Computable Integrability.
Chapter 1: General notions and ideas*

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*This is a preliminary version of the Chapter 1 of a planned book "Computable Integrability."

And on Sunday mornings, every Sunday morning, all the year round, while he is closed to the outer world, and every night after ten, he goes into his bar parlour, bearing a glass of gin faintly tinged with water, and having placed this down, he locks the door and examines the blinds, and even looks under the table. And then, being satisfied of his solitude, he unlocks the cupboard and a box in the cupboard and a drawer in that box, and produces three volumes bound in brown leather, and places them solemnly in the middle of the table. The covers are weather-worn and tinged with an algal green for once they sojourned in a ditch and some of the pages have been washed blank by dirty water. The landlord sits down in an armchair, fills a long clay pipe slowly – gloating over the books the while. Then he pulls one towards him and opens it, and begins to study it – turning over the leaves backwards and forwards.

His brows are knit and his lips move painfully. "Hex, little two up in the air, cross and a fiddle-de-dee. Lord! what a one he was for intellect!"

Presently he relaxes and leans back, and blinks through his smoke across the room at things invisible to other eyes. "Full of secrets," he says. "Wonderful secrets!"

"Once I get the haul of them – Lord!"

"I wouldn't do what he did; I'd just – well!" He pulls at his pipe.

So he lapses into a dream, the undying wonderful dream of his life.

"Invisible man", H.G. Wells

1 Introduction

In our first Chapter we are going to present some general notions and ideas of modern theory of integrability trying to outline its computability aspects. The reason of this approach is that though this theory was wide and deeply developed in the last few decades, its results are almost unusable for non-specialists in the area due to its complexity as well as due to some specific jargon unknown to mathematicians working in other areas. On the other hand, a lot of known results are completely algorithmical and can be used as a base for developing some symbolic programm package dealing with the problems of integrability. Creation of such a package will be of a great help not only at the stage of formulating of some new hypothesis but also as a tool to get new systematization and classification results, for instance, to get complete lists of integrable equations with given properties as it was done already for PDEs with known symmetries [9].

Some results presented here are quite simple and can be obtained by any student acquainted with the basics of calculus (in these cases direct derivation is given) while some ideas and results demand deep knowing of a great deal of modern mathematics (in this cases only formulations and references are given). Our main idea is not to present here the simplest subjects of integrability theory but to give its general description in simplest possible form in order to give a reader a feeling what has been done already and what could/should be done further in this area. Most of the subjects mentioned here will be discussed in details in the next Chapters.

We will use the word "integrability" as a generalization of the notion "exactly solvable" for differential equations. Possible definitions of differential operator will be discussed as well as some definitions of integrability itself. Numerous examples presented here are to show in particular that it is reasonable not only to use different notions of integrability for different differential equations but sometimes it proves to be very useful to regard **one equation** using various definitions of integrability, depending on what properties of the equation are under the study. Two classical approaches to classification of integrable equations - conservation laws and Lie symmetries - are also briefly presented and a few examples are given in order to demonstrate deep difference between these two notions, specially in case of PDEs. Two interesting semi-integrable systems are introduced showing one more aspect of integrability theory - some equations though not integrable in any strict sense, can be treated as "almost" integrable due to their intrinsic properties.

2 Notion of differential operator

There are many ways to define linear differential first-order operator D, starting with Leibnitz formula for product differentiating

$$D(ab) = a^{'}b + ab^{'}.$$

This definition leads to

$$D \cdot a = a^{'} + aD, \quad D^2 a = aD^2 + 2a^{'}D + a^{''}, ...,$$

$$D^n \cdot a = \sum_k \begin{pmatrix} n \\ k \end{pmatrix} D^k(a) D^{n-k} \tag{1}$$

where

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n(n-1)...(n-k+1)}{1 \cdot 2 \cdots k} \quad \text{with} \quad \begin{pmatrix} n \\ 0 \end{pmatrix} = 1$$

are binomial coefficients. Thus, **one** linear differential first-order operator and its powers are defined. Trying to define a composition of two linear operators, we get already quite cumbersome formula

$$D_1 D_2 a = D_1(D_2(a) + aD_2) = D_1(a)D_2 + D_1 D_2(a) + D_2(a)D_1 + aD_1 D_2, \tag{2}$$

which can be regarded as a **definition** of a factorizable linear differential second-order operator. Notice that though each of D_1 and D_2 satisfies Leibnitz rule, their composition $D_1 D_2$ **does not!** An important notion of commutator of two operators $[D_1 D_2 - D_2 D_1]$ plays a role of special multiplication[1]. In case of **non-factorizable** second-order operator an attempt to generalize Leibnitz rule leads to a very complicated Bourbaki-like constructions which we are not going to present here. All these problems appear due to coordinateness of this approach.

On the other hand, a linear differential operator being written in coordinate form as

$$L = \sum_j f_j \partial_j$$

leads to

$$L = \sum_{|\alpha| \leq m} f_\alpha \partial^\alpha, \quad \partial^\alpha = \partial^{\alpha_1} \cdots \partial^{\alpha_n}, \quad |\alpha| = \alpha_1 + ... + \alpha_n$$

[1] see Ex.1

which is **definition** of LPDO of order m with n independent variables. Composition of two operators L and M is defined as

$$L \circ M = \sum f_\alpha \partial^\alpha \sum g_\beta \partial^\beta = \sum h_\gamma \partial^\gamma$$

and coefficients h_γ are to be found from formula (1).

Now, notion of linear differential **equation** (LDE) can be introduced in terms of the kernel of differential operator, i.e.

$$\mathrm{Ker}(L) := \{\varphi | \sum_{|\alpha| \le m} f_\alpha \partial^\alpha \varphi = 0\}.$$

Let us regard as illustrating example second-order LODO with one independent variable x:

$$L = f_0 + f_1 \partial + f_2 \partial^2, \quad \partial := \frac{d}{dx}.$$

Notice that if two functions φ_1 and φ_2 belong to its kernel $\mathrm{Ker}(L)$, then

$$c_1\varphi_1 + c_2\varphi_2 \in \mathrm{Ker}(L),$$

i.e. $\mathrm{Ker}(L)$ is a linear vector space over constants´ field (normally, it is $\mathbb{R}$ or $\mathbb{C}$).

Main theorem about ODEs states the existence and uniqueness of the Cauchy problem for any ODE, i.e. one-to-one correspondence between elements of the kernel and initial data. In our case Cauchy data

$$\varphi|_{x=x_0} = \varphi^0, \quad \partial\varphi|_{x=x_0} = \varphi^1$$

form a two-dimensional vector space and, correspondingly, dimension of kernel is equal 2, $dim(\mathrm{Ker}(L)) = 2$. Any two functions $\varphi_1, \varphi_2 \in \mathrm{Ker}(L)$ form its basis if Wronskian $< \varphi_1, \varphi_2 >$ is non-vanishing:

$$< \varphi_1, \varphi_2 > := \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} \neq 0$$

while an arbitrary function $\psi \in \mathrm{Ker}(L)$ has to satisfy following condition:

$$< \varphi_1, \varphi_2, \psi > := \begin{vmatrix} \varphi_1 & \varphi_2 & \psi \\ \varphi_1' & \varphi_2' & \psi' \\ \varphi_1'' & \varphi_2'' & \psi'' \end{vmatrix} = 0.$$

Now we can construct immediately differential operator with a given kernel as

$$L(\psi) =< \varphi_1, \varphi_2, \psi > .$$

For instance, if we are looking for an LODE with solutions $\sin x$ and $\sqrt{x}$, then corresponding LODE has form

$$\psi''(1 - \frac{1}{2}\tan x) + \tan x\psi' - \frac{1}{2x}\psi - \frac{3}{4}\frac{1}{x^2}\psi = 0.$$

Coming back to LODO of order m, we re-write formula for the kernel in the form

$$L(\psi) = \frac{< \varphi_1, ..., \varphi_m, \psi >}{< \varphi_1, ..., \varphi_m >} \tag{3}$$

which provides that high-order coefficient $f_m = 1$. It is done just for our convenience and we will use this form further.

Now, some constructive definition of linear differential operator was given and importance of its kernel was demonstrated. Corresponding differential equation was defined in terms of this kernel and construction of operator with a given kernel was described. All this is **not possible** for nonlinear operator because in this case manifold of solutions has much more complicated structure then just linear vector space - simply speaking, the reason of it is that in this case linear combination of solutions is not a solution anymore. Due to this reason some other notions are to be used to study properties of nonlinear operators - symmetries, conservation laws and, of course, as the very first step - change of variables transforming a nonlinear operator into a linear one. We will discuss all this in the next sections.

3 Notion of integrability

- **3.1. Solution in elementary functions:**

  $$\boxed{y'' + y = 0.}$$

  General solution of this equation belongs to the class of trigonometrical functions, $y = a\sin(x+b)$, with arbitrary const a, b. In order to find this solution one has to notice that this equation is LODE with constant coefficients which possess fundamental system of solutions, all of the form $e^{\lambda x}$ where λ is a root of characteristic polynomial.

- **3.2. Solution *modulo* class of functions:**

3.2.1. Integrability in quadratures

$$\boxed{y'' = f(y).}$$

In order to integrate this equation let us notice that $y''y' = f(y)y'$ which leads to

$$\frac{1}{2}y'^2 = \int f(y)dy + const = F(y)$$

and finally

$$dx = \frac{dy}{\sqrt{2F(y)}},$$

which describes differential equation with **separable variables**. In case when $F(y)$ is a polynomial of third or fourth degree, this is **definition of elliptic integral** and therefore the initial nonlinear ODE is integrable in elliptic functions. Particular case when polynomial $F(y)$ has multiple roots might leads to a particular solution in elementary functions. Let us regard, for instance, equation $y'' = 2y^3$ and put $const = 0$, then $y'^2 = y^4$ and $y = 1/x$, i.e. we have ONE solution in the class of *rational functions*. General solution is written out in terms of *elliptic functions*. Conclusion: equation is **integrable** in the class of elliptic functions and **not integrable** in the class of rational functions.

The case when $F(y)$ is a not-specified smooth function, is called **integrability in quadratures**. This is collective name was introduced in classic literature of 18th century and used in a sense of "closed-form" or "explicit form" of integrability while analytic theory of special functions was developed much later.

Notice that as a first step in finding of solution, the order n of initial ODE was diminished to $n-1$, in our case $n = 2$. Of course, this is **not possible** for any arbitrary differential equation. This new ODE is called **first integral** or **conservation law** due its physical meaning in applications, for instance, our example can be reformulated as Newton second law of mechanics and its first integral corresponds to energy conservation law.

3.2.2. Painleve transcendent P1

$$\boxed{y'' = y^2 + x.}$$

This equation defines first **Painleve transcendent**. About this equation it was proven that it has no solutions in classes of elementary or special functions. On the other hand, it is also proven that **Painleve transcendent** is a meromorphic function with known special qualitative properties ([3]).

This example demonstrates us the intrinsical difficulties when defining the notion of integrability. Scientific community has no general opinion about integrability of Painleve transcendent. Those standing on the classical positions think about it as about non-integrable equation. Those who are working on different applicative problems of theoretical physics involving the use of Painleve transcendent look at it as at some new special function (see also Appendix 1).

- **3.3. Solution *modulo* inexplicit function:**

$$\boxed{u_t = 2uu_x.}$$

This equation describes so called **shock waves** and its solutions are expressed in terms of inexplicit function. Indeed, let us rewrite this equation in the new independent variables $\tilde{t} = t$, $\tilde{x} = u$ and dependent one $\tilde{u} = x$, i.e. now $x = \theta(t, u)$ is a function on t, u. Then

$$d\tilde{t} = dt, \quad dx = \theta_t dt + \theta_u du, \quad u_t|_{dx=0} = -\frac{\theta_t}{\theta_u}, \quad u_x|_{dt=0} = \frac{1}{\theta_u}$$

and

$$-\frac{\theta_t}{\theta_u} = 2u\frac{1}{\theta_u} \Rightarrow -\theta_t = 2u \Rightarrow -\theta = 2ut - \varphi(u)$$

and finally $x + 2tu = \varphi(u)$ where $\varphi(u)$ is **arbitrary function** on u. Now, we have finite answer but no explicit form of dependence $u = u(x, t)$. Has the general solution been found? The answer is that given some initial conditions, i.e. $t = 0$, we may define solution as $u = \varphi^{-1}(x)$ where is φ^{-1} denotes **inverse function** for φ.

- **3.4. Solution *modulo* change of variables (C-Integrability):**

$$\boxed{\psi_{xy} + \alpha\psi_x + \beta\psi_y + \psi_x\psi_y = 0.}$$

This equation is called **Thomas equation** and it could be made linear with a change of variables. Indeed, let $\psi = log\theta$ for some positively defined function θ:

$$\psi_x = (log\theta)_x = \frac{\theta_x}{\theta}, \quad \psi_y = (log\theta)_y = \frac{\theta_y}{\theta},$$

$$\psi_{xy} = (\frac{\theta_x}{\theta})_y = \frac{\theta_{xy}\theta - \theta_y\theta_x}{\theta^2}, \quad \psi_x\psi_y = \frac{\theta_x\theta_y}{\theta^2}$$

and substituting this into Thomas equation we get finally linear PDE

$$\boxed{\theta_{xy} + \alpha\theta_x + \beta\theta_y = 0.}$$

Suppose for simplicity that $\beta = 0$ and make once more change of variables: $\theta = \phi e^{k_1 y}$, then

$$\theta_x = \phi_x e^{k_1 y}, \quad \theta_{xy} = \phi_{xy} e^{k_1 y} + k_1 \phi_x e^{k_1 y} \quad \text{and} \quad \phi_{xy} + (k_1 + \alpha)\phi_x = 0,$$

and finally

$$\partial_x(\phi_y + (k_1 + \alpha)\phi) = 0,$$

which yields to

$$\phi_y - k_2\phi = f(y) \quad \text{with} \quad k_2 = -(k_1 + \alpha)$$

and arbitrary function $f(y)$. Now general solution can be obtained by the method of variation of a constant. As a first step let us solve homogeneous part of this equation, i.e. $\phi_y - k_2\phi = 0$ and $\phi(x, y) = g(x)e^{k_2 y}$ with arbitrary $g(x)$. As a second step, suppose that $g(x)$ is function on x, y, i.e. $g(x, y)$, then initial equation takes form

$$(g(x, y)e^{k_2 y})_y - k_2 g(x, y)e^{k_2 y} = f(y),$$

$$g(x, y)_y e^{k_2 y} + k_2 g(x, y)e^{k_2 y} - k_2 g(x, y)e^{k_2 y} = f(y),$$

$$g(x, y)_y e^{k_2 y} = f(y), \quad g(x, y) = \int f(y)e^{-k_2 y}dy + h(x),$$

and finally the general solution of Thomas equation with $\alpha = 0$ can be written out as

$$\phi(x, y) = g(x, y)e^{k_2 y} = e^{k_2 y}(\int f(y)e^{-k_2 y}dy + h(x)) = \hat{f}(y) + e^{k_2 y}h(x)$$

with two arbitrary functions $\hat{f}(y)$ and $h(x)$.

- **3.5. Solution *modulo* Fourier transform (F-Integrability):**

$$\boxed{u_t = 2uu_x + \varepsilon u_{xx}}$$

where ε is a constant. This equation is called **Burgers equation** and it differs from Thomas equation studied above where change of variables was local in a sense that solution in each point does not depend on the solution in some other points of definition domain, i.e. local in (x, y)-space. This equation can be transformed into

$$u_t = 2uu_x + u_{xx}$$

by the change of variables $\tilde{x} = \varepsilon x$, $u = \varepsilon \tilde{u}$ and $\tilde{t} = \varepsilon^2 t$. We will use this form of Burgers equation skipping tildes in order to simplify the calculations below. To find solution of Burgers equation one has to use Fourier transform which obviously is nonlocal, i.e. here solution is local only in k-space. In order to demonstrate it let us integrate it using notation

$$\int u dx = v,$$

then after integration

$$v_t = v_x^2 + v_{xx} = e^{-v}(e^v)_{xx}$$

and change of variables $w = e^v$, Burgers equation is reduced to the **heat equation**

$$w_t = w_{xx}$$

which is linear. Therefore, solutions of Burgers equation could be obtained from solutions of heat equation by the change of variables

$$u = v_x = \frac{w_x}{w}$$

as

$$w(x, 0) = \int_{-\infty}^{\infty} \exp(ikx)\hat{w}(k)dk \Rightarrow u(x, t) = \frac{\int \exp(ikx - k^2 t)\hat{w}(k)ikdk}{\int \exp(ikx - k^2 t)\hat{w}(k)dk}$$

where $w(x, 0)$ is initial data and $\hat{w}(k)$ is a function called **its Fourier transform** and it can be computed as

$$\hat{w}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx)w(x, 0)dx.$$

In fact, it is well-known that **any** linear PDE with constant coefficients on an infinite line can be solved using as standard basis $\{e^{ikx}|k \in \mathbb{R}\}$ because they are eigenfunctions of these operators. Thus, Thomas equation with $\alpha = 0$ where the general solution was found explicitly, is **an exception** while heat equation demonstrates the general situation.

- **3.6. Solution *modulo* IST (S-Integrability):**

$$\boxed{u_t = 6uu_x + u_{xxx}.}$$

This equation is called **Korteveg-de Vries (KdV) equation** and it is nonlinear PDE with nonconstant coefficients. In this case, choosing set of functions $\{e^{ikx}\}$ as a basis is not helpful anymore: Fourier transform does not simplify the initial equation and only generates an infinite system of ODEs on Fourier coefficients.

On the other hand, some new basis can be found which allows to reduce KdV with rapidly decreasing initial data, $u \to 0$, $x \to \pm\infty$, to the linear equation and to solve it. This new basis can be constructed using solutions of **linear Schrödinger equation**

$$\boxed{\psi_{xx} + k^2\psi = u\psi}$$

where function u is called **potential** due to its origin in quantum mechanics. Solutions of linear Schrödinger equation are called **Jost functions**, $\psi^{\pm}(t, x, k)$, with asymptotic boundary conditions:

$$\psi^{\pm}(t, x, k; u(x, t))e^{-\pm i(kx+k^3t)} \to 1, \quad x \to \pm\infty.$$

Jost function $\phi(x, k) = \psi^{+}(t, x, k; u(x, t))e^{-i(kx+k^3t)}$ is defined by the integral equation

$$\phi(x, k) = 1 + \int_x^{\infty} \frac{1 - \exp[2k(x - x')]}{2k} u(x')\phi(x', k)dx'$$

with t playing role of a parameter. Second Jost function is defined analogously with integration over $[-\infty, x]$. Notice that asymptotically for $x \to \pm\infty$ linear Schrödinger equation

$$\psi_{xx} + k^2\psi = u\psi \quad \text{is reduced to} \quad \psi_{xx} + k^2\psi = 0$$

as in case of Fourier basis $\{e^{ikx}|k \in \mathbb{R}\}$. It means that asymptotically their solutions do coincide and, for instance, any solution of linear Schrödinger equation

$$\psi \sim c_1 e^{i(kx+k^3t)} + c_2 e^{-i(kx+k^3t)} \quad \text{for} \quad x \to \infty.$$

It turns out that solutions of KdV can be regarded as potentials of linear Schrödinger equation, i.e. following system of equations

$$\begin{cases} u_t = 6uu_x + u_{xxx}, \\ \psi_{xx} + k^2\psi = u\psi \end{cases}$$

is consistent and any solution of KdV can be written out as an expansion of Jost functions which in a sense are playing role of exponents e^{ikx} in Fourier basis [13].

On the other hand, there exists a major difference between these two basis: Fourier basis is written out in explicit form *via* one function while Jost basis is written out in inexplicit form *via* two functions with different asymptotic properties on the different ends of a line. The crucial fact here is that two Jost functions are connected by simple algebraic equation:

$$\psi^-(x,k,t) = a(k)\psi^+(x,-k,t) + b(k)e^{ik^3t}\psi^+(x,k,t)$$

while it allows us to construct rational approximation of Jost functions for given $a(k)$ and $b(k)$ and, correspondingly, general solution of KdV. The problem of reconstruction of function u according to $a(k), b(k)$ is called **inverse scattering problem** and this method, correspondingly, **inverse scattering transform (IST)**.

- **3.7. Solution *modulo* Dressing method (D-Integrability):**

$$\boxed{iu_t = u_{xx} \pm |u|^2 u.}$$

This equation is called **nonlinear Schrödinger equation** (NLS) and it is very important in many physical applications, for instance, in nonlinear optics. Dressing method is generalization of IST and in this case role of auxiliary linear equation (it was linear Schrödinger equation, second order ODE, in the previous case) plays a system of two linear first order ODEs [15]:

$$\begin{cases} \psi_x^{(1)} = \lambda\psi^{(1)} + u\psi^{(2)} \\ \psi_x^{(2)} = -\lambda\psi^{(2)} + v\psi^{(1)} \end{cases}$$

where $v = \pm\bar{u}$. It turns out that system of equations

$$\begin{cases} iu_t = u_{xx} \pm |u|^2 u \\ \psi_x^{(1)} = \lambda\psi^{(1)} + u\psi^{(2)} \\ \psi_x^{(2)} = -\lambda\psi^{(2)} + v\psi^{(1)} \end{cases}$$

is consistent and is equivalent to Riemann-Hilbert problem. Solutions of this last system are called **matrix Jost functions** and any solution of NLS can be written out as an expansion of matrix Jost functions which in a sense are playing role of exponents e^{ikx} in Fourier basis [12]. Detailed presentation of Dressing method will be given in some of our further Chapters.

4 Approach to classification

Our list of definitions is neither full nor exhaustive, moreover one equation can be regarded as integrable due to a few different definitions of integrability. For instance, equation for shock waves from § **3.3**, $u_t = 2uu_x$, is a particular form of Burgers equation from § **3.5**, $u_t = 2uu_x + \varepsilon u_{xx}$ with $\varepsilon = 0$, and it can be linearized as above, i.e. it is not only integrable in terms of inexplicit function but also C-integrable and F-integrable, with general solution

$$u(x,t) = \frac{\int \exp(ikx - k^2 t)\hat{u}(\varepsilon k) ik dk}{\int \exp(ikx - k^2 t)\hat{u}(\varepsilon k) dk}.$$

What form of integrability is chosen for some specific equation depends on what properties of it we are interested in. For instance, the answer in the form of inexplicit function shows immediately dependence of solution form on initial conditions - graphically presentation of inverse function $u = \varphi^{-1}(x)$ can be obtained as mirrored image of $x = \varphi(u)$. To get the same information from the formula above is a very nontrivial task. On the other hand, the general formula is the only known tool to study solutions with singularities. This shows that definitions of integrability do not suit to serve as a basis for classification of integrable systems and some more intrinsic ways should be used to classify and solve them. Below we present briefly two possible classification bases - conservation laws and Lie symmetries.

4.1 Conservation laws

Some strict and reasonable definition of a **conservation law** (which is also called **first integral** for ODEs) is not easy to give, even in case of ODEs. As most general definitions one might regard

$$\frac{d}{dt}F(\vec{y}) = 0 \quad \text{for ODE} \quad \frac{d}{dt}\vec{y} = f(\vec{y}), \quad \vec{y} = (y_1, ..., y_n), \quad f(\vec{y}) = (f_1, ..., f_n) \tag{4}$$

and

$$\frac{d}{dt}\int G(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}...)dxdy... = 0 \quad \text{for PDE} \quad \partial_t u = g(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, ...). \tag{5}$$

Obviously, without putting some restrictions on function F or G these definitions are too general and do not even point out some specific class of differential equations. For instance, let us take **any** second order ODE, due to well-known theorem on ODEs solutions we can write its general solution in a form

$$F(t, y, a, b) = 0$$

where a, b are two independent parameters (defined by initial conditions). Theorem on inexplicit function gives immediately

$$a = F_1(t, y, b) \quad \forall b \quad \text{and} \quad b = F_2(t, y, a) \quad \forall a,$$

i.e. any second order ODE has 2 independent conservation laws and obviously, by the same way n independent conservation laws can be constructed for ODE of order n. For instance, in the simplest case of second order ODE with constant coefficients, general solution and its first derivative have form

$$\begin{cases} y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y' = c_1 \lambda_1 e^{\lambda_1 x} + c_2 \lambda_2 e^{\lambda_2 x} \end{cases}$$

and multiplying y by λ_1 and λ_2 we get equations on c_1 and c_2 correspondingly:

$$\begin{cases} \lambda_2 y - y' = c_1(\lambda_2 - \lambda_1)e^{\lambda_1 x} \\ \lambda_1 y - y' = c_2(\lambda_1 - \lambda_2)e^{\lambda_2 x} \end{cases} \Rightarrow \begin{cases} \lambda_2 x + \hat{c_2} = log(y' - \lambda_1 y) \\ \lambda_1 x + \hat{c_1} = log(y' - \lambda_2 y) \end{cases}$$

and two conservation laws are written out explicitly:

$$\hat{c_2} = log(y' - \lambda_1 y) - \lambda_2 x, \quad \hat{c_1} = log(y' - \lambda_2 y) - \lambda_1 x.$$

To find these conservation laws without knowing of solution is more complicated task then to solve equation itself and therefore they give no additional information about equation. This is the reason why often only polynomial or rational conservation laws are regarded - they are easier to find and mostly they describe qualitative properties of the equation which are very important for applications (conservation of energy, momentum, etc.)

On the other hand, conservation laws, when known, are used for construction of ODEs solutions. Indeed, let us rewrite Eq.(4) as

$$\frac{d}{dt}F(\vec{y}) = (f_1\partial_1 + f_2\partial_2 + ... + f_n\partial_n)F = \mathcal{L}(F) = 0,$$

i.e. as an equation in partial derivatives $\mathcal{L}(F) = 0$. Such an equation has $(n-1)$ **independent** particular solutions $(\varphi_1, \varphi_2, ..., \varphi_{n-1})$ if its Jacobian matrix has maximal rank

$$rank\frac{\partial(\varphi_1, \varphi_2, ..., \varphi_{n-1})}{\partial(y_1, y_2, ..., y_{n-1})} = n-1, \tag{6}$$

with notation

$$\frac{\partial(\varphi_1, \varphi_2, ..., \varphi_{n-1})}{\partial(y_1, y_2, ..., y_{n-1})} = \begin{pmatrix} \frac{\partial\varphi_1}{\partial y_1} & \frac{\partial\varphi_1}{\partial y_2} & \cdots & \frac{\partial\varphi_1}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\varphi_n}{\partial y_1} & \frac{\partial\varphi_n}{\partial y_2} & \cdots & \frac{\partial\varphi_n}{\partial y_n} \end{pmatrix}.$$

Now we can write out the general solution as $F(\varphi_1, \varphi_2, ..., \varphi_{n-1})$ with arbitrary function F and initial ODE can be reduced to

$$\frac{dz}{dt} = f(z), \quad \frac{dz}{f(z)} = dt$$

and solved explicitly in quadratures (see § **3.2**).

4.2 Symmetry properties

In order to give definition of canonical form for n-order ODE

$$y^{(n)} = F(x, y, y', ..., y^{(n-1)}) \tag{7}$$

let us first introduce vector

$$\vec{y} = \begin{bmatrix} x \\ y \\ y' \\ ... \\ y^{(n-1)} \end{bmatrix}$$

which is called **vector of dynamical variables** with all its coordinates regarded as independent, and its first derivative

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} 1 \\ y^{'} \\ y^{''} \\ \dots \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} 1 \\ y^{'} \\ y^{''} \\ \dots \\ F \end{bmatrix}$$

with respect to some new independent variable t such that $dt = dx$, then the equation

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} 1 \\ y^{'} \\ y^{''} \\ \dots \\ F \end{bmatrix} \tag{8}$$

is called **canonical form** of an ODE. This canonical form is also called **dynamical system**, important fact is that dimension of dynamical system is $(n+1)$ for n-order ODE.

Definition 4.1. Dynamical system

$$\frac{d\vec{y}}{d\tau} = g(\vec{y}) \tag{9}$$

is called **a symmetry** of another dynamical system

$$\frac{d\vec{y}}{dt} = f(\vec{y}),$$

if

$$\frac{d}{d\tau}(\frac{d\vec{y}}{dt}) = \frac{d}{dt}(\frac{d\vec{y}}{d\tau}) \quad \Leftrightarrow \quad \frac{d}{d\tau}(f(\vec{y}) = \frac{d}{dt}(g(\vec{y})) \tag{10}$$

holds. Symmetry g of dynamical system f is called **trivial** if $g = \text{const} \cdot f$.

Obviously, Eq.(10) gives necessary condition of compatibility of this two dynamical systems. It can be proven that this condition is also sufficient. Therefore, construction of each Lie symmetry with group parameter τ is equivalent to a construction of some ODE which have τ as independent variable and is consistent with a given ODE. System of ODEs obtained this way, when being written in canonical form is called **dynamical system connected to the element of Lie symmetry algebra**.

We will discuss it in all details later (also see [11]), now just pointing out the fact that Lie algebra can be generated not only by Lie transformation group (normally used for finding of solutions) but also by the set of dynamical systems (8) (normally used for classification purposes).

Two following elementary theorems show interesting interconnections between conservation laws and symmetries in case of ODEs.

Theorem 4.1 *Let dynamical system*

$$\frac{d\vec{y}}{d\tau} = g(\vec{y})$$

is a symmetry of another dynamical system

$$\frac{d\vec{y}}{dt} = f(\vec{y})$$

and

$$\frac{d}{dt}F(\vec{y}) = (f_1\partial_1 + f_2\partial_2 + ... + f_n\partial_n)F = \mathcal{L}(F) = 0$$

is a conservation law. Then Fg is symmetry as well.

▶ Indeed, let us introduce

$$\mathcal{M} = (g_1\partial_1 + g_2\partial_2 + ... + g_n\partial_n),$$

then consistency condition Eq.(10) is written out as

$$\mathcal{L} \circ \mathcal{M} = \mathcal{M} \circ \mathcal{L} \qquad (11)$$

and after substituting $F\mathcal{M}$ instead of $\mathcal{M}$ we get on the left hand of (11)

$$\mathcal{L}(F\mathcal{M}) = \mathcal{L}(F)\mathcal{M} + F\mathcal{L}\mathcal{M} = F\mathcal{L}\mathcal{M} = F\mathcal{M}\mathcal{L},$$

and on the right hand of

$$(F\mathcal{M})\mathcal{L} = F\mathcal{M}\mathcal{L}.$$

▮

Corollary 4.2 *Let n-th order ODE of the form (7)* has a symmetry

$$\frac{d\vec{g}}{dt} = \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \dots \\ g_n \end{bmatrix},$$

then g_0 is a conservation law and consequently without loss of generality we may put $g_0 = 1$, if $g_0 \neq 0$.

Theorem 4.3 *An ODE of arbitrary order n having of $(n-1)$ independent conservation laws (i.e. complete set), has no nontrivial symmetries consistent with conservation laws.*

▶ Indeed, using all conservation laws we can reduce original n-order ODE into first order ODE

$$\frac{da}{dt} = f(a)$$

and look for symmetries in the form

$$\frac{da}{d\tau} = g(a).$$

Then

$$\frac{d}{d\tau}(\frac{da}{dt}) = f^{'}(a)g(a), \quad \frac{d}{dt}(\frac{da}{d\tau}) = g^{'}(a)f(a)$$

and finally

$$\frac{f^{'}}{f(a)} = \frac{g^{'}}{g(a)} \quad \Rightarrow \quad ln(\frac{f}{g}) = \text{const},$$

i.e. functions $f(a)$ and $g(a)$ are proportional. ▮

Simply speaking, Eq.(9) defines a one-parameter transformation group with parameter τ which conserves the form of original equation. The very important achievement of Lie was his first theorem giving constructive procedure for obtaining such a group. It allowed him to classify integrable differential equations and to solve them. The simplest example of such a classification for second order ODEs with two symmetries is following: each of them can be transformed into one of the four types

(I) $y^{''} = h(y^{'})$ **(II)** $y^{''} = h(x)$, **(III)** $y^{''} = \frac{1}{x}h(y^{'})$, **(IV)** $y^{''} = h(x)y^{'}$, where h denotes an arbitrary smooth function while explicit form of corresponding transformations was also written out by Lie.

Symmetry approach can also be used for PDEs. For instance, first-order PDE for shock waves $u_t = 2uu_x$ (§ **3.3**) has following Lie symmetry algebra $u_\tau = \varphi(u)u_x$ where any smooth function $\varphi = \varphi(u)$ defines one-parameter Lie symmetry group with corresponding choice of parameter τ. Indeed, direct check gives immediately

$$(u_t)_\tau = (2\varphi u_x)u_x + 2u(\varphi u_{xx} + \varphi^{'} u_x^2)$$

and

$$(u_\tau)_t = \varphi^{'} u_x 2uu_x + \varphi(2u_x^2 + 2uu_{xx})$$

while relation $(u_t)_\tau = (u_\tau)_t$ yields to the final answer (here notation $\varphi^{'} \equiv \varphi_u$ was used). In order to construct dynamic system for this equation, let us introduce dynamical independent variables as

$$u, \quad u_1 = u_x, \quad u_2 = u_{xx}, \quad ...$$

and dynamical system as

$$\frac{d}{dt}\begin{bmatrix} u \\ u_1 \\ u_2 \\ ... \end{bmatrix} = \begin{bmatrix} 2uu_1 \\ 2uu_2 + 2u_1^2 \\ 2uu_3 + 6u_1u_2 \\ ... \end{bmatrix}.$$

This system can be transformed into finite-dimensional system using characteristics method [10].

For PDE of order $n > 1$ analogous dynamic system turns out to be always infinite and only particular solutions are to be constructed but no general solutions. Infinite-dimensional dynamical systems of this sort are not an easy treat and also choice of dynamical variables presents sometimes a special problem to be solved, therefore even in such an exhaustive textbook as Olver's [11] these systems are not even discussed. On the other hand, practically all known results on classification of integrable nonlinear PDEs of two variables have been obtained using this approach (in this context the notion of F-Integrability is used as it was done to integrate Burgers equation, **§ 3.5.**) For instance, in [16] for a PDE of the form

$$u_{xy} = f(x, y)$$

with arbitrary smooth function f on the right hand it was proven that this PDE is integrable and has symmetries **iff** right part has one of the following forms: e^u, $\sin u$ or $c_1 e^u + c_2 e^{-2u}$ with arbitrary constants c_1, c_2. Another interesting result was presented in [10] where all PDEs of the form

$$u_t = f(x, u, u_x, u_{xx})$$

have been classified. Namely, a PDE of this form is integrable and has symmetries **iff** if it can be linearized by some special class of transformation. General form of transformation is written out explicitly.

4.3 Examples

Few examples presented here demonstrate different constellations of symmetries (SYM), conservation laws (CL) and solutions (SOL) for a given equation(s).

4.3.1: SYM +, CL +, SOL +.

Let us regard a very simple equation

$$y^{''} = 1,$$

then its dynamical system can be written out as

$$\frac{d\vec{y}}{dt} = \begin{bmatrix} 1 \\ y^{'} \\ F \end{bmatrix} = \begin{bmatrix} 1 \\ y^{'} \\ 1 \end{bmatrix} \quad \text{with} \quad dt = dx$$

and its general solution is $y = \frac{1}{2}x^2 + c_1 x + c_2$ with two constants of integration. In order to construct conservation laws, we need to resolve formula for solution with respect to the constants c_1, c_2:

$$c_1 = y^{'} - x, \quad c_2 = y + \frac{1}{2}x^2 - xy'.$$

Now, we look for solutions $F(\vec{y})$ of the equation

$$\frac{d}{dt}F(\vec{y}) = (\partial_x + y^{'}\partial_y + \partial_{y'})F = \mathcal{L}(F) = 0$$

with $\mathcal{L} = \partial_x + y^{'}\partial_y + \partial_{y'}$. Direct check shows that $F = y^{'} - x$ and $F = y + \frac{1}{2}x^2 - xy'$ are functionally independent solutions of this equation. Moreover, general solution is an **arbitrary function** of two variables

$$F = F(y^{'} - x, y + \frac{1}{2}x^2 - xy'),$$

for example,

$$F = (y^{'} - x)^2 - 2(y + \frac{1}{2}x^2 - xy') = y^{'} - 2y.$$

On the other hand, if there are no restriction on the function F, the conservation laws may take some quite complicated form, for instance,

$$F = Arcsin(y^{'} - x)/(y + \frac{1}{2}x^2 - xy')^{0.93}.$$

Now, that dynamical system, conservation laws and general solution of the original equation have been constructed, let us look for its symmetry:

$$g(\vec{y}) : \quad \frac{d\vec{y}}{d\tau} = g(\vec{y}), \quad g(\vec{y}) = (g_1, g_2, g_3).$$

Demand of compatibility

$$\frac{d}{d\tau}(f(\vec{y}) = \frac{d}{dt}(g(\vec{y})) \quad \text{is equivalent to} \quad \mathcal{L}(g_1) = \mathcal{L}(g_3) = 0, \quad \mathcal{L}(g_2) = g_3,$$

and it can be proven that any linear combination of two vectors $(1, 0, 0)$ and $(0, x, 1)$ with (some) scalar coefficients provides solution of compatibility problem. Thus, Lie symmetry group corresponding to the vector $(1, 0, 0)$ is shift in x while the second vector $(0, x, 1)$ corresponds to summing y with particular solutions of homogeneous equation.

4.3.2: SYM −, CL +, SOL +.

Let us regard as a system of ODEs

$$\begin{cases} n_1 \frac{da_1}{dt} = (n_2 - n_3)a_2 a_3 \\ n_2 \frac{da_2}{dt} = (n_3 - n_1)a_1 a_3 \\ n_3 \frac{da_3}{dt} = (n_1 - n_2)a_1 a_2 \end{cases} \tag{12}$$

with variables a_i and constants n_i. This system is well-known in physical applications - it describes three-wave interactions of atmospheric planetary waves, dynamics of elastic pendulum or swinging string, etc. Conservation laws for this system are:

- Energy conservation law is obtained by multiplying the i-th equation by a_i, $i = 1, 2, 3$ and adding all three of them:

$$n_1 a_1^2 + n_2 a_2^2 + n_3 a_3^2 = \text{const}\,.$$

- Enstrophy conservation law is obtained by multiplying the i-th equation by $n_i a_i$, $i = 1, 2, 3$ and adding all three of them:

$$n_1^2 a_1^2 + n_2^2 a_2^2 + n_3^2 a_3^2 = \text{const}\,.$$

Using these two conservation laws one can easily obtain expressions for a_2 and a_3 in terms of a_1. Substitution of these expressions into the first equation of Sys.(12) gives us differential equation on a_1:

$$(y^{'})^2 = f(y) \quad \text{with} \quad y = a_1$$

whose explicit solution is one of Jacobian elliptic functions while a_2 and a_3 are two other Jacobian elliptic functions [8]. In fact, Sys.(12) is often regarded as one of possible definitions of Jacobian elliptic functions.

Notice that from **Theorem 4.3** one can conclude immediately that all symmetries consistent with conservation laws, are trivial.

4.3.3: SYM +, CL ±, SOL +.

Let us regard heat equation

$$\partial_t u = \partial_{xx} u, \tag{13}$$

which generates solutions of Burgers equation (see **§ 3.5**). Direct check shows that Eq.(13) is invariant due to transformations $x = \tilde{x}\tau$ and $t = \tilde{t}\tau^2$ with any constant τ, i.e. dilation transformations constitute Lie symmetry group for Eq.(13). Moreover, Eq.(13) is integrable and its solution

$$u = \int \exp(ikx - k^2 t)\hat{u}(k)dk \quad \text{with} \quad \hat{u}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx)u(x,0)dx$$

is obtained by Fourier transformation.

This example demonstrates also some very peculiar property - heat equation (as well as Burgers equation) has **only one conservation law**:

$$\frac{d}{dt}\int_{-\infty}^{\infty} u dx = 0.$$

Nonexistence of any other conservation laws is proven, for instance, in [10].

5 Semi-integrability

5.1 Elements of integrability

Dispersive evolution PDEs on compacts is our subject in this subsection. In contrast to standard mathematical classification of LPDO into hyperbolic, parabolic and elliptic operators there exists some other classification - into dispersive and non-dispersive operators - which is successfully used in theoretical physics and **is not complementary** to mathematical one (for details see [8]). Let regard LPDE with constant coefficients in a form

$$P(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}) = 0$$

where t is time variable and x is space variable, and suppose that a **linear wave**

$$\psi(x) = \tilde{A} \exp i(kx - \omega t)$$

with constant amplitude $\tilde{A}$, wave number k and frequency ω is its solution. After substituting a linear wave into initial LPDE we get $P(-i\omega, ik) = 0$, which means that k and ω are connected in some way: there exist some function f such, that $f(\omega, k) = 0$.

This connection is called **dispersion relation** and solution of the dispersion relation is called **dispersion function**, $\omega = \omega(k)$. If condition

$$\frac{\partial^2 \omega}{\partial k^2} \neq 0$$

holds, then initial LPDE is called **evolution dispersive equation** and it obviously is completely defined by dispersive function. All these definitions can be easily reformulated for a case of more space variables, namely $x_1, x_2, ..., x_n$. In this case linear wave takes form

$$\psi(x) = \tilde{A} \exp i(\vec{k}\vec{x} - \omega t)$$

with **wave vector** $\vec{k} = (k_1, ...k_n)$ and space-like variable $\vec{x} = (x_1, ..., x_n)$. Then

$$P(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}) = 0,$$

dispersion function can be computed from $P(-i\omega, ik_1, ..., ik_n) = 0$ and the condition of non-zero second derivative of the dispersion function takes a matrix form:

$$|\frac{\partial^2\omega}{\partial k_i \partial k_j}| \neq 0.$$

Notice now that solutions of linear evolution dispersive PDE are known **by definition** and the reasonable question here is: what can be found about solutions of **nonlinear** PDE

$$\mathcal{L}(\psi) = \mathcal{N}(\psi)$$

with dispersive linear part $\mathcal{L}(\psi)$ and some nonlinearity $\mathcal{N}(\psi)$?

Nonlinear PDEs of this form play major role in the theory of wave turbulence and in general there is no final answer to this question. Case of **weak turbulence**, i.e. when nonlinearity $\mathcal{N}(\psi)$ is regarded small in a sense that wave amplitudes A are small enough (smallness of an amplitude can be strictly defined), is investigated in much more details. Two qualitatively different cases have to be regarded:

1. coordinates of wave vector are real numbers, $\{\vec{k} = (k_1, ...k_n) | k_i \in \mathbb{R}\}$ (corresponds to infinite space domain);

2. coordinates of wave vector are integer numbers, $\{\vec{k} = (k_1, ...k_n) | k_i \in \mathbb{Z}\}$ (corresponds to compact space domain).

In the first case method of **wave kinetic equation** has been developed in 60-th (see, for instance, [5]) and applied for many different types evolution PDEs. Kinetic equation is approximately equivalent to initial nonlinear PDE but has more simple form allowing direct numerical computations of each wave amplitudes in a given domain of wave spectrum. Wave kinetic equation is an averaged equation imposed on a certain set of correlation functions and it is in fact one limiting case of the quantum Bose-Einstein equation while the Boltzman kinetic equation is its other limit. Some statistical assumptions have been used in order to obtain kinetic equations and limit of its applicability then is a very complicated problem which should be solved separately for each specific equation [14].

In the second case, **exact solutions** in terms of elliptic functions have been found [7]. More precisely, it is proven that solving of initial nonlinear PDE **can be reduced** to solving a few small systems of ODEs of the form

$$\begin{cases} \dot{A}_1 = \mathfrak{C}_1 A_2 A_3, \\ \dot{A}_2 = \mathfrak{C}_2 A_1 A_3, \\ \dot{A}_3 = \mathfrak{C}_3 A_1 A_2, \end{cases} \tag{14}$$

in case of quadratic nonlinearity,

$$\begin{cases} \dot{A}_1 = \mathfrak{C}_1 A_2 A_3 A_4, \\ \dot{A}_2 = \mathfrak{C}_2 A_1 A_3 A_4, \\ \dot{A}_3 = \mathfrak{C}_3 A_1 A_2 A_4, \\ \dot{A}_4 = \mathfrak{C}_4 A_1 A_2 A_3, \end{cases}$$

in case of cubic nonlinearity and so on. Notice, that in contrast to a linear wave with a constant amplitude $\tilde{A} \neq \tilde{A}(t, \vec{x})$, waves in nonlinear PDE have amplitudes A_i **depending on time**. It means that solutions of initial nonlinear PDE have characteristic wave form as in linear case but wave amplitudes are Jacobian elliptic functions on time, $cn(T), dn(T)$ and $sn(T)$. Notice that Sys.(14) has been studied in **§ 4.3.2** and its conservation laws were found. Exact solutions of Sys.(14) are

$$\begin{cases} A_1 = b_1 cn(T/t_0 - \lambda), \\ A_2 = b_2 dn(T/t_0 - \lambda), \\ A_3 = b_3 sn(T/t_0 - \lambda), \end{cases}$$

and constants b_i, t_0, λ are written out explicitly as functions of initial values of waves´ amplitudes (see [8] for details).

These systems of ODEs providing exact solutions of initial nonlinear evolution PDE are called **elements of integrability**. Some constructive procedure, **Clipping method**, has been developed [6] allowing to find all elements of integrability for a given evolution PDE.

5.2 Levels of integrability

Let us formulate classical three-body problem whose integrability attracted attention of many investigators beginning with Lagrange. Computing the mutual gravitational interaction of three masses is surprisingly difficult to solve and only two integrable cases were found. For simplicity we regard three-body problem with all masses equal, then equations of motion take form

$$\begin{cases} \frac{d^2 z_1}{dt^2} = z_{12} f_{12} + z_{13} f_{13} \\ \frac{d^2 z_2}{dt^2} = z_{21} f_{12} + z_{23} f_{23} \\ \frac{d^2 z_3}{dt^2} = z_{31} f_{13} + z_{32} f_{23} \end{cases} \tag{15}$$

where z_j is a complex number, $z_j = x_j + iy_j$, describing coordinates of j-th mass on a plane, f_{jk} is a given function depending on the distance between j-th and k-th masses (physically it is attraction force) while following notations are used: $z_{jk} = z_j - z_k$ and $f_{jk} = f(|z_{jk}|^2)$.

This system admits following conservation laws:

- **Velocity of center of masses is constant**
  Summing up all three equations, we get

$$\frac{d^2}{dt^2}(z_1 + z_2 + z_3) = 0.$$

  This equality allows us to choose the origin of coordinate system in such a way that

$$z_1 + z_2 + z_3 = 0$$

  which simplifies all further calculations significantly. That is the reason why till the end of this section this coordinate system is used. Physically it means that coordinate system is connected with masses center.

- **Conservation of energy**
  Multiplying j-th equation by $\bar{z}_j^{'}$, summing up all three equations and adding complex conjugate, we obtain on the left

$$\sum_{j=1}^{3}(\bar{z}_j^{'} z_j^{''} + z_j^{'} \bar{z}_j^{''}) = \frac{d}{dt}\sum_{j=1}^{3} z_j^{'} \bar{z}_j^{'}.$$

  i.e. left hand describes derivative of kinetic energy.

  On the right we have derivative of potential energy $\mathcal{U}$:

$$\frac{d}{dt}\mathcal{U} = \frac{d}{dt}(F(|z_{12}|^2) + F(|z_{13}|^2) + F(|z_{23}|^2)) \quad \text{with notation} \quad F^{'} = f$$

  and finally energy conservation law takes form

$$\sum_{j=1}^{3} z_j^{'} \bar{z}_j^{'} = F(|z_{12}|^2) + F(|z_{13}|^2) + F(|z_{23}|^2) + \text{const}$$

- **Conservation of angular momentum**

By differentiating of angular momentum

$$\mathrm{Im}\sum_{j=1}^{3} z_j^{'}\bar{z}_j$$

with respect to t, we get

$$\mathrm{Im}\left(\sum |z_j^{'}|^2 + f_{12}z_{12}\bar{z}_{12} + f_{13}z_{13}\bar{z}_{13} + f_{23}z_{23}\bar{z}_{23}\right) =$$

$$\mathrm{Im}\left(f_{12}|z_{12}|^2 + f_{13}|z_{13}|^2 + f_{23}|z_{23}|^2\right) = 0,$$

while force f is some real-valued function.

In general case there are no other conservation laws and the problem is not integrable. On the other hand, one may look for some periodical solutions of Sys.(15) and try to deduce the necessary conditions of periodicity. Importance of the existence of periodical solutions was pointed out already by Poincare and is sometimes even regarded as a **definition** of integrability - just as opposite case for a chaos.

Theorem 5.1. *If $f_{ij} > 0, \quad \forall i, j$ (so-called repulsive case), then Sys.(15) has no periodical solutions.*

▶ Indeed, in case of periodical solution, magnitude of inertia momentum

$$\mathcal{Z} := |z_{12}|^2 + |z_{13}|^2 + |z_{23}|^2$$

should have minimums and maximums as sum of distances between three masses. On the other hand,

$$\frac{1}{2}\frac{d^2}{dt^2}(|z_{12}|^2+|z_{13}|^2+|z_{23}|^2) = |\dot{z}_{12}|^2+|\dot{z}_{13}|^2+|\dot{z}_{23}|^2+f_{12}|z_{12}|^2+f_{13}|z_{13}|^2+f_{23}|z_{23}|^2 > 0,$$

which contradicts to the fact that function $\mathcal{Z}$ has to have different signs in the points of minimum and maximum. ∎

One interesting case - **Poincare case** - though does not lead to integrable reduction of Sys.(15), give quite enlightening results and allows to regard this case as "almost" integrable. In this case there exists one more conservation law - conservation of inertia momentum

$$|z_{12}|^2 + |z_{13}|^2 + |z_{23}|^2 = \mathrm{const}$$

and it is possible due to a special choice of function $f_{jk} = 1/|z_{jk}|^4$ which allows us to reduce initial system to the ODE of the form

$$\mathcal{B}'' = a(\mathcal{B}')^3 + b(\mathcal{B}')^2 + c\mathcal{B}' + d. \tag{16}$$

in new polar coordinates. This equation describes a geometrical place of points on the plane, i.e. some plane curve $\mathcal{B}$, providing solutions of initial system. The curve $\mathcal{B}$ is of figure-eight form and can not be described by any known algebraic curve. On the other hand, it can be approximated with desirable accuracy, for instance, by lemniscate

$$x^4 + \alpha x^2 y^2 + \beta y^4 = x^2 - y^2.$$

Very comprehensive collection of results and graphics one can find in [4]

There exists hypothesis that **the only periodical solution** of Eq.(16) is this eight-form curve (not proven). Existence theorem for non-equidistant periodical solutions is proven for a wide class of functions f (in variational setting).

The simplest possible case of periodical solution can be obtained if one of z_i is equal to zero[2] (obviously, the problem is reduced to two-body problem). Two more complicated classical integrable cases with periodical solutions for particular choice of z-s are known:

- **Lagrange case**: $|z_{12}| = |z_{13}| = |z_{23}|$.
  It means that distances between three masses are equal as well as all corresponding attraction forces, and the masses are moving along a circle. Lagrange case is also called equidistance case. In this case Sys.(15) can be reduced to ODE $y'' = f(y)$ and solved in quadratures.

- **Calogero case**: all z_j are real and $f_{jk} = 1/|z_{jk}|^4$ for $j, k = 1, ...n$
  It means that all masses are moving along a line (in fact, along a real axes) and this is generalization of Euler case of three-body problem which after appropriate change of variables Sys.(15) takes form

$$\begin{cases} \frac{d^2x_1}{dt^2} = \frac{1}{(x_1-x_2)^3} + \frac{1}{(x_1-x_3)^3} \\ \frac{d^2x_2}{dt^2} = -\frac{1}{(x_1-x_2)^3} + \frac{1}{(x_2-x_3)^3} \\ \frac{d^2x_3}{dt^2} = -\frac{1}{(x_1-x_3)^3} - \frac{1}{(x_2-x_3)^3} \end{cases}$$

[2]see Ex.6

Euler´s system was generalized in [1], [2] to

$$\frac{d^2x_j}{dt^2} = \frac{\partial \mathcal{U}}{\partial x_j} \quad \text{with} \quad \mathcal{U} = \sum_{i<j} \frac{1}{(x_i - x_j)^2} \quad \text{and} \quad j = 1, ..., n.$$

All n independent conservation laws were found and it was proven that the system is integrable.

6 Summary

In our first Chapter we introduced a notion of differential operator and gave few different definitions of integrable differential equation. It was shown that some of them can be equivalent for a given equation and it is reasonable to choose an appropriate one depending on what properties of the equation are under the study. Some interesting and physically important examples of "almost" integrable systems were described. Very intrinsic question on interconnections of conservation laws and symmetries was also discussed.

Simplest possible example of nonlinear equation - famous Riccati equation which is ODE of first order with quadratic nonlinearity. We will use this equation in our next Chapter, **Chapter 2: Riccati equation**, in order to demonstrate many properties of differential equations described above. Riccati equation will also be very useful for introduction of some new notions like singularities of solutions, integrability tests, etc.

7 Exercises for Chapter 1

1. Using (2) prove that $D_3 = D_2D_1 - D_1D_2$ satisfies Leibnitz rule.

2. Prove that operator

$$\mathcal{L} = \sum a_k x^k \partial_x^k$$

can be transformed into an operator with **constant** coefficients by the following change of variables: $x = e^t$. (Euler)

3. Transform equation $u_t = \varphi(u)u_x$ into $v_t = vv_x$ by appropriate change of variables.

4. Prove that LODE with constant coefficients has conservation laws of the form

$$\frac{y^{'} - \lambda_1 y}{\lambda_2} e^{-\lambda_3 x} = \text{const}\,.$$

5. For potential energy $\mathcal{U}$ from § **5.2** prove that (Lagrange-Jacobi identity)

$$x_1 \frac{\partial \mathcal{U}}{\partial x_1} + y_1 \frac{\partial \mathcal{U}}{\partial y_1} + x_2 \frac{\partial \mathcal{U}}{\partial x_2} + y_2 \frac{\partial \mathcal{U}}{\partial y_2} + x_3 \frac{\partial \mathcal{U}}{\partial x_3} + y_3 \frac{\partial \mathcal{U}}{\partial y_3} = 2(f_{12}|z_{12}|^2 + f_{13}|z_{13}|^2 + f_{23}|z_{23}|^2).$$

6. For particular case $z_1 = 0$ and our choice of coordinate system solve Sys.(15) explicitly.

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