

How One Can Play with Sums

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Abstract

There are implementations of the celebrated Gosper algorithm (1978) and Zeilberger algorithm (1991) on almost any computer algebra platform. Inspired by Karr's indefinite summation algorithm (1981) I developed an algorithm based on difference fields which can handle indefinite and definite summation problems in a very general setting including Gosper's and Zeilberger's algorithms. The algorithm is available in form of a package developed in the computer algebra system Mathematica. The usage of my summation package `Sigma` will be illustrated by a concrete example.

1 Introduction

Karr developed an algorithm for **indefinite** summation [Kar81, Kar85] based on the theory of difference fields [Coh65]. He introduced so called $\Pi\Sigma$ -fields, in which first order linear difference equations can be solved in full generality. This algorithm cannot only deal with series of hypergeometric terms, like Gosper's algorithm [Gos78, PS95a, PS95b], series with q -hypergeometric terms, like [PR97], or holonomic series, like Chyzak's algorithm [CS98], but also with series of terms where for example the harmonic numbers can appear in the denominator. Karr's algorithm is, in a sense, the summation counterpart of Risch's algorithm [Ris69, Ris70] for indefinite integration.

Inspired by this algorithm I developed a summation algorithm based on difference field theory [Sch99, Sch00a, Sch00b, Sch01] in the computer algebra system Mathematica. Based on Bronstein's denominator bounding [Bro00], I was able to streamline Karr's ideas. Moreover, I implemented a user interface that dispenses the user from working explicitly with difference fields. Instead, the user can handle all summation problems in terms of sums and products.

My summation package `Sigma` cannot only handle indefinite summation problems but also can find closed forms of definite multisums. Although Karr's original summation algorithm

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was already capable of carrying out creative telescoping [Zei90], nobody has noticed this possibility until now. With creative telescoping one can compute a recurrence which has a given definite sum as a solution; therefore one can verify automatically a given definite sum identity.

Additionally, I have generalized Karr's algorithm such that linear difference equations of any order can be solved in any given $\Pi\Sigma$ -field. Hence one can find solutions of recurrences and thus not only prove definite sum identities, but even discover closed forms of definite sums in a very general setting.

In order to find solutions of a given difference equation, in many cases one has to extend the underlying difference field. Inspired by results in [AP94, HS99], I focus in [Sch01] on problems how one can find appropriate difference field extensions, namely sum extensions and so called d'Alembertian extensions, to find solutions for a recurrence. In this context, my indefinite summation algorithm plays a major role to simplify this solutions further. These aspects will be made clear by proving the following multisum identity:

Theorem 1. For nonnegative integers n we have

$$\sum_{k=0}^{2n} (-1)^k k H_k \binom{2n}{k}^3 = \frac{1}{6} (1 + 3n H_n + 6n H_{2n} - 3n H_{3n}) (-1)^n \frac{(3n)!}{n!^3} - \frac{1}{6} (-1)^n \frac{n!^3 (6n)!}{(2n)!^3 (3n)!}$$

where $H_k = \sum_{i=1}^k \frac{1}{i}$ are the k -th harmonic numbers.

In Section 2 the main goal is to demonstrate the basic strategy [PWZ96] how one can find automatically a closed form of a definite sum. Finally in Section 3 I will not only prove Theorem 1 but even will find the right hand side of the multisum identity.

2 A strategy to handle definite sums

In the following some functions of the Mathematica package `Sigma` are explained by a simple definite summation problem. After loading the summation package

```
In[1]:= << Sigma
```

```
      Sigma - A summation package by Carsten Schneider
```

we are able to insert the summation problem:

```
In[2]:= mySum = SigmaSum[(i - 1) i^3
```

```
      SigmaBinomial[2n, i]^3 SigmaPower[-1, i]/(2n - i + 1)^3, {i, 0, 2n}]
```

$$\text{Out[2]} = \sum_{i=0}^{2n} \left(\frac{(-1+i) i^3 \left(\binom{2n}{i} \right)^3 (-1)^i}{(1-i+2n)^3} \right)$$

The functions `SigmaSum` and `SigmaProduct` are used to define sums and products. There are several other functions available, like `SigmaHNumber`, `SigmaBinomial` or `SigmaPower` to define harmonic numbers, binomials or powers. Additional functions are provided to introduce new objects.

Finding a recurrence

For this definite summation problem we are now able to find a closed form. In a first step we generate a recurrence that is satisfied by `mySum`. The idea how to compute a recurrence is based on Zeilberger's creative telescoping method [Zei90].

```
In[3]:= rec = GenerateRecurrence[mySum]
```

```
Out[3]= {18 (1 + 3 n) (2 + 3 n) (1 + 6 n)
          (5 + 6 n) (13 + 18 n + 6 n^2) SUM[n] + 12 (90 + 814 n +
          2543 n^2 + 3864 n^3 + 3126 n^4 + 1296 n^5 + 216 n^6) SUM[1 + n] +
          2 (1 + n) (2 + n) (3 + 2 n)^2 (1 + 6 n + 6 n^2) SUM[2 + n] ==
          8 (288 + 3465 n + 18272 n^2 +
          55648 n^3 + 98548 n^4 + 97016 n^5 + 48384 n^6 + 9408 n^7) (-1)^(2 n)}
```

Since $(-1)^{2n} = 1$ for all integers n , we can replace $(-1)^{2n}$ by 1 in our recurrence.

```
In[4]:= rec = rec /. {(-1)^(2 n) -> 1};
```

Solving the recurrence

Next we solve the recurrence in terms of objects given in the recurrence.

```
In[5]:= SolveRecurrence[rec[[1]], SUM[n]]
```

```
Out[5]= {{1, 2 n}}
```

Unfortunately the algorithm delivers only a particular inhomogeneous solution $2n$, which is not a sufficient solution, since it does not have the same initial values as the original summation problem.

Therefore we try to find product extensions which deliver us solutions of the homogeneous version of the recurrence `rec` by calling the function `FindProductExtensions`. This function uses M. Petkovšek's package `Hyper` [Pet92, Pet94, PWZ96] which has to be loaded first.

```
In[6]:= << Hyper;
```

This package is able to find all hypergeometric solutions of a linear recurrence with polynomial coefficients. We obtain the following product extensions:

```
In[7]:= FindProductExtensions[rec, SUM[n], Solutions -> All]
```

I use M. Petkovšek's package `Hyper` to find product extensions.

```
Out[7]= {∏_{i=2}^n (- 3 (-2 + 3 i) (-1 + 3 i) / ((-1 + i) i)), ∏_{i=1}^n (- 3 (-5 + 6 i) (-1 + 6 i) / ((-1 + 2 i)^2))}
```

Next we solve the recurrence `rec` by using the two product extensions in a rewritten form:

```
In[8]:= tower = {((3 n)! (-1)^n / ((n)!)^3)_n, (((n)!)^3 (6 n)! (-1)^n / ((2 n)!)^3 (3 n)!)_n};
```

```
In[9]:= recSol = SolveRecurrence[rec[[1]], SUM[n], Tower -> tower]
```

11.53 Second

```
Out[9]= {{0, n ((3 n)! (-1)^n / ((n)!)^3)_n}, {0, (((n)!)^3 (6 n)! (-1)^n / ((2 n)!)^3 (3 n)!)_n}, {1, 2 n}}
```

This has to be interpreted as follows: our algorithm delivers two linear independent solutions of the homogeneous version of the recurrence, namely

$$(-1)^n n \frac{(3n)!}{n!^3} \qquad (-1)^n \frac{n!^3 (6n)!}{(2n)!^3 (3n)!},$$

and one particular solution $2n$ of the inhomogeneous recurrence itself.

Finding the linear combination

Finally the closed form of `mySum` is that linear combination of the homogeneous solutions plus the inhomogeneous solution which has exactly the same initial values as `mySum`. This is also computed automatically:

`In[10]:= FindLinearCombination[recSol, mySum, 2]`

$$\text{Out[10]} = 2n - n \left(\frac{(3n)! \cdot (-1)^n}{(n)!^3} \right)_n.$$

Summarizing we were able to compute the right hand side of the identity

$$\sum_{i=0}^{2n} \frac{(i-1)i^3(-1)^i \binom{2n}{i}^3}{(2n-i+1)^3} = n(2 - (-1)^n \frac{(3n)!}{n!^3}). \quad (1)$$

Please note that this example can be solved by just using the hypergeometric packages [PS95a] and [Pet92].

3 A multisum example

In the following we try to find a closed form representation of the multisum in Theorem 1 by using the summation package.

$$\text{In[11]:= mySum} = \sum_{k=0}^{2n} \left(k \mathbf{H}_k \left(\left(\binom{2n}{k} \right)^3 (-1)^k \right)_k \right);$$

3.1 Finding a recurrence

As already illustrated in the first example, we try to compute a recurrence for the definite sum `mySum`.

Creative telescoping in a difference field setting

Since our algorithm, using Zeilberger's creative telescoping method, is based on a very general difference field setting, we are able to compute a recurrence for this multisum.

In[12]:= **GenerateRecurrence**[mySum, RecOrder \rightarrow 4]

2016.48 Second

```
Out[12]= {81 (1 + n) (10 + 117 n + 441 n2 + 648 n3 + 324 n4)2
(579023679111696+
6203096595284292 n + 30574972749055508 n2 +
92475481987210701 n3 + 192864735750636284 n4 +
295166120513347017 n5 + 344113220933469194 n6 +
312890401572444600 n7 + 225181229898272112 n8 +
129339961859979540 n9 + 59474372437202472 n10 +
21854565707771808 n11 + 6372893337871680 n12 +
1455288215784768 n13 + 254598040577664 n14 +
32934777209856 n15 + 2967155877888 n16 +
166161051648 n17 + 4353564672 n18) SUM[n]+
108 (- 8911086594732000+
59568685520321800 n + 16380227867435099780 n2 +
185672492904312930710 n3 + 1271723758536088957353 n4 +
6026151073985872712073 n5 + 21197749937538020891079 n6 +
57793321639546981142298 n7 + 125693551925945528389705 n8 +
222521457681141044963341 n9 +
325368258856450491542511 n10 +
397108616509050749048718 n11 +
407622807225028518763356 n12 +
353729663174629500044400 n13 +
260330393614389288503220 n14 +
16270980603775713128520 n15 +
86335405854765454150272 n16 + 38809363531072919958144 n17 +
14720133478715210657664 n18 + 4681642828665855843072 n19 +
1237296059054356451328 n20 + 268300933294762027008 n21 +
46890597952821408768 n22 + 6437495043769780224 n23 +
668002856934260736 n24 + 49220844925353984 n25 +
2293562354761728 n26 + 50779978334208 n27) SUM[1 + n]+
18 (2 + n) (3 + 2 n) (- 8228295571986000 - 29467353203684820 n+
1381518393267116428 n2 + 19978139922191293573 n3 +
139144387971971638219 n4 + 625542630805627460455 n5 +
2017285686440215860490 n6 + 4933055970124372861135 n7 +
9465689765373655917267 n8 + 14579998008141370748253 n9 +
18312629998410321364656 n10 + 18961209332586432771048 n11 +
16304139557770332127212 n12 + 11695416700671314908740 n13 +
7013537868185350191792 n14 + 3515617464514069708512 n15 +
1469465760759532649280 n16 + 509652781805658910464 n17 +
145518011266651170048 n18 + 33806212169624059392 n19 +
6282436535103246336 n20 + 910948598145469440 n21 +
99231835717287936 n22 + 7633845045411840 n23 +
369565397876736 n24 + 8463329722368 n25) SUM[2 + n]+
12 (2 + n) (3 + n) (3 + 2 n) (5 + 2 n)
(- 64001714143920 - 503422860673228 n+
4002975025720952 n2 + 79747990756043705 n3 +
565678480977551301 n4 + 2447100392628223047 n5 +
7404218627394040182 n6 + 16709317348234374364 n7 +
29191436701822318447 n8 + 40425384732611573230 n9 +
45074461215631426464 n10 + 4087846323256991732 n11 +
30338483534960452020 n12 + 18477110572629289128 n13 +
9232514580951306000 n14 + 3772738135947714336 n15 +
1252587607610477760 n16 + 334329670014178176 n17 +
70597472266909440 n18 + 11513259270314496 n19 +
1397288190984192 n20 + 118711550287872 n21 +
6295254515712 n22 + 156728328192 n23) SUM[3 + n]+
(2 + n) (3 + n)2 (4 + n)2 (3 + 2 n) (5 + 2 n)
(7 + 2 n)3 (- 945554940 - 7607976456 n + 35254988575 n2 +
756814949687 n3 + 4816720182041 n4 + 17947420546069 n5 +
45372683784936 n6 + 83005099177032 n7 +
113701841575020 n8 + 118788006388788 n9 +
95405698339488 n10 + 58876332512544 n11 +
27669385543104 n12 + 9716847158592 n13 +
2466213765120 n14 + 426750114816 n15 +
44986834944 n16 + 2176782336 n17) SUM[4 + n] ==
0}
```

Here we succeed in finding a recurrence of order 4, but we are not able to find a recurrence with smaller order.

Creative telescoping and sum extensions

Now we could go on, as illustrated in the previous section, to solve the recurrence and finally to combine the computed solutions in order to derive a closed form evaluation. We will not proceed in this way; instead we try to compute again a recurrence but this time we are looking for additional sum extensions such that the recurrence order is smaller. This feature is activated by setting the option `SimplifyByExt` \rightarrow `DepthNumber`.

In[13]:= rec = GenerateRecurrence[mySum, SimplifyByExt → DepthNumber,
 RecOrder → 2]

157.97 Second

$$\begin{aligned}
 \text{Out[13]} = & \{ -36 (1+n) (2+n) (1+2n) (3+2n) (1+3n) (2+3n) \\
 & (1+6n) (5+6n) (1+6n+6n^2) (13+18n+6n^2) \text{SUM}[n] - \\
 & 24 (1+n) (2+n) (1+2n) (3+2n) (1+6n+6n^2) (90+814n+ \\
 & 2543n^2+3864n^3+3126n^4+1296n^5+216n^6) \text{SUM}[1+n] - \\
 & 4 (1+n)^2 (2+n)^2 (1+2n) (3+2n)^3 (1+6n+6n^2)^2 \text{SUM}[2+n] == \\
 & 2 \left((-19512 - 448728n - 4422462n^2 - 24996138n^3 - \right. \\
 & \quad 91349700n^4 - 227427644n^5 - 376226464n^6 - \\
 & \quad 308925516n^7 + 319086320n^8 + 1617697256n^9 + \\
 & \quad 3088351728n^{10} + 3851758512n^{11} + 3453843392n^{12} + \\
 & \quad 2288224320n^{13} + 1119909888n^{14} + 396032256n^{15} + \\
 & \quad 96095232n^{16} + 14349312n^{17} + 995328n^{18}) (-1)^{2n} + \\
 & \quad (-14802 - 376587n - 3834063n^2 - 21159534n^3 - 71496792n^4 - \\
 & \quad 157297032n^5 - 232167060n^6 - 231571656n^7 - \\
 & \quad 153801504n^8 - 65046240n^9 - 15816384n^{10} - 1679616n^{11}) \\
 & \quad \left. \sum_{\ell_1=0}^{2n} \left(\frac{(-1+\ell_1) \ell_1^3 \left(\binom{2n}{\ell_1} \right)^3 (-1)^{\ell_1}}{(1+2n-\ell_1)^3} \right) + \right. \\
 & \quad (-26016 - 715824n - 8970272n^2 - 68124912n^3 - 352009200n^4 - \\
 & \quad 1316397856n^5 - 3697583664n^6 - 7984118976n^7 - \\
 & \quad 13441452832n^8 - 17772262080n^9 - 18480846528n^{10} - \\
 & \quad 15046225664n^{11} - 9482866944n^{12} - 4533055488n^{13} - \\
 & \quad 1588110336n^{14} - 384380928n^{15} - 57397248n^{16} - 3981312n^{17}) \\
 & \quad \left. \left. \sum_{\ell_1=0}^{2n} \left(\frac{(-1+\ell_1) \ell_1^3 \left(\binom{2n}{\ell_1} \right)^3 (-1)^{\ell_1}}{(1+2n-\ell_1)^3 (2+2n-\ell_1)^3 (3+2n-\ell_1)^3} \right) \right) \right\}
 \end{aligned}$$

We found two sum extensions *automatically* which allowed us to compute a recurrence of order 2. Please note that the computation time is remarkably smaller, namely 158 seconds, in comparison to the standard approach, namely 2016 seconds.

Here we point out that identity (1) in Section 2 gives us a closed form for the first sum extension. Similar to identity (1) we can find a closed form for the second sum extension and obtain the following identity:

$$\begin{aligned}
 \sum_{i=0}^{2n} \frac{(i-1)i^3(-1)^i \binom{2n}{i}^3}{(2n-i+1)^3(2n-i+2)^3(2n-i+3)^3} &= (-1)^n \frac{(3n)!}{n!^3} \frac{3(1+3n)(2+3n)}{8(1+n)^4(1+2n)^3} \\
 &+ \frac{-3-15n-12n^2+17n^3+38n^4+28n^5+8n^6}{4(1+n)^2(1+2n)^3}.
 \end{aligned}$$

Therefore we are allowed to substitute the two sums by their right hand side evaluations and we obtain the following recurrence:

$$\begin{aligned}
\text{ln[14]} := \text{rec} &= 4 (1 + n) (2 + n) (1 + 2 n) (3 + 2 n) \\
& (1 + 6 n + 6 n^2) (-9 (1 + 3 n) (2 + 3 n) (1 + 6 n) (5 + 6 n) \\
& (13 + 18 n + 6 n^2) \text{SUM}[n] - 6 (90 + 814 n + 2543 n^2 + \\
& 3864 n^3 + 3126 n^4 + 1296 n^5 + 216 n^6) \text{SUM}[1 + n] - \\
& (1 + n) (2 + n) (3 + 2 n)^2 (1 + 6 n + 6 n^2) \text{SUM}[2 + n]) == \\
& -\frac{1}{1 + n} \left(6 (6504 + 144754 n + 1384851 n^2 + 7537254 n^3 + 26070977 n^4 + \right. \\
& 60620448 n^5 + 97542252 n^6 + 109802520 n^7 + 86051628 n^8 + \\
& \left. 45881424 n^9 + 15822864 n^{10} + 3172608 n^{11} + 279936 n^{12}) \left(\frac{(3 n)! (-1)^n}{((n)!)^3} \right)_n \right)
\end{aligned}$$

3.2 Solving the recurrence

In the following we will solve the recurrence. Similar to Section 2 we can use the function `FindProductExtensions` in order to find the two product extensions

$$\text{ln[15]} := \text{tower} = \left\{ \left(\frac{(3 n)! (-1)^n}{((n)!)^3} \right)_n, \left(\frac{((n)!)^3 (6 n)! (-1)^n}{((2 n)!)^3 (3 n)!} \right)_n \right\};$$

which deliver us two linear independent solutions of the homogeneous version of the recurrence:

$$\text{ln[16]} := \text{recSol} = \text{SolveRecurrence}[\text{rec}, \text{SUM}[n], \text{Tower} \rightarrow \text{tower}]$$

$$\text{Out[16]} = \left\{ \left\{ 0, \left(\frac{(n)!^3 (6 n)! (-1)^n}{(2 n)!^3 (3 n)!} \right)_n \right\}, \left\{ 0, n \left(\frac{(3 n)! (-1)^n}{(n)!^3} \right)_n \right\} \right\}$$

Unfortunately we do not find a solution of the inhomogeneous recurrence itself. By setting the option `NestedSumExt` $\rightarrow \infty$ we find *all* nested sum solutions expressed in terms of objects given in the recurrence and of the two product extensions given in `tower`.

$$\begin{aligned}
\text{ln[17]} := \text{recSol} &= \text{SolveRecurrence}[\text{rec}, \text{SUM}[n], \text{NestedSumExt} \rightarrow \infty, \\
& \text{Tower} \rightarrow \text{tower}, \text{AlgebraicRelationInSumSolutions} \rightarrow \text{True}]
\end{aligned}$$

19.78 Second

$$\begin{aligned}
\text{Out[17]} = & \left\{ \left\{ 0, \left(\frac{(n)!^3 (6 n)! (-1)^n}{(2 n)!^3 (3 n)!} \right)_n \right\}, \left\{ 0, n \left(\frac{(3 n)! (-1)^n}{(n)!^3} \right)_n \right\}, \right. \\
& \left\{ 1, \frac{1}{6} \left(\frac{(n)!^3 (6 n)! (-1)^n}{(2 n)!^3 (3 n)!} \right)_n \sum_{\iota_1=1}^n \left(\left(\iota_1 (1 - 6 \iota_1 + 6 \iota_1^2) \left(\frac{(3 \iota_1)! (-1)^{\iota_1}}{(\iota_1)!^3} \right)_{\iota_1} \right. \right. \right. \\
& \left. \left. \sum_{\iota_2=2}^{\iota_1} \left((9360 - 64710 \iota_2 + 63189 \iota_2^2 + 413410 \iota_2^3 - \right. \right. \right. \\
& \left. \left. \left. 1436799 \iota_2^4 + 2117172 \iota_2^5 - 1737846 \iota_2^6 + \right. \right. \right. \\
& \left. \left. \left. 826740 \iota_2^7 - 213840 \iota_2^8 + 23328 \iota_2^9 \right) / \right. \right. \\
& \left. \left. \left. ((-1 + \iota_2) \iota_2 (-3 + 2 \iota_2) (-5 + 3 \iota_2) \right. \right. \right. \\
& \left. \left. \left. (-4 + 3 \iota_2) (13 - 18 \iota_2 + 6 \iota_2^2) (1 - 6 \iota_2 + 6 \iota_2^2) \right) \right) \right) / \\
& \left. \left. \left((-1 + 2 \iota_1)^2 (-2 + 3 \iota_1) (-1 + 3 \iota_1) \left(\frac{(\iota_1)!^3 (6 \iota_1)! (-1)^{\iota_1}}{(2 \iota_1)!^3 (3 \iota_1)!} \right)_{\iota_1} \right) \right\} \right\}
\end{aligned}$$

Note that the found sum is indefinite. More generally, these nested sums of indefinite sums are included in the class of d'Alembertian solutions, which are introduced in [AP94]; further results can be found in [Sin91, Sch01].

Solving the recurrence with quantifier elimination

As already mentioned above, the found sum is indefinite and therefore we can apply our **indefinite** summation algorithm in order to simplify that double nested sum further. I consider it as an essential step to eliminate algebraic relations and to simplify nested sums further to simpler nested sums. Therefore, by default, this is done automatically. In the previous attempt I had to force my algorithm not to simplify the found indefinite sum further by setting the option `AlgebraicRelationInSumSolutions` \rightarrow `True`. Omitting this option, we obtain the following inhomogeneous solution which is expressed just by a single nested sum:

```
In[18]:= recSol = SolveRecurrence[rec[[1]], SUM[n],
      NestedSumExt  $\rightarrow$   $\infty$ , Tower  $\rightarrow$  tower]
31.09 Second
Out[18]= { {0,  $\left(\frac{((n!)^3 (6n)! (-1)^n}{((2n!)^3 (3n)!)}\right)_n$  }, {0, n  $\left(\frac{(3n)! (-1)^n}{((n!)^3}\right)_n$  },
      {1, - $\left(n \left(\frac{(3n)! (-1)^n}{((n!)^3}\right)_n\right.$ 
       $\left(\begin{aligned} &60 - 887n + 4948n^2 - 13599n^3 + 19512n^4 - 13932n^5 + \\ &3888n^6 + (2 - 25n + 117n^2 - 258n^3 + 270n^4 - 108n^5) \\ &\sum_{\iota_1=2}^n ((9360 - 64710\iota_1 + 63189\iota_1^2 + \\ &413410\iota_1^3 - 1436799\iota_1^4 + 2117172\iota_1^5 - \\ &1737846\iota_1^6 + 826740\iota_1^7 - 213840\iota_1^8 + 23328\iota_1^9) / \\ &((-1 + \iota_1)\iota_1(-3 + 2\iota_1)(-5 + 3\iota_1)(-4 + 3\iota_1) \\ &(13 - 18\iota_1 + 6\iota_1^2)(1 - 6\iota_1 + 6\iota_1^2))) \end{aligned}\right)} / \\ \left. (12(-1 + 2n)(-2 + 3n)(-1 + 3n)(1 - 6n + 6n^2)) \right\}$ 
```

Solving the recurrence with simple sums

Furthermore, by setting the option `SimpleSumRepresentation` \rightarrow `True`, the sum extension is expressed by simpler sums:

In[19]:= **recSol = SolveRecurrence[rec, SUM[n], NestedSumExt → ∞,**
Tower → tower, SimpleSumRepresentation → True]

45.83 Second

$$\text{Out[19]} = \left\{ \left\{ 0, \left(\frac{(n)!^3 (6n)! (-1)^n}{(2n)!^3 (3n)!} \right)_n \right\}, \left\{ 0, n \left(\frac{(3n)! (-1)^n}{(n)!^3} \right)_n \right\}, \right. \\
\left. \left\{ 1, \frac{1}{6(-1+2n)(-2+3n)(-1+3n)} \right. \right. \\
\left. \left(\left(\frac{(3n)! (-1)^n}{(n)!^3} \right)_n \left(-12 + 91n - 245n^2 + 261n^3 - \right. \right. \right. \\
90n^4 + (-10n + 65n^2 - 135n^3 + 90n^4) \sum_{\iota_1=2}^n \left(\frac{1}{-1+\iota_1} \right) + \\
(-12n + 78n^2 - 162n^3 + 108n^4) \sum_{\iota_1=2}^n \left(\frac{1}{-3+2\iota_1} \right) + \\
(6n - 39n^2 + 81n^3 - 54n^4) \sum_{\iota_1=2}^n \left(\frac{1}{-5+3\iota_1} \right) + \\
\left. \left. \left. \left. (6n - 39n^2 + 81n^3 - 54n^4) \sum_{\iota_1=2}^n \left(\frac{1}{-4+3\iota_1} \right) \right) \right) \right) \right\} \right\}$$

Solving the recurrence with standard objects

Finally we notice that these sum extensions can be expressed by the Harmonic numbers. Therefore we can solve the recurrence directly by using the following sum and product extensions:

$$\text{In[20]:= tower} = \left\{ \left(\frac{(3n)! (-1)^n}{(n)!^3} \right)_n, \left(\frac{((n)!)^3 (6n)! (-1)^n}{((2n)!)^3 (3n)!} \right)_n, \mathbf{H}_n, \mathbf{H}_{2n}, \mathbf{H}_{3n} \right\};$$

In[21]:= **recSol = SolveRecurrence[rec, SUM[n], NestedSumExt → ∞, Tower → tower]**

41.52 Second

$$\text{Out[21]} = \left\{ \left\{ 0, \left(\frac{(n)!^3 (6n)! (-1)^n}{(2n)!^3 (3n)!} \right)_n \right\}, \left\{ 0, n \left(\frac{(3n)! (-1)^n}{(n)!^3} \right)_n \right\}, \right. \\
\left. \left\{ 1, \frac{1}{6} (1 + 3n H_n + 6n H_{2n} - 3n H_{3n}) \left(\frac{(3n)! (-1)^n}{(n)!^3} \right)_n \right\} \right\}$$

3.3 Finding a closed form

In the end, the closed form of **mySum** is that linear combination of the homogeneous solutions plus the inhomogeneous solution which has exactly the same initial values as **mySum**.

In[22]:= **result = FindLinearCombination[recSol, mySum, 2]**

0.16 Second

$$\text{Out[22]} = \frac{1}{6} (1 + 3n H_n + 6n H_{2n} - 3n H_{3n}) \left(\frac{(3n)! (-1)^n}{(n)!^3} \right)_n - \frac{1}{6} \left(\frac{(n)!^3 (6n)! (-1)^n}{(2n)!^3 (3n)!} \right)_n$$

References

- [AP94] S. A. Abramov and M. Petkovšek. D'Alembertian solutions of linear differential and difference equations. In J. von zur Gathen, editor, *Proc. ISSAC'94*, pages 169–174. ACM Press, Baltimore, 1994.
- [Bro00] M. Bronstein. On solutions of linear ordinary difference equations in their coefficient field. *J. Symbolic Comput.*, 29(6):841–877, June 2000.
- [Coh65] R. M. Cohn. *Difference Algebra*. Interscience Publishers, John Wiley & Sons, 1965.
- [CS98] F. Chyzak and B. Salvy. Non-commutative elimination in ore algebras proves multivariate identities. *J. Symbolic Comput.*, 26(2):187–227, 1998.
- [Gos78] R. W. Gosper. Decision procedures for indefinite hypergeometric summation. *Proc. Nat. Acad. Sci. U.S.A.*, 75:40–42, 1978.
- [HS99] P. A. Hendriks and M. F. Singer. Solving difference equations in finite terms. *J. Symbolic Comput.*, 27(3):239–259, 1999.
- [Kar81] M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
- [Kar85] M. Karr. Theory of summation in finite terms. *J. Symbolic Comput.*, 1:303–315, 1985.
- [Pet92] M. Petkovšek. Hypergeometric solutions of linear recurrences with polynomial coefficients. *J. Symbolic Comput.*, 14(2-3):243–264, 1992.
- [Pet94] M. Petkovšek. A generalization of Gosper's algorithm. *Discrete Math.*, 134(1-3):125–131, 1994.
- [PR97] P. Paule and A. Riese. A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated approach to q-hypergeometric telescoping. In M. Ismail and M. Rahman, editors, *Special Functions, q-Series and Related Topics*, volume 14, pages 179–210. Fields Institute Toronto, AMS, 1997.
- [PS95a] P. Paule and M. Schorn. A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities. *J. Symbolic Comput.*, 20(5-6):673–698, 1995.
- [PS95b] P. Paule and V. Strehl. Symbolic summation - some recent developments. In J. Fleischer et al., editor, *Computer Algebra in Science and Engineering - Algorithms, Systems, and Applications*, pages 138–162. World Scientific, Singapore, 1995.
- [PWZ96] M. Petkovšek, H. S. Wilf, and D. Zeilberger. *A = B*. A. K. Peters, Wellesley, MA, 1996.
- [Ris69] R. Risch. The problem of integration in finite terms. *Trans. Amer. Math. Soc.*, 139:167–189, 1969.

- [Ris70] R. Risch. The solution to the problem of integration in finite terms. *Bull. Amer. Math. Soc.*, 76:605–608, 1970.
- [Sch99] C. Schneider. An implementation of Karr’s summation algorithm in Mathematica. Technical Report SFB 99-15, Johannes Kepler University Linz, August 1999.
- [Sch00a] C. Schneider. An implementation of Karr’s summation algorithm in Mathematica. *Sém. Lothar. Combin.*, S43b:1–10, 2000.
- [Sch00b] C. Schneider. Symbolic summation in difference fields. Contributed talk at the “10th SIAM Conference on Discrete Mathematics”, Minneapolis, Minnesota, June 2000.
- [Sch01] C. Schneider. Symbolic summation in difference fields. Technical Report 01-17, RISC-Linz, J. Kepler University, November 2001. PhD Thesis.
- [Sin91] M. F. Singer. Liouvillian solutions of linear differential equations with Liouvillian coefficients. *J. Symbolic Comput.*, 11(3):251–274, 1991.
- [Zei90] D. Zeilberger. A fast algorithm for proving terminating hypergeometric identities. *Discrete Math.*, 80(2):207–211, 1990.