construct the simplest subsidiary conditions with respect to certain predefined criteria, such as lowest degree or dependence on a given set of variables.

The work of Wu Wen-tsün [Wu 1978], [Wu 1984] has sparked a lot of interest in automated geometry theorem proving. He has developed a decision algorithm for a certain class of geometry problems. The class of problems he considers consists, intuitively speaking, of those geometry problems that can be translated into algebraic equations. The statements that can be expressed in the geometry considered by Wu (Wu's geometry, for short) are those whose hypotheses and conclusion can be expressed as polynomial equations in the coordinates of the points occurring in the geometric construction. This, of course, after some coordinate system has been fixed. The ground field, i.e. the field from which these coordinates are taken, has to be an algebraically closed field; this even if both the hypotheses and the conclusion can be described by polynomials whose coefficients lie in a smaller field. Basically, Wu's geometry allows to talk about incidence, parallelism, perpendicularity, cocircularity, congruence, etc., but not about "betweenness", because no order predicate is available. Kapur [Kapur 1986a] shows that the satisfiability of any quantifier-free formula involving elements of the ground field, variables ranging over the ground field, the function symbols $+, -, \times$, the predicate symbol $=$ and Boolean connectives is equivalent to the satisfiability of a finite set of polynomial equations.

We give a formal specification of the geometry theorem proving problem. Let K be a field, $\overline{K}$ the algebraic closure of K, and $\mathcal{R}$ the polynomial ring in the indeterminates $x_1, \ldots, x_\nu$ over K, i.e. $\mathcal{R} = K[x_1, \ldots, x_\nu]$. Whenever we speak of polynomials we mean elements of $\mathcal{R}$. In [Wu 1984] Wu poses the following problem:

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P_{Wu}:

given: polynomials $f_1, \ldots, f_m, f$

decide: does there exist a polynomial s such that
(1) $(\forall \bar{x} \in \overline{K}^{\nu})\ (f_1(\bar{x}) = \ldots = f_m(\bar{x}) = 0\ \wedge\ s(\bar{x}) \neq 0 \implies f(\bar{x}) = 0)$
and
(2) $(\exists \bar{x} \in \overline{K}^{\nu})\ (f_1(\bar{x}) = \ldots = f_m(\bar{x}) = 0\ \wedge\ s(\bar{x}) \neq 0)$?
If so, find such an s.

The polynomial s is supposed to describe the subsidiary condition of the geometric statement or the degeneracy condition of the geometric figure under consideration, e.g. $s(\bar{x}) \neq 0$ might state that a triangle does not collapse into a line segment. Sometimes it seems natural to use a finite number $s_1, \ldots, s_n$ of subsidiary conditions, replacing $s(\bar{x})$ in P_{Wu} by $s_1(\bar{x}) \neq 0 \wedge \ldots \wedge s_n(\bar{x}) \neq 0$, thus getting a modified problem. However, it can easily be seen that a single subsidiary condition s is sufficient. The factors of s satisfy the modified problem, and if $s_1, \ldots, s_n$ satisfy the modified problem, then their product $s_1 \cdot \ldots \cdot s_n$ safisfies P_{Wu}.

Wu's decision algorithm for this problem has been partially implemented by himself and by Chou [Chou 1985]. Many interesting theorems have been proved by these implementations.

Different approaches to geometry theorem proving, based on the computation of Gröbner bases [Buchberger 65], [Buchberger 85], have been reported. In [Chou,Schelter 1986] Gröbner bases over the field generated by the independent variables of a geometric construction are employed. Kapur [Kapur 1986a,b] describes a refutational theorem prover, based on Rabinowitsch's trick for proving Hilbert's Nullstellensatz. Kutzler and Stifter [Kutzler,Stifter 1986a,b] describe various ways of applying Gröbner bases to this problem, one of which is centered on the computation of a basis for the module of syzygies of the geometrical hypotheses and conclusion. We consider this idea and develop it into a full decision procedure for P_{Wu}.

Of course it would be of interest to keep the subsidiary condition in P_{Wu} as simple as possible. Referring to his approach Kapur [Kapur 1986b] claims that *"conditions found using this approach are often simpler and weaker than the ones reported using Wu's method or reported by an earlier version of Kutzler & Stifter's paper as well as Chou & Schelter based on the Gröbner basis method."* Our algorithm *GEO* is able to compute the "simplest possible" subsidiary conditions by giving a complete overview of the possible subsidiary conditions. Reasonable criteria for "simplest possible" might be "of as low a degree as possible" or "involving only certain variables".

Every set of polynomials F generates a polynomial ideal $ideal(F)$, consisting of all linear combinations of elements of F with coefficients in $\mathcal{R}$. If F is a finite set $\{f_1, \ldots, f_m\}$, then we write $ideal(f_1, \ldots, f_m)$ for the ideal generated by F. F is a *basis* for $ideal(F)$. Every polynomial ideal (in $\mathcal{R}$) admits a finite basis, and certain finite bases are called *Gröbner bases* [Buchberger 1965,1985]. The radical of a polynomial ideal I, $radical(I)$, consists of all polynomials f such that some power f^p is in I. $radical(I)$ is again a polynomial ideal. For a given sequence $G = (g_1, \ldots, g_n)$ of polynomials we say that another sequence of polynomials $S = (s_1, \ldots, s_n)$ is a *syzygy* of G iff $\sum_{i=1}^{n} s_i g_i = 0$. The set of all syzygies of a given sequence G form a module over $\mathcal{R}$, the *module of syzygies.*

Lemma 1: *Let $P = (f_1, \ldots, f_m, f)$ be an instance of P_{Wu}.*

(i) Those polynomials $s \in \mathcal{R}$, which satisfy part (1) of the instance P of P_{Wu}, constitute an ideal N_P in $\mathcal{R}$.

(ii) For every $s \in N_P$ there exist $s_1, \ldots, s_m \in \mathcal{R}$ and $k \in \mathbb{N}$, such that $(s_1, \ldots, s_m, s^k \cdot f^{k-1})$ is a syzygy of $(f_1, \ldots, f_m, f)$, i.e. $s_1 \cdot f_1 + \cdots + s_m \cdot f_m + s^k \cdot f^{k-1} \cdot f = 0$.

(iii) If $(s_1, \ldots, s_m, s^k \cdot f^{k-1})$, $k \in \mathbb{N}$, is a syzygy of $(f_1, \ldots, f_m, f)$, then $s \in N_P$.

(iv) If ~~we let~~
$S_P = \{s \mid (s_1, \ldots, s_m, s)$ a syzygy of $(f_1, \ldots, f_m, f)$ for some $s_1, \ldots, s_m\}$,
then $N_P = \{s \in \mathcal{R} \mid s^k \cdot f^{k-1} \in S_P$ for some $k \geq 1\}$.

Proof: (i) Suppose both s_1 and s_2 solve part (1) of P. Now let t_1, t_2 be arbitrary polynomials, and let $\bar{x} \in \overline{K}^\nu$ be such that $f_1(\bar{x}) = \ldots = f_m(\bar{x})$ and $(t_1 s_1 + t_2 s_2)(\bar{x}) = t_1(\bar{x}) \cdot s_1(\bar{x}) + t_2(\bar{x}) \cdot s_2(\bar{x}) \neq 0$. Then either $s_1(\bar{x}) \neq 0$ or $s_2(\bar{x}) \neq 0$. W.l.o.g. assume that $s_1(\bar{x}) \neq 0$. But then $f(\bar{x}) = 0$, since s_1 is a solution of part (1) of P. So also $t_1 s_1 + t_2 s_2$ is a solution of part (1) of P.

(ii) Since $s \in N_P$, we know that $s \cdot f$ vanishes on every common zero of $f_1, \ldots, f_m$ in $\overline{K}$. That, however, means that $s \cdot f$ is in the radical of $ideal(f_1, \ldots, f_m)$, and a power of $s \cdot f$, say $s^k \cdot f^k$, $k \in \mathbb{N}$, is in $ideal(f_1, \ldots, f_m)$. Therefore, for some $s_1, \ldots, s_m \in \mathcal{R}$,

$$s_1 \cdot f_1 + \cdots + s_m \cdot f_m + s^k \cdot f^k = 0,$$

i.e. $(s_1, \ldots, s_m, s^k \cdot f^{k-1})$ is a syzygy of $(f_1, \ldots, f_m, f)$.

(iii) $s_1 \cdot f_1 + \cdots + s_m \cdot f_m + s^k \cdot f^k = 0$, so for every $\bar{x} \in \overline{K}^\nu$

$$f_1(\bar{x}) = \ldots = f_m(\bar{x}) = 0 \ \wedge \ s(\bar{x}) \neq 0 \implies f(\bar{x}) = 0.$$

(iv) By combination of (ii) and (iii). ∎

It is well known how the Gröbner basis algorithm can be used to compute a finite basis for the module of syzygies of a finite sequence of polynomials [Buchberger 1985], [Winkler 1986]. So for every instance P of P_{Wu} one can compute a finite basis for the ideal S_P. What is left to do is to extract N_P from S_P.

Definition: *Let f be a polynomial and I a polynomial ideal. Then the radical of I at f is defined as $radical(I, f) = \{s \mid s^k \cdot f^{k-1} \in I$* for some $k \in \mathbb{N}\}$. ∎

For any polynomial ideal I and polynomial f, $radical(I, f)$ is an ideal. Furthermore, by Lemma 1 we clearly have $N_P = radical(S_P, f)$ for any instance $P = (f_1, \ldots, f_m, f)$ of P_{Wu}. So extracting N_P from S_P really means computing a finite basis for $radical(S_P, f)$. In order to compute such a finite basis for $radical(I, f)$, we use the fact

$$f \cdot radical(I, f) = radical(I) \cap ideal(f).$$

The well known elimination property of Gröbner bases [Trinks 1978], [Buchberger 1985] allows to compute a finite basis of the intersection of two ideals I, J as

$$I \cap J = ideal((t-1)I \cup tJ) \cap \mathcal{R}.$$

So we get the following algorithm for computing $radical(I, f)$:

Algorithm $RADICAL$ (in: polynomials $f_1, \ldots, f_m, f$,
out: B, a finite basis for $radical(ideal(f_1, \ldots, f_m), f)$);

(1) Let $I = ideal(f_1, \ldots, f_m)$. Compute a basis C for $radical(I)$ in $\mathcal{R}$;

(2) Compute a Gröbner basis C' for $ideal((t-1)C \cup \{tf\})$ in $\mathcal{R}[t]$ with respect to a term ordering which orders primarily according to the exponent of t;

(3) The elements of C', which do not depend on t, are a basis for $radical(I) \cap ideal(f) = f \cdot radical(I, f)$. Call this set of polynomials C'';

(4) Set $B = \{g \mid g = h/f$ for some $h \in C''\}$ ■

The only problematic step in $RADICAL$ is (1), the computation of a basis of the radical of I. An algorithm is given in [Kandri-Rody 1984]. Kandri-Rody's algorithm requires the computation of Gröbner bases and characteristic sets, and factorization of polynomials over algebraic extensions of the ground field K, which is generally considered a complex problem.

Now we have the machinery for constructing a finite basis for N_P, the set of all solutions of part (1) in P_{Wu}, and hence we have a complete overview of the solutions of (1). The remaining question is, whether there is a solution of (1), which also satisfies (2).

Lemma 2: *Let P be an instance of P_{Wu}, B a finite basis for N_P.*
(i) If there is a polyomial in N_P which satisfies (2), then there is a polynomial in the basis B which satisfies (2).
(ii) If B is a Gröbner basis for N_P with respect to the term ordering $<$, B' is the set of those $b \in B$ satisfying part (2) of P, and $t = \min\{lpp(b) \mid b \in B'\}$, then for every solution s of P, $lpp(s) \geq t$.

Proof: (i) Let $P = (f_1, \ldots, f_m, f)$ and $B = \{b_1, \ldots, b_n\}$. Assume that no basis polyomial b_i, $1 \leq i \leq n$, satisfies (2), i.e.

$$(\forall \bar{x} \in \overline{K}^{\nu})\ (f_1(\bar{x}) = \ldots = f_m(\bar{x}) = 0 \Longrightarrow b_i(\bar{x}) = 0) \quad \text{for all } 1 \leq i \leq n.$$

Then also for every linear combination $s = \sum_{i=1}^{n} h_i b_i$ we have

$$(\forall \bar{x} \in \overline{K}^{\nu})\ (f_1(\bar{x}) = \ldots = f_m(\bar{x}) = 0 \Longrightarrow s(\bar{x}) = 0),$$

so no $s \in N_P$ satisfies (2).

(ii) Let s be a solution of part (1) of P. $s \in N_P$, so s is reducible to 0 w.r.t. B. Let $C \subseteq B$ be the set of elements of B used in this reduction. Then $lpp(b) \leq lpp(s)$ for every $b \in C$. If no $b \in C$ satisfies part (2) of P, then neither does s. ■

Lemma 2 (ii) states that by choosing the term ordering $<$ appropriately, one can compute solutions of P_{Wu} satisfying certain additional constraints. A Gröbner basis for N_P with respect to a graduated ordering contains a solution of lowest degree of P, if any such solution exists. A Gröbner basis for N_P with respect to a lexicographic ordering $x_1 < \ldots < x_m < \ldots < x_n$ contains a solution depending only on $x_1, \ldots, x_m$, if such a solution exists. The variables $x_1, \ldots, x_m$ could be the "independent" variables (see [Kutzler,Stifter 1986b]) of the geometric construction. So one can ask the question whether there is a nondegeneracy condition depending only on the independent variables. The two orderings can, of course, be combined by ordering the power products in $x_1, \ldots, x_m$ by some ordering $<_1$, e.g. according to the degree, and also the power products in $x_{m+1}, \ldots, x_n$ by some ordering $<_2$. Then a term ordering $<$ can be constructed by

$$u_1u_2 < t_1t_2 :\Longleftrightarrow u_2 <_2 t_2 \vee (u_2 = t_2 \wedge u_1 <_1 t_1),$$

where u_1, t_1 are power products over $x_1, \ldots, x_m$ and u_2, t_2 power products over $x_{m+1}, \ldots, x_n$. This ordering will lead to a subsidiary condition of lowest degree involving only the independent variables $x_1, \ldots, x_m$.

In their report [Chou,Yang 1986] Chou and Yang consider the problem statement P_{Wu} and claim: *"The algebraic problem in this formulation is well defined. However, the polynomial s sometimes has nothing to do with nondegenerate conditions in geometry. To make things worse, this formulation is unsound from the geometric point of view."* They go on to stress their point by an example. We will deal with this example and the criticism of P_{Wu} in Example 2 below.

Algorithm *GEO* (**in:** polynomials $f_1, \ldots, f_m, f$,
out: s, a solution of the instance $P = (f_1, \ldots, f_m, f)$ of P_{Wu},
if such a solution exists,
or "no");

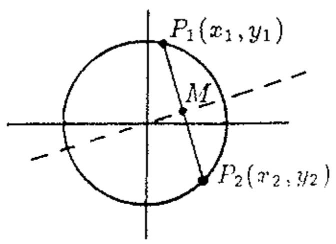
(1) Let $I = ideal(f_1, \ldots, f_m)$. Compute a finite basis for S_P, the ideal generated by the last component of the syzygies of $(f_1, \ldots, f_m, f)$. Call the algorithm RADICAL to compute a finite basis B for $N_P = radical(S_P, f)$;

(2) Check the polynomials b in B for $b \notin radical(I)$. This can be done by a Gröbner basis computation [Buchberger 1985]. *If a basis polynomial b is found which is not in the radical of I, then set $s = b$ and stop. Otherwise output "no".* ■

The algorithm *GEO* is complete for P_{Wu}, i.e. if a given instance P of P_{Wu} has a solution, then *GEO* finds one.

Example 1: We use the decision algorithm *GEO* to prove that

"if P_1 and P_2 are two points on a circle and M is the midpoint of P_1 and P_2 then the line through M and perpendicular to P_1P_2 contains the center of the circle".



So the hypotheses of the given instance P of P_{Wu} are
f_1: $x_1^2 + y_1^2 - x_2^2 - y_2^2$
(P_1 and P_2 are points on a circle with center (0,0))
f_2: $a(x_2 - x_1) + b(y_2 - y_1)$
($\begin{pmatrix} a \\ b \end{pmatrix}$ is perpendicular to P_1P_2)

and the conclusion is
f: $a(y_1 + y_2) - b(x_1 + x_2)$
(the line $y = \frac{b}{a}x$ contains M, the midpoint of P_1 and P_2)

First we compute a basis for the ideal S_P, i.e. the third component of the module of syzygies of (f_1, f_2, f). A Gröbner basis for $ideal(f_1, f_2, f)$ in $\mathbb{Q}[a, b, x_1, x_2, y_1, y_2]$ w.r.t. the lexicographic ordering with $a < b < x_1 < x_2 < y_1 < y_2$ is

$$\{f_1,\ f_2,\ f,\ f_3 = aby_1 - \frac{1}{2}b^2x_2 - \frac{1}{2}a^2x_2 - \frac{1}{2}b^2x_1 + \frac{1}{2}a^2x_1\},$$

so the syzygies derived from the reduction of the S–polynomials of these basis elements to 0 constitute a basis for the module of syzygies of (f_1, f_2, f_3, f). For the module of syzygies of (f_1, f_2, f) we get [Buchberger 1985] the basis

$$(-b, y_2 + y_1, x_1 - x_2),$$

$$(-a, x_2 + x_1, y_2 - y_1),$$

$$(0, ay_2 + ay_1 - bx_2 - bx_1, -by_2 + by_1 - ax_2 + ax_1),$$

$$(2aby_1 - b^2x_2 - a^2x_2 - b^2x_1 + a^2x_1, ay_2^2 - ay_1^2 + ax_2^2 - ax_1^2, -by_2^2 + by_1^2 - bx_2^2 + bx_1^2).$$

So $S_P = ideal(x_2 - x_1, y_2 - y_1)$.

Next we apply the algorithm *RADICAL* to S_P and f. S_P is radical. For computing $S_P \cap ideal(f)$ we determine a Gröbner basis for $ideal((z-1)S_P, zf)$ in $\mathbb{Q}[a, b, x_1, x_2, y_1, y_2, z]$, getting

$$\begin{aligned}
&x_2z - x_1z - x_2 + x_1,\\
&y_2z - y_1z - y_2 + y_1,\\
&ay_2z + ay_1z - bx_2z - bx_1z,\\
&ay_1z - bx_1z + \frac{1}{2}ay_2 - \frac{1}{2}ay_1 - \frac{1}{2}bx_2 + \frac{1}{2}bx_1,\\
&ax_2y_2 - ax_1y_2 + ax_2y_1 - ax_1y_1 - bx_2^2 + bx_1^2 = (x_2 - x_1)\cdot f,\\
&ay_2^2 - bx_2y_2 - bx_1y_2 - ay_1^2 + bx_2y_1 + bx_1y_1 = (y_2 - y_1)\cdot f.
\end{aligned}$$

Intersecting this basis with $\mathbb{Q}[a, b, x_1, x_2, y_1, y_2]$ and dividing by f we get the basis $\{x_2 - x_1, y_2 - y_1\}$ for $radical(S_P, f)$. This completes step (1) of *GEO*.

Neither $x_2 - x_1$ nor $y_2 - y_1$ is in the radical of $ideal(f_1, f_2)$, so both are solutions of the geometric problem instance P.

That means the theorem holds if either the x–coordinates or the y–coordinates of the two points P_1 and P_2 differ from one another, i.e. P_1 and P_2 do not collapse to a single point. ■

Example 2: This is the example used in [Chou,Yang 1986] to support the claim that the polynomial s computed as a solution of P_{Wu} may have nothing to do with a subsidiary condition for the geometric problem.

The goal is to prove that

"every triangles is isosceles",

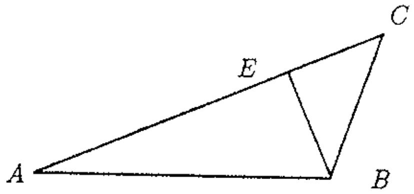
which, of course, is not a geometric theorem. Chou & Yang observe, however, that there is a formulation of this problem as an instance of P_{Wu}, which admits a subsidiary condition s.

The algebraic formulation they use is the following: let ABC be a triangle, and BE the altitude from B. Show that $AB \equiv CB$. As coordinates for the points they choose $A = (0, 0)$, $B = (y_1, 0)$, $C = (y_4, y_5)$, and $E = (y_2, y_3)$. Now the hypotheses can be translated into the algebraic equations

$$h_1 = y_3 y_5 + (y_2 - y_1) y_4 = 0 \qquad BE \perp AC$$
$$h_2 = -y_2 y_5 + y_3 y_4 = 0 \qquad E \text{ is on } AC$$

and the conclusion into the equation

$$g = -y_5^2 - y_4^2 + 2 y_1 y_4 = 0 \qquad AB \equiv CB.$$



Chou & Yang observe that $s = y_3^2 + y_2^2 - y_1 y_2$ satisfies both conditions in P_{Wu}. In fact, Kapur's theorem prover confirms the "theorem" under the subsidiary condition s. Chou & Yang now state that *"Thus under this formulation we can prove that "every" triangle is isosceles"* and they take this as evidence of their claim that P_{Wu} is *"unsound"*.

In our oppinion, the controversy stems from the fact that the dependent variables y_2, y_3 are not explicitly excluded from the subsidiary condition. If one wants to consider only such subsidiary conditions, which do not involve the dependent variables (which is reasonable from a geometric point of view), then this can be achieved by a suitable ordering of the power products, e.g. a lexicographic ordering based on

$$\underbrace{y_1 < y_4 < y_5}_{\text{indep. var.}} < \underbrace{y_2 < y_3}_{\text{dep. var.}}.$$

Now the algorithm *GEO* is able to detect that there exists no subsidiary condition involving only the independent variables y_1, y_4, y_5. Actually also Kapur [Kapur 1986b] mentions the possibility of recognizing that there is no such subsidiary condition in a remark following Theorem 2.

Let us apply the algorithm *GEO* to the geometric problem in the formulation above, where h_1, h_2 are the hypotheses and g is the conclusion. In step (1) we get

$$\begin{aligned}\{b_1 &= y_4 y_3 - y_5 y_2,\\ b_2 &= y_5^2 y_2 + y_4^2 y_2 - y_1 y_4^2,\\ b_3 &= y_3^2 + y_2^2 - y_1 y_2,\\ b_4 &= y_5 y_3 + y_4 y_2 - y_1 y_4\}\end{aligned}$$

as a basis for S_P, the ideal generated by the last component of the syzygies of (h_1, h_2, g). S_P is radical, so we just have to compute the intersection $S_P \cap ideal(g)$ and divide by g. This leads to the basis $\{b_1, b_2, b_3, b_4\}$ for N_P.

In step (2) we detect that $b_3 \notin radical(h_1, h_2)$, but there exists no possible subsidiary condition involving only the independent variables y_1, y_4, y_5. ∎

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