Focus Windows:
A New Technique for Proof Presentation

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■ Proving, Solving, Computing and *Theorema*

■ Symbolic Computation = Automated Proving, Solving, Computing

■

B.B.: Invited talk at Multiparadigm Logic Programming Workshop, Sep. 15-16, 1996:

> "Proving, Solving, Computing (Simplifying):
> A Language Frame Based on Mathematica."

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The three aspects of "symbolic computation" (automating mathematics):

Given Φ_x, a formula with free variable x:

- proving: show that Φ_x is true for all x (in a given theory T).
- solving: find an x such that Φ_x is true (in a given theory T).
- simplifying: find a "simpler" Ψ_x such that Φ_x and Ψ_x are equivalent (in a given theory T).

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In other words:

- proving: handle $\underset{x}{\forall}\ \Phi_x$ (in a given theory T).
- solving: handle $\underset{x}{\exists}\ \Phi_x$ (in a given theory T).
- simplifying: handle $\underset{x}{\lambda}\,\Phi_x$ (in a given theory T).

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Note that the whole thing becomes challenging, interesting and practically useful only if we take into account the dependence of the proving, solving, and simplifying problem on the underlying theory. This theory may change many times during one proving, solving, or simplifying algorithm:

Proving (solving, simplifying) in T is often reduced to proving in some other P, solving in some other S, simplifiying in some other C. (This reduction is recursive.)

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Example: Proving boolean combinations of equalities over the complex numbers can be reduced to the solvability of certain systems of multivariate polynomial equations over the complex numbers. The latter problem, by a certain simplification procedure (Groebner bases computation) can be reduced to detecting whether 1 is in the system.

Example: Proving certain formulae over real functions can be reduced, by the PCS method, to proving certain formulae over the real numbers. The latter problem can be reduced, by Collins' simplification method, to evaluating formulae on finitely many rational points.

Example: General first order proving in the empty theory can be reduced, by the resolution method, to the solvability of systems of clauses in the term algebra. The latter problem, by the unification algorithm - wich is a general simplification algorithm - can be reduced to detecting whether the empty clause is in the system.

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Summarizing: Symbolic computation is a magma of provers, solvers, simplifyers for various theories.

Theorema

A system that tries to provide a growing magma of provers, solvers, simplifiers for various theories in one language and software frame.

Programmed in Mathematica but not relying on built-in knowledge of Mathematica.

An emphasis on the automatic generation of "human readable" proofs.

This talk: not on a particular prover, solver, or simplifyer but on the presentation of proofs.

The Problem

Understanding Proofs vs. Checking Proofs

Inventing and studying proofs has two aspects:

- inventing / understanding the key ideas
- checking the correctness

Both aspects are indispensable.

In this talk, I am only concerned with checking the correctness.

Checking Fully Displayed Proofs

Fully displayed proofs (in linear or any other form of presentation) are hard to check because

- the formulae relevant in a proof step, typically, are scattered at quite distant places in the proof text
- and, hence, one must "jump back and forth" and memorize formulae many times when checking proofs.

Note that this phenomenon is independent of the particular technique of displaying proofs.

Example

Remark

In a short talk, it is difficult to give good examples because the phenomenon becomes pronounced only in long proofs.

Definitions

We now give a couple of definitions in *Theorema* syntax:

$$\textbf{Definition}[\text{"is relation"}, \text{any}[R], \\ \text{is–relation}[R] :\Leftrightarrow (R \subseteq X \times X)]$$

'X' is a global constant.

$$\textbf{Definition}\Big[\text{"is reflexive"}, \text{any}[R], \\ \text{is–reflexive}[R] :\Leftrightarrow \underset{x \in X}{\forall} \langle x, x \rangle \in R\Big]$$

$$\textbf{Definition}\Big[\text{"is subsets set"}, \text{any}[P], \\ \text{is–subsets–set}[P] :\Leftrightarrow \underset{p \in P}{\forall} p \subseteq X\Big]$$

$$\textbf{Definition}\Big[\text{"is all nonempty"}, \text{any}[P], \\ \text{is–all–nonempty}[P] :\Leftrightarrow \Big(\underset{p \in P}{\forall} p \neq \{\}\Big)\Big]$$

$$\textbf{Definition}\Big[\text{"class"}, \text{any}[R, x], \\ \text{class}[R, x] := \Big\{y \underset{y \in X}{|} \langle y, x \rangle \in R\Big\}\Big]$$

$$\textbf{Definition}\Big[\text{"factor set"}, \text{any}[R], \\ \text{factor–set}[R] := \Big\{\text{class}[R, x] \underset{x}{|} x \in X\Big\}\Big];$$

Lemma

Lemma["example", any[R],
(is–relation[R] $\wedge$ is–reflexive[R]) $\Rightarrow$ (is–subsets–set[factor–set[R]] $\wedge$ is–all–nonempty[factor–set[R]])]

Prover Call

Prove[Lemma["example"], using $\rightarrow$ ⟨Definition["factor set"], Definition["class"], Definition["is relation"],
Definition["is reflexive"], Definition["is subsets set"], Definition["is all nonempty"]⟩,
by $\rightarrow$ SetTheoryPCSProver]

This call to *Theorema* generates, fully automatically, the following proof (including the intermediate explanatory English phrases):

Proof

Prove:

(Lemma (example))

$$\underset{R}{\forall}\, (\text{is–relation}[R] \wedge \text{is–reflexive}[R] \Rightarrow \text{is–subsets–set}[\text{factor–set}[R]] \wedge \text{is–all–nonempty}[\text{factor–set}[R]]),$$

under the assumptions:

(Definition (factor set)) $\underset{R}{\forall}\left(\text{factor–set}[R] := \left\{\text{class}[R, x] \underset{x}{\mid} x \in \mathrm{X}\right\}\right)$,

(Definition (class)) $\underset{R,x}{\forall}\left(\text{class}[R, x] := \left\{y \underset{y}{\mid} y \in \mathrm{X} \wedge \langle y, x\rangle \in R\right\}\right)$,

(Definition (is relation)) $\underset{R}{\forall}\,(\text{is–relation}[R] \;:\Leftrightarrow\; R \subseteq \mathrm{X} \times \mathrm{X})$,

(Definition (is reflexive)) $\underset{R}{\forall}\left(\text{is–reflexive}[R] \;:\Leftrightarrow\; \underset{x}{\forall}\,(x \in \mathrm{X} \Rightarrow \langle x, x\rangle \in R)\right)$,

(Definition (is subsets set)) $\underset{P}{\forall}\left(\text{is–subsets–set}[P] \;:\Leftrightarrow\; \underset{p}{\forall}\,(p \in P \Rightarrow p \subseteq \mathrm{X})\right)$,

(Definition (is all nonempty)) $\underset{P}{\forall}\left(\text{is–all–nonempty}[P] \;:\Leftrightarrow\; \underset{p}{\forall}\,(p \in P \Rightarrow (p \neq \{\}))\right)$.

We assume

(1) $\text{is–relation}[R_0] \wedge \text{is–reflexive}[R_0]$,

and show

(2) $\text{is–subsets–set}[\text{factor–set}[R_0]] \wedge \text{is–all–nonempty}[\text{factor–set}[R_0]]$.

We prove the individual conjunctive parts of (2):

Proof of (2.1) $\text{is–subsets–set}[\text{factor–set}[R_0]]$:

Formula (2.1), using (Definition (factor set)), is implied by:

$$\text{is–subsets–set}\left[\left\{\text{class}[R_0, x] \underset{x}{\mid} x \in X\right\}\right],$$

which, using (Definition (class)), is implied by:

$$\text{is–subsets–set}\left[\left\{\left\{y \underset{y}{\mid} y \in X \wedge \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} x \in X\right\}\right],$$

which, using (Definition (is subsets set)), is implied by:

(3) $\underset{p}{\forall}\left(p \in \left\{\left\{y \underset{y}{\mid} y \in X \wedge \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} x \in X\right\} \Rightarrow p \subseteq X\right)$.

We assume

(4) $p_0 \in \left\{\left\{y \underset{y}{\mid} y \in X \wedge \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} x \in X\right\}$,

and show

(5) $p_0 \subseteq X$.

From what we already know follows:

From (4) we know that we can choose an appropriate value such that

(6) $y1_0 \in X$,

(7) $p_0 = \left\{y \underset{y}{\mid} y \in X \bigwedge \langle y, y1_0\rangle \in R_0\right\}$.

For proving (5) we choose

(13) $p1_0 \in p_0$,

and show:

(14) $p1_0 \in X$.

Formula (13), by (7), implies:

(17) $p1_0 \in \{\underset{y}{y \mid} y \in X \bigwedge \langle y, y1_0\rangle \in R_0\}$.

From what we already know follows:

From (17) we can infer

(18) $p1_0 \in X \bigwedge \langle p1_0, y1_0\rangle \in R_0$.

Formula (14) is true because it is identical to (18.1).

Proof of (2.2) is–all–nonempty[factor–set[R_0]]:

Formula (2.2), using (Definition (factor set)), is implied by:

$\text{is–all–nonempty}\left[\{\underset{x}{\text{class}[R_0, x] \mid} x \in X\}\right]$,

which, using (Definition (class)), is implied by:

$\text{is–all–nonempty}\left[\{\underset{x}{\{\underset{y}{y \mid} y \in X \wedge \langle y, x\rangle \in R_0\} \mid} x \in X\}\right]$,

which, using (Definition (is all nonempty)), is implied by:

(19) $\underset{p}{\forall}\left(p \in \{\underset{x}{\{\underset{y}{y \mid} y \in X \wedge \langle y, x\rangle \in R_0\} \mid} x \in X\} \Rightarrow (p \neq \{\})\right)$.

We assume

(20) $p_1 \in \{\underset{x}{\{\underset{y}{y \mid} y \in X \wedge \langle y, x\rangle \in R_0\} \mid} x \in X\}$,

and show

(21) $p_1 \neq \{\}$.

From what we already know follows:

From (20) we know that we can choose an appropriate value such that

(22) $y5_0 \in X$,

(23) $p_1 = \{\underset{y}{y \mid} y \in X \bigwedge \langle y, y5_0\rangle \in R_0\}$.

Formula (21) means that we have to show that

(29) $\underset{p2}{\exists}(p2 \in p_1)$.

Formula (29), using (23), is implied by:

(30) $\underset{p2}{\exists}\left(p2 \in \left\{y \underset{y}{\mid} y \in \mathrm{X} \bigwedge \langle y, y5_0\rangle \in R_0\right\}\right).$

In order to prove (30) we have to show:

(31) $\underset{p2}{\exists}\,(p2 \in \mathrm{X} \bigwedge \langle p2, y5_0\rangle \in R_0).$

Now, let $p2 := y5_0$. Thus, for proving (31) it is sufficient to prove:

(32) $y5_0 \in \mathrm{X} \bigwedge \langle y5_0, y5_0\rangle \in R_0.$

We prove the individual conjunctive parts of (32):

Proof of (32.1) $y5_0 \in \mathrm{X}$:

Formula (32.1) is true because it is identical to (22).

Proof of (32.2) $\langle y5_0, y5_0\rangle \in R_0$:

Formula (1.2), by (Definition (is reflexive)), implies:

(42) $\underset{x}{\forall}\,(x \in \mathrm{X} \Rightarrow \langle x, x\rangle \in R_0).$

Formula (32.2), using (42), is implied by:

(43) $y5_0 \in \mathrm{X}.$

Formula (43) is true because it is identical to (22).

□

■ The Problem

Design a presentation technique for proofs that, in each proof step, has all the relevant formulae "in the focus of the reader".

A Solution

A Note

Note that no solution is possible for proofs on paper or blackboard.

However, when proofs are available as data objects in machines (as this is the case with proofs generated by automated theorem provers like *Theorema*), a solution is near at hand.

Solution

In a post-processing step, we go over the proof object and generate a sequence of "focus windows".

Proof Objects and Proof Situations

Proof object: nested tree of proof situations.

Proof situation: consists of

- current goal formula
- current knowledge base
- indication of inference rule used together with reference to the relevant formulae in the knowledge base.

Focus Windows

The focus window of a proof step displays the current goal and the relevant formulae.

Displaying a focus window proceeds in two phases:

- attention phase: the current goal and the relevant formulae are displayed
- transformation phase: in addition, the resulting new formulae (goals and / or new knowledge) are displayed.

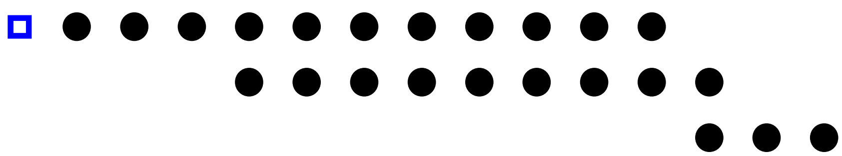
Example

Prove[Lemma["example"],
using → ⟨Definition["factor set"], Definition["class"], Definition["is relation"],
Definition["is reflexive"], Definition["is subsets set"], Definition["is all nonempty"]⟩,
by → SetTheoryPCSProver,
showBy → FocusWindows]

This *Theorema* call, with the option 'showBy' set to 'FocusWindows', produces now the following alternating sequence of attention and transformation windows, each of wich should be easy to study because each attention window contains all the information relevant in the respective proof step and the successive transformation then shows, in addition, the next formulae generated in this step.

Initialization Window

Tree representation



Current goal:

(Lemma (example))

$\underset{R}{\forall}$ (is–relation[R] ∧ is–reflexive[R] ⇒ is–subsets–set[factor–set[R]] ∧ is–all–nonempty[factor–set[R]])

Assumptions:

(Definition (class)) $\underset{R,x}{\forall}\left(\text{class}[R, x] := \left\{y \underset{y}{\mid} y \in X \wedge \langle y, x\rangle \in R\right\}\right)$

(Definition (factor set)) $\underset{R}{\forall}\left(\text{factor–set}[R] := \left\{\text{class}[R, x] \underset{x}{\mid} x \in X\right\}\right)$

(Definition (is all nonempty)) $\underset{P}{\forall}\left(\text{is–all–nonempty}[P] :\Leftrightarrow \underset{p}{\forall}\,(p \in P \Rightarrow (p \neq \{\}))\right)$

(Definition (is reflexive)) $\underset{R}{\forall}\left(\text{is–reflexive}[R] \;:\Leftrightarrow\; \underset{x}{\forall}\,(x \in \mathrm{X} \Rightarrow \langle x, x\rangle \in R)\right)$

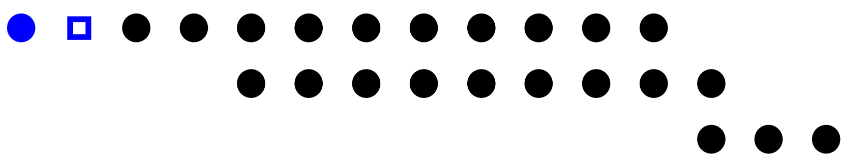
(Definition (is relation)) $\underset{R}{\forall}\,(\text{is–relation}[R] \;:\Leftrightarrow\; R \subseteq \mathrm{X} \times \mathrm{X})$

(Definition (is subsets set)) $\underset{P}{\forall}\left(\text{is–subsets–set}[P] \;:\Leftrightarrow\; \underset{p}{\forall}\,(p \in P \Rightarrow p \subseteq \mathrm{X})\right)$

Next Previous Done

- **Attention Window**

`Tree representation`



Current goal:

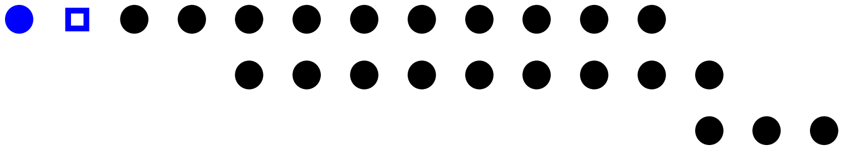
(Lemma (example))

$\underset{R}{\forall}\,(\text{is–relation}[R] \wedge \text{is–reflexive}[R] \Rightarrow \text{is–subsets–set}[\text{factor–set}[R]] \wedge \text{is–all–nonempty}[\text{factor–set}[R]])$

Next Previous Done

- **Transformation Window**

`Tree representation`



Old goal:

(Lemma (example))

$\underset{R}{\forall}$ (is–relation[R] ∧ is–reflexive[R] ⇒ is–subsets–set[factor–set[R]] ∧ is–all–nonempty[factor–set[R]])

New goal:

(2) is–subsets–set[factor–set[R_0]] ∧ is–all–nonempty[factor–set[R_0]]

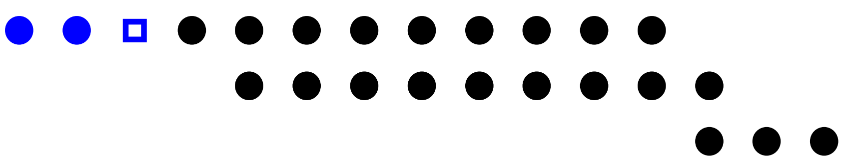
New assumptions:

(1) is–relation[R_0] ∧ is–reflexive[R_0]

Next Previous Done

- **Attention Window**

Tree representation



Current goal:

(2) is–subsets–set[factor–set[R_0]] ∧ is–all–nonempty[factor–set[R_0]]

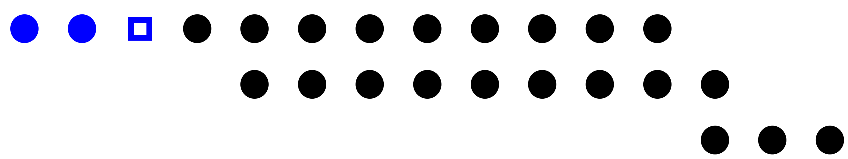
Assumptions:

(1) is–relation[R_0] ∧ is–reflexive[R_0]

Next Previous Done

- **Transformation Window**

Tree representation



Current goal:

(2) is–subsets–set[factor–set[R_0]] ∧ is–all–nonempty[factor–set[R_0]]

Assumptions:

(1) is–relation[R_0] ∧ is–reflexive[R_0]

New assumptions:

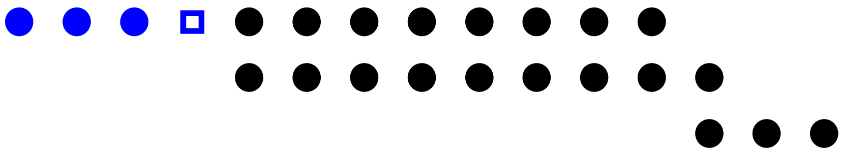
(1.1) is–relation[R_0]

(1.2) is–reflexive[R_0]

Next Previous Done

■ Attention Window

Tree representation



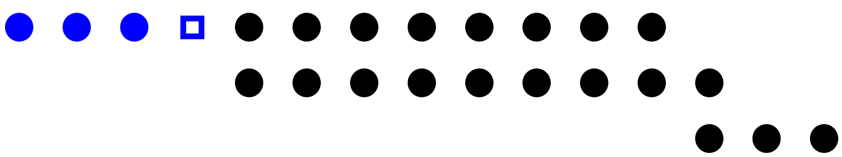
Current goal:

(2) is–subsets–set[factor–set[R_0]] ∧ is–all–nonempty[factor–set[R_0]]

Next Previous Done

■ Transformation Window

Tree representation



Branch 1

Old goal:

(2) is–subsets–set[factor–set[R_0]] ∧ is–all–nonempty[factor–set[R_0]]

New goal:

(2.1) is–subsets–set[factor–set[R_0]]

Next Previous Done

Branch 2

Old goal:

(2) is–subsets–set[factor–set[R_0]] ∧ is–all–nonempty[factor–set[R_0]]

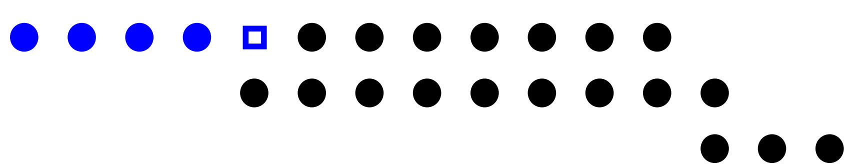
New goal:

(2.2) is–all–nonempty[factor–set[R_0]]

Next Previous Done

- **Attention Window**

Tree representation



Current goal:

(2.1) is–subsets–set[factor–set[R_0]]

Assumptions:

(Definition (class)) $\forall_{R,x}\left(\text{class}[R,x] := \left\{ y \mid_{y} y \in X \wedge \langle y, x\rangle \in R \right\}\right)$

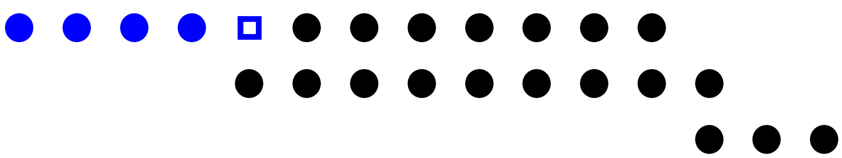
(Definition (factor set)) $\forall_{R}\left(\text{factor–set}[R] := \left\{ \text{class}[R,x] \mid_{x} x \in X \right\}\right)$

(Definition (is subsets set)) $\forall_{P}\left(\text{is–subsets–set}[P] :\Leftrightarrow \forall_{p} (p \in P \Rightarrow p \subseteq X)\right)$

Next Previous Done

- **Transformation Window**

Tree representation



Old goal:

(2.1) is–subsets–set[factor–set[R_0]]

New goal:

(3) $\forall_{p}\left(p \in \left\{\left\{ y \mid_{y} y \in X \wedge \langle y, x\rangle \in R_0 \right\} \mid_{x} x \in X\right\} \Rightarrow p \subseteq X\right)$

Assumptions:

(Definition (class)) $\underset{R,x}{\forall}\left(\text{class}[R, x] := \left\{y \underset{y}{\mid} y \in \text{X} \wedge \langle y, x\rangle \in R\right\}\right)$

(Definition (factor set)) $\underset{R}{\forall}\left(\text{factor–set}[R] := \left\{\text{class}[R, x] \underset{x}{\mid} x \in \text{X}\right\}\right)$

(Definition (is subsets set)) $\underset{P}{\forall}\left(\text{is–subsets–set}[P] :\Leftrightarrow \underset{p}{\forall}(p \in P \Rightarrow p \subseteq \text{X})\right)$

Next Previous Done

- **Attention Window**

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Tree representation
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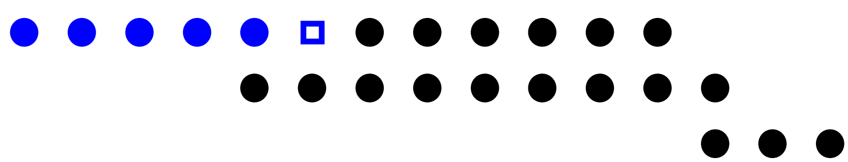


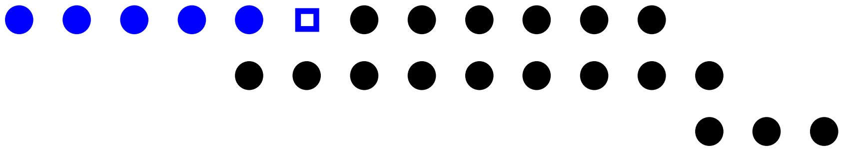
Current goal:

(3) $\underset{p}{\forall}\left(p \in \left\{\left\{y \underset{y}{\mid} y \in \text{X} \wedge \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} x \in \text{X}\right\} \Rightarrow p \subseteq \text{X}\right)$

Next Previous Done

▪ Transformation Window

`Tree representation`



Old goal:

(3) $\underset{p}{\forall}\left(p \in \left\{\left\{y \underset{y}{\mid} y \in \mathrm{X} \wedge \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} x \in \mathrm{X}\right\} \Rightarrow p \subseteq \mathrm{X}\right)$

New goal:

(5) $p_0 \subseteq \mathrm{X}$

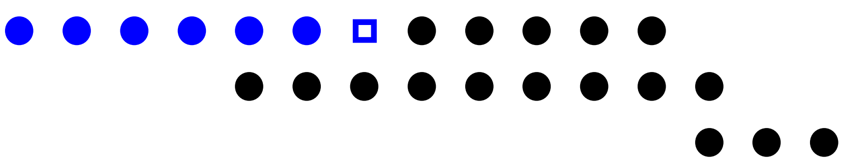
New assumptions:

(4) $p_0 \in \left\{\left\{y \underset{y}{\mid} y \in \mathrm{X} \wedge \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} x \in \mathrm{X}\right\}$

Next Previous Done

▪ Attention Window

`Tree representation`



Current goal:

(5) $p_0 \subseteq \mathrm{X}$

Assumptions:

(4) $p_0 \in \left\{\left\{y \underset{y}{\mid} y \in \mathrm{X} \land \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} x \in \mathrm{X}\right\}$

Next Previous Done

■ **Transformation Window**

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Tree representation
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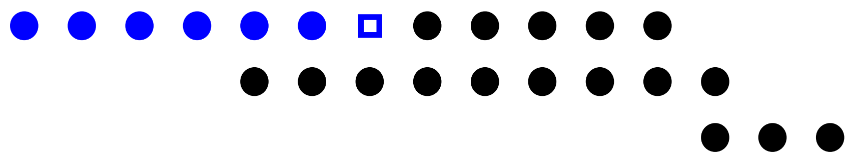


Current goal:

(5) $p_0 \subseteq \mathrm{X}$

Assumptions:

(4) $p_0 \in \left\{\left\{y \underset{y}{\mid} y \in \mathrm{X} \land \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} x \in \mathrm{X}\right\}$

New assumptions:

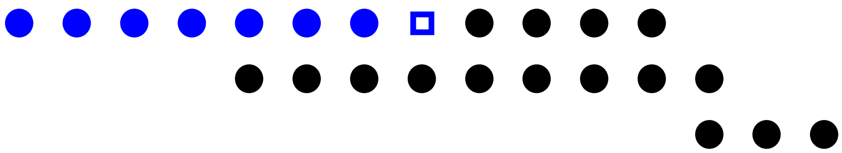
(6) $y1_0 \in \mathrm{X}$

(7) $p_0 = \{y \underset{y}{|} \; y \in \mathrm{X} \bigwedge \langle y, y1_0 \rangle \in R_0\}$

Next Previous Done

- **Attention Window**

`Tree representation`



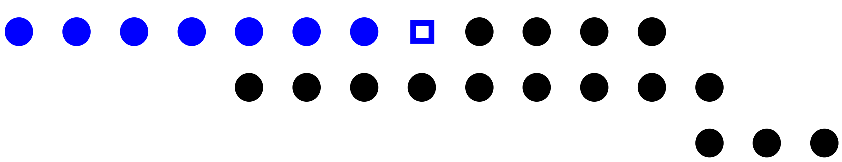
Current goal:

(5) $p_0 \subseteq \mathrm{X}$

Next Previous Done

- **Transformation Window**

`Tree representation`



Old goal:

(5) $p_0 \subseteq X$

New goal:

(14) $p1_0 \in X$

New assumptions:

(13) $p1_0 \in p_0$

Next Previous Done

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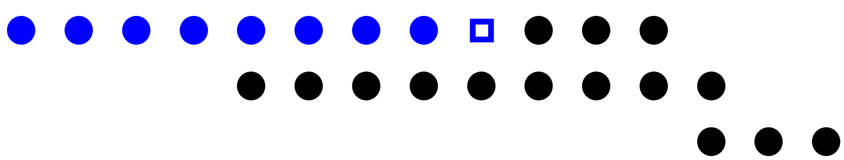
`Tree representation`



Current goal:

(14) $p1_0 \in X$

Assumptions:

(13) $p1_0 \in p_0$

(7) $p_0 = \{ y \underset{y}{|} y \in \mathrm{X} \wedge \langle y, y1_0 \rangle \in R_0 \}$

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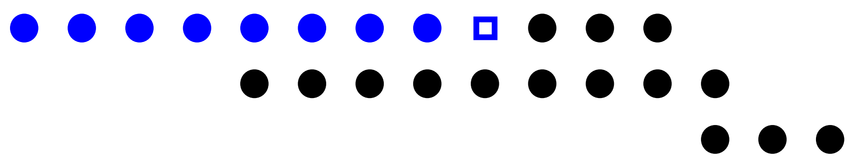


Current goal:

(14) $p1_0 \in \mathrm{X}$

Assumptions:

(13) $p1_0 \in p_0$

(7) $p_0 = \{ y \underset{y}{|} y \in \mathrm{X} \wedge \langle y, y1_0 \rangle \in R_0 \}$

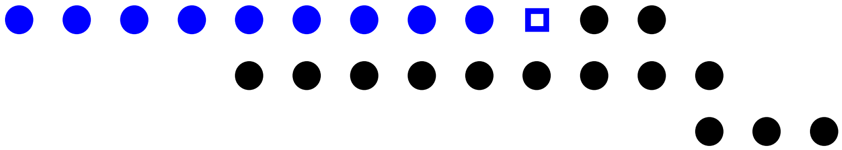
New assumptions:

(17) $p1_0 \in \{ y \underset{y}{|} y \in \mathrm{X} \wedge \langle y, y1_0 \rangle \in R_0 \}$

Next Previous Done

- **Attention Window**

`Tree representation`



Current goal:

(14) $p1_0 \in \mathrm{X}$

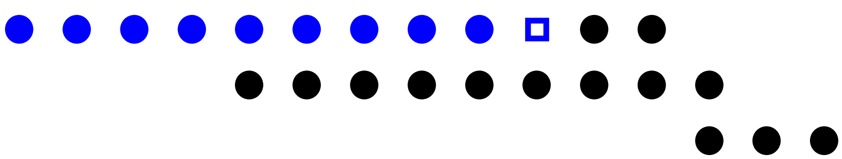
Assumptions:

(17) $p1_0 \in \left\{y \underset{y}{\mid} y \in \mathrm{X} \bigwedge \langle y, y1_0 \rangle \in R_0\right\}$

Next Previous Done

- **Transformation Window**

`Tree representation`



Current goal:

(14) $p1_0 \in \mathrm{X}$

Assumptions:

(17) $p1_0 \in \{\underset{y}{y \mid}\; y \in \mathrm{X} \bigwedge \langle y, y1_0 \rangle \in R_0\}$

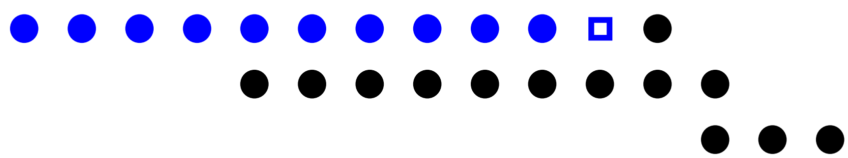
New assumptions:

(18) $p1_0 \in \mathrm{X} \bigwedge \langle p1_0, y1_0 \rangle \in R_0$

Next Previous Done

- **Attention Window**

`Tree representation`



Current goal:

(14) $p1_0 \in \mathrm{X}$

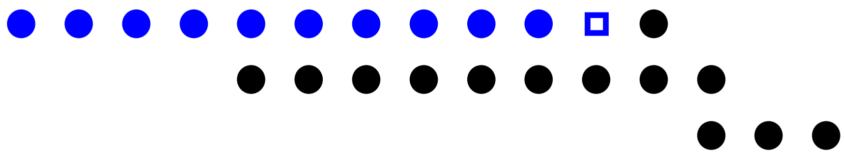
Assumptions:

(18) $p1_0 \in \mathrm{X} \bigwedge \langle p1_0, y1_0 \rangle \in R_0$

Next Previous Done

- **Transformation Window**

`Tree representation`



Current goal:

(14) $p1_0 \in \mathrm{X}$

Assumptions:

(18) $p1_0 \in \mathrm{X} \bigwedge \langle p1_0, y1_0 \rangle \in R_0$

New assumptions:

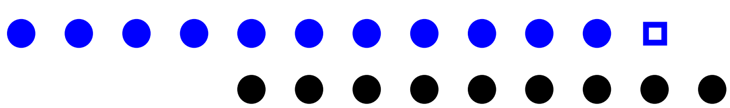
(18.1) $p1_0 \in \mathrm{X}$

(18.2) $\langle p1_0, y1_0 \rangle \in R_0$

Next Previous Done

- **Attention Window**

`Tree representation`



Current goal:

(14) $p1_0 \in X$

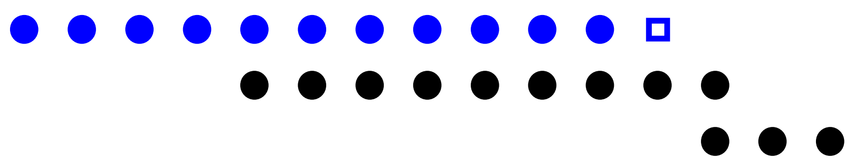
Assumptions:

(18.1) $p1_0 \in X$

Next Previous Done

- **Transformation Window**

Tree representation



Current goal:

(14) $p1_0 \in X$

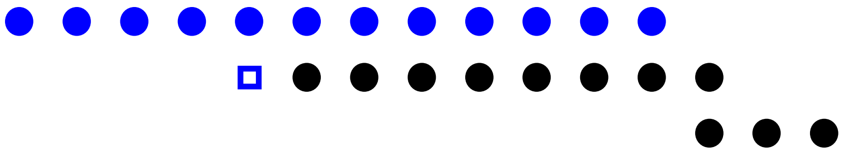
Assumptions:

(18.1) $p1_0 \in X$

Next Previous Done

- **Attention Window**

Tree representation



Current goal:

(2.2) is–all–nonempty[factor–set[R_0]]

Assumptions:

(Definition (class)) $\underset{R,x}{\forall}\left(\text{class}[R,\, x] := \left\{ y \underset{y}{\mid} y \in \mathrm{X} \wedge \langle y,\, x\rangle \in R \right\}\right)$

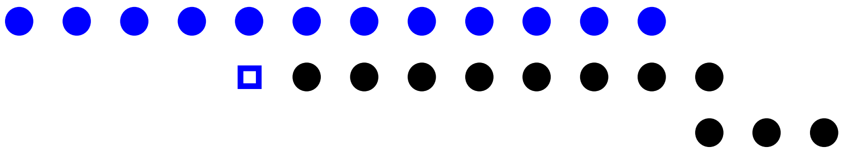
(Definition (factor set)) $\underset{R}{\forall}\left(\text{factor–set}[R] := \left\{ \text{class}[R,\, x] \underset{x}{\mid} x \in \mathrm{X} \right\}\right)$

(Definition (is all nonempty)) $\underset{P}{\forall}\left(\text{is–all–nonempty}[P] :\Leftrightarrow \underset{p}{\forall}\, (p \in P \Rightarrow (p \neq \{\}))\right)$

Next Previous Done

- **Transformation Window**

Tree representation



Old goal:

(2.2) is–all–nonempty[factor–set[R_0]]

New goal:

(19) $\underset{p}{\forall}\left(p \in \left\{\left\{\underset{y}{y \mid} \; y \in \mathrm{X} \wedge \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} \; x \in \mathrm{X}\right\} \Rightarrow (p \neq \{\})\right)$

Assumptions:

(Definition (class)) $\underset{R,x}{\forall}\left(\text{class}[R, x] := \left\{\underset{y}{y \mid} \; y \in \mathrm{X} \wedge \langle y, x\rangle \in R\right\}\right)$

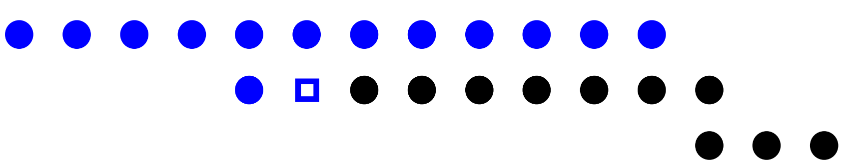
(Definition (factor set)) $\underset{R}{\forall}\left(\text{factor–set}[R] := \left\{\text{class}[R, x] \underset{x}{\mid} \; x \in \mathrm{X}\right\}\right)$

(Definition (is all nonempty)) $\underset{P}{\forall}\left(\text{is–all–nonempty}[P] \; :\Leftrightarrow \; \underset{p}{\forall}\,(p \in P \Rightarrow (p \neq \{\}))\right)$

Next Previous Done

■ **Attention Window**

`Tree representation`



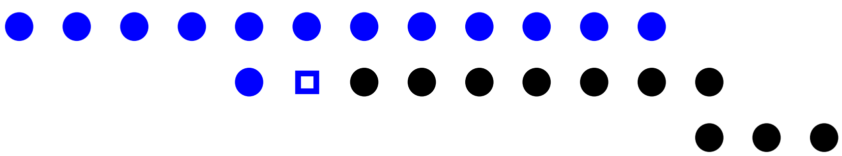
Current goal:

(19) $\underset{p}{\forall}\left(p \in \left\{\left\{y \underset{y}{\mid} y \in \mathrm{X} \wedge \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} x \in \mathrm{X}\right\} \Rightarrow (p \neq \{\})\right)$

Next Previous Done

- **Transformation Window**

Tree representation



Old goal:

(19) $\underset{p}{\forall}\left(p \in \left\{\left\{y \underset{y}{\mid} y \in \mathrm{X} \wedge \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} x \in \mathrm{X}\right\} \Rightarrow (p \neq \{\})\right)$

New goal:

(21) $p_1 \neq \{\}$

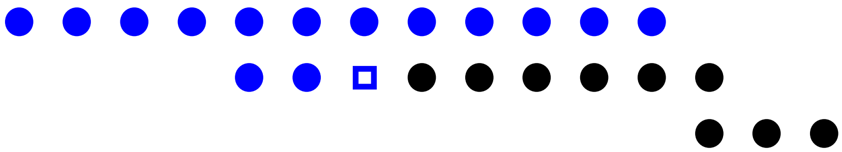
New assumptions:

(20) $p_1 \in \left\{\left\{y \underset{y}{\mid} y \in \mathrm{X} \wedge \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} x \in \mathrm{X}\right\}$

Next Previous Done

- **Attention Window**

Tree representation



Current goal:

(21) $p_1 \neq \{\}$

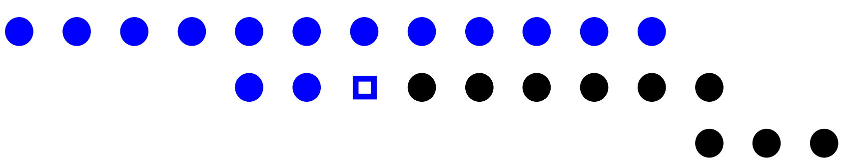
Assumptions:

(20) $p_1 \in \left\{\left\{y \underset{y}{\mid} y \in \mathrm{X} \wedge \langle y, x\rangle \in R_0\right\} \underset{x}{\mid} x \in \mathrm{X}\right\}$

Next Previous Done

- **Transformation Window**

Tree representation



Current goal:

(21) $p_1 \neq \{\}$

Assumptions:

(20) $p_1 \in \left\{\left\{y \underset{y}{|} y \in \mathrm{X} \wedge \langle y, x\rangle \in R_0\right\} \underset{x}{|} x \in \mathrm{X}\right\}$

New assumptions:

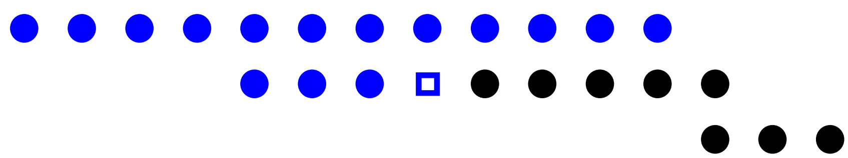
(22) $y5_0 \in \mathrm{X}$

(23) $p_1 = \left\{y \underset{y}{|} y \in \mathrm{X} \bigwedge \langle y, y5_0\rangle \in R_0\right\}$

Next Previous Done

- **Attention Window**

`Tree representation`



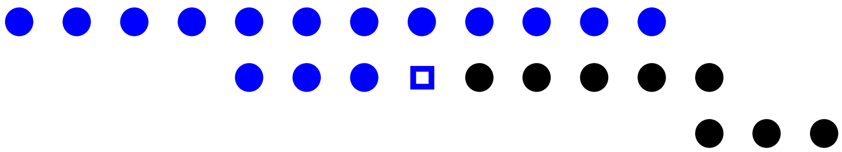
Current goal:

(21) $p_1 \neq \{\}$

Next Previous Done

- **Transformation Window**

`Tree representation`



Old goal:

(21) $p_1 \neq \{\}$

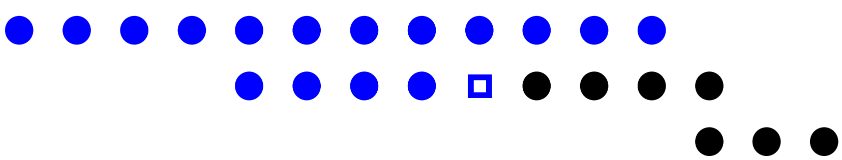
New goal:

(29) $\underset{p2}{\exists}\,(p2 \in p_1)$

Next Previous Done

- **Attention Window**

`Tree representation`



Current goal:

(29) $\underset{p2}{\exists}(p2 \in p_1)$

Assumptions:

(23) $p_1 = \left\{ y \underset{y}{|} y \in \mathrm{X} \bigwedge \langle y, y5_0 \rangle \in R_0 \right\}$

Next Previous Done

- **Transformation Window**

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Tree representation
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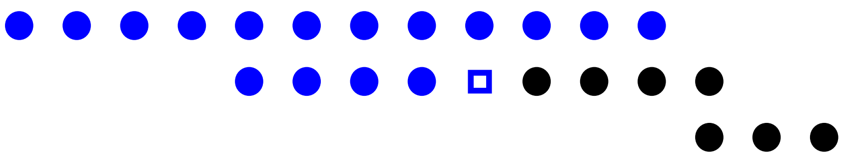


Old goal:

(29) $\underset{p2}{\exists}(p2 \in p_1)$

New goal:

(30) $\underset{p2}{\exists}\left(p2 \in \left\{ y \underset{y}{|} y \in \mathrm{X} \bigwedge \langle y, y5_0 \rangle \in R_0 \right\}\right)$

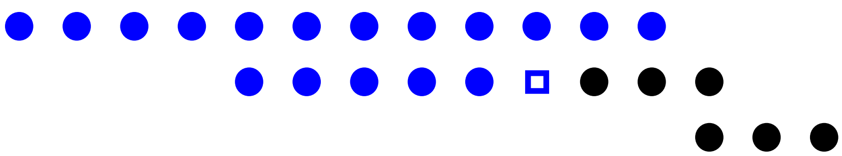
Assumptions:

(23) $p_1 = \left\{ y \underset{y}{|} y \in \mathrm{X} \bigwedge \langle y, y5_0 \rangle \in R_0 \right\}$

Next Previous Done

- **Attention Window**

`Tree representation`



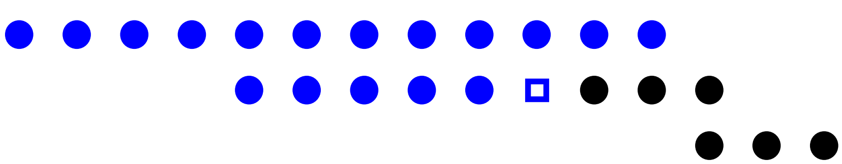
Current goal:

(30) $\underset{p2}{\exists}\left(p2 \in \left\{\underset{y}{y \mid y \in X \bigwedge \langle y, y5_0\rangle \in R_0}\right\}\right)$

Next Previous Done

- **Transformation Window**

`Tree representation`



Old goal:

(30) $\underset{p2}{\exists}\left(p2 \in \left\{y \underset{y}{\mid} y \in \mathrm{X} \bigwedge \langle y, y5_0\rangle \in R_0\right\}\right)$

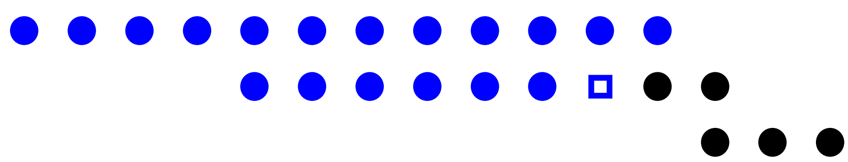
New goal:

(31) $\underset{p2}{\exists}\left(p2 \in \mathrm{X} \bigwedge \langle p2, y5_0\rangle \in R_0\right)$

Next Previous Done

- **Attention Window**

`Tree representation`



Current goal:

(31) $\underset{p2}{\exists}\left(p2 \in \mathrm{X} \bigwedge \langle p2, y5_0\rangle \in R_0\right)$

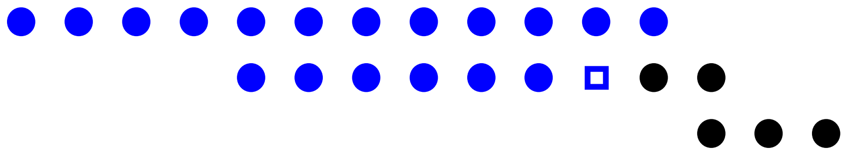
Assumptions:

(22) $y5_0 \in \mathrm{X}$

Next Previous Done

■ Transformation Window

Tree representation



Old goal:

(31) $\underset{p2}{\exists}\,(p2 \in \mathrm{X} \bigwedge \langle p2,\, y5_0 \rangle \in R_0)$

New goal:

(32) $y5_0 \in \mathrm{X} \bigwedge \langle y5_0,\, y5_0 \rangle \in R_0$

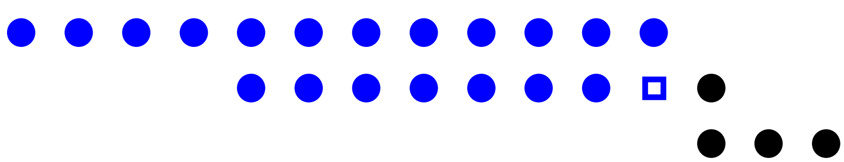
Assumptions:

(22) $y5_0 \in \mathrm{X}$

Next Previous Done

■ Attention Window

Tree representation



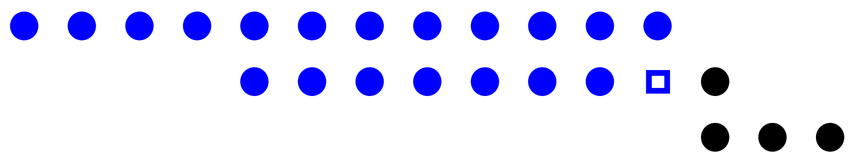
Current goal:

(32) $y5_0 \in \mathrm{X} \bigwedge \langle y5_0,\ y5_0 \rangle \in R_0$

Next Previous Done

■ Transformation Window

Tree representation



Branch 1

Old goal:

(32) $y5_0 \in \mathrm{X} \bigwedge \langle y5_0,\ y5_0 \rangle \in R_0$

New goal:

(32.1) $y5_0 \in \mathrm{X}$

Next Previous Done

Branch 2

Old goal:

(32) $y5_0 \in \mathrm{X} \bigwedge \langle y5_0,\, y5_0 \rangle \in R_0$

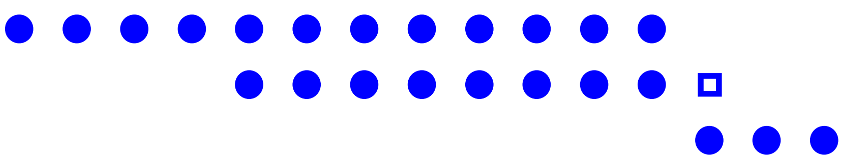
New goal:

(32.2) $\langle y5_0,\, y5_0 \rangle \in R_0$

Next Previous Done

■ Attention Window

Tree representation



Current goal:

(32.1) $y5_0 \in \mathrm{X}$

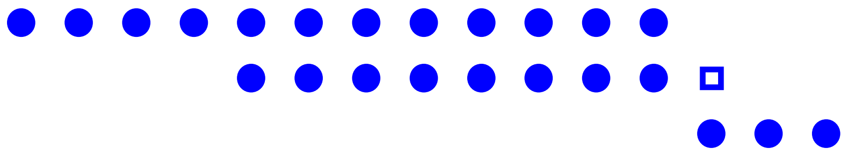
Assumptions:

(22) $y5_0 \in \mathrm{X}$

Next Previous Done

■ Transformation Window

Tree representation



Current goal:

(32.1) $y5_0 \in X$

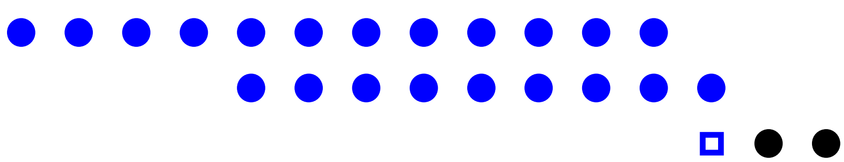
Assumptions:

(22) $y5_0 \in X$

Next Previous Done

■ Attention Window

Tree representation



Current goal:

(32.2) $\langle y5_0, y5_0 \rangle \in R_0$

Assumptions:

(1.2) is–reflexive$[R_0]$

(Definition (is reflexive)) $\underset{R}{\forall}\left(\text{is–reflexive}[R] :\Leftrightarrow \underset{x}{\forall}(x \in \mathrm{X} \Rightarrow \langle x, x\rangle \in R)\right)$

Next Previous Done

■ **Transformation Window**

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Tree representation
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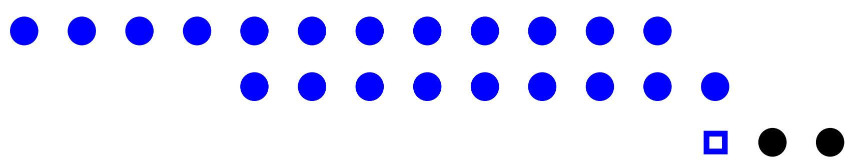


Current goal:

(32.2) $\langle y5_0, y5_0\rangle \in R_0$

Assumptions:

(1.2) is–reflexive$[R_0]$

(Definition (is reflexive)) $\underset{R}{\forall}\left(\text{is–reflexive}[R] :\Leftrightarrow \underset{x}{\forall}(x \in \mathrm{X} \Rightarrow \langle x, x\rangle \in R)\right)$

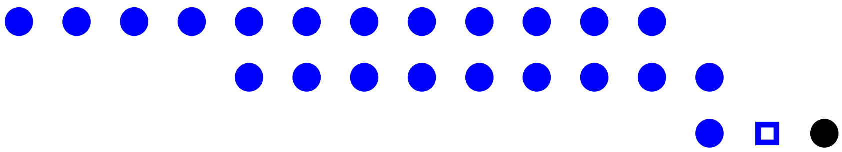
New assumptions:

(42) $\underset{x}{\forall}(x \in \mathrm{X} \Rightarrow \langle x, x\rangle \in R_0)$

Next Previous Done

- **Attention Window**

`Tree representation`



Current goal:

(32.2) $\langle y5_0, y5_0\rangle \in R_0$

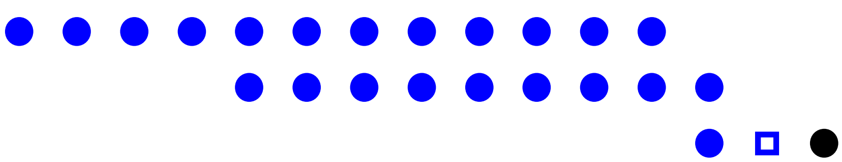
Assumptions:

(42) $\underset{x}{\forall}\,(x \in \mathrm{X} \Rightarrow \langle x, x\rangle \in R_0)$

Next Previous Done

- **Transformation Window**

`Tree representation`



Old goal:

(32.2) $\langle y5_0, y5_0 \rangle \in R_0$

New goal:

(43) $y5_0 \in \mathrm{X}$

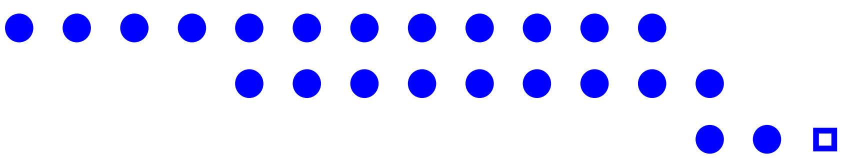
Assumptions:

(42) $\underset{x}{\forall} (x \in \mathrm{X} \Rightarrow \langle x, x \rangle \in R_0)$

Next Previous Done

- **Attention Window**

Tree representation



Current goal:

(43) $y5_0 \in \mathrm{X}$

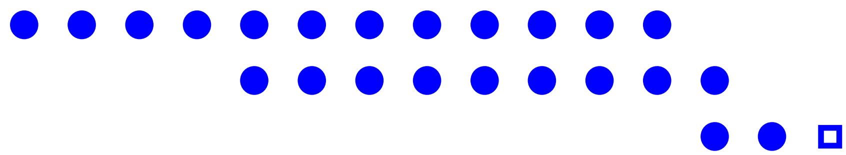
Assumptions:

(22) $y5_0 \in \mathrm{X}$

Next Previous Done

Transformation Window

Tree representation



Current goal:

(43) $y5_0 \in \mathrm{X}$

Assumptions:

(22) $y5_0 \in \mathrm{X}$

Next Previous Done

Conclusion

- Next version of the implementation of the Focus Windows technique in *Theorema*: Will combine the usual linear presentation of proofs with "focus windows on demand".
- We will also implement a feature by which the size of the font in a proof window will decrease to such an extent that the entire text in a focus window always fits into the physical screen.
- We learned that the focus windows proof presentation is also a very useful tool in the phase of developing and debugging new automated provers.

- The notion of focus window also has theoretical bearing:

  It is another way of characterizing the inference rules of special inference systems (for special mathematical theories).

  Also, focus windows contain exactly the information necessary in the "cascading technique" (= analyze failing proofs and invent useful lemmata) of *Theorema*.

References

The following papers on *Theorema,* the focus window technique, and various issues referred to in the talk (e.g. the cascading technique and theory exploration) can be downloaded from my home page:

http://www.risc.uni-linz.ac.at/people/buchberg/

B. Buchberger.
Focus Windows: A New Technique for Presenting Mathematical Proofs (in Automated Theorem Proving Systems).
Theorema Technical Report, 2000-01-30, Research Institute for Symbolic Computation, Johannes Kepler University, A4040 Linz, Austria, 19 pages.

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In: Proceedings of SYNASC 2000 (2nd International Workshop on Symbolic and Numeric Algorithms in Scientific Computing), Oct.4-6, 2000, Timisoara, Rumania, (T.Jebelean, V.Negru, A.Popovici eds.), Analele Universitatii Din Timisoara, Ser. Matematica-Informatica, Vol.XXXVIII, Fasc.2, 2000, pp.9-32.

B. Buchberger.
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Science Faculty, September 27-28, 2001.

B. Buchberger.
Logicographic Symbols: A New Feature in Theorema.
In: Symbolic Computation - New Horizons (Proceedings of the 4th International Mathematica Symposium,Tokyo Denki University, Chiba Campus, Japan, June 25-27, 2001), Tokyo Denki University Press, 2001, pp.23-30.

B. Buchberger, C. Dupre, T. Jebelean, F. Kriftner, K. Nakagawa, D. Vasaru, W. Windsteiger.
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In: Symbolic Computation and Automated Reasoning (Proceedings of CALCULEMUS 2000, Symposium on the Integration of Symbolic Computation and Mechanized Reasoning, August 6-7, 2000, St. Andrews, Scotland, M. Kerber and M. Kohlhase eds.), A.K. Peters, Natick, Massachusetts, pp. 98-113. ISBN 1-56881-145-4.