Example of Using a *Theorema* Prover (the PCS Prover)

```
Definition["limit:", any[f, a],
    limit[f, a] ⟺ ∀_{ϵ>0} ∃_N ∀_{n≥N} |f[n] − a| < ϵ]
```

```
Proposition["limit of sum", any[f, a, g, b],
    (limit[f, a] ∧ limit[g, b])  ⇒  limit[f + g, a + b]]
```

```
Definition["+:", any[f, g, x],
    (f + g)[x] = f[x] + g[x]]
```

```
Lemma["|+|", any[x, y, a, b, δ, ϵ],
    (|(x + y) − (a + b)| < (δ + ϵ))  ⟸  (|x − a| < δ ∧ |y − b| < ϵ)]
```

```
Lemma["max", any[m, M1, M2],
    m ≥ max[M1, M2]  ⇒  (m ≥ M1 ∧ m ≥ M2)]
```

```
Theory["limit",
    Definition["limit:"]
    Definition["+:"]
    Lemma["|+|"]
    Lemma["max"]
]
```

The PCS prover: A heuristic proof method (by Bruno Buchberger 2000) for predicate logic.

Generates "natural" proofs.

For formulae with alternating quantifiers.

The proof of the above theorem (and hundreds of other theorems in analysis) can now be generated completely automatically by calling the PCS prover:

```
Prove[Proposition["limit of sum"], using → Theory["limit"], by → PCS]
```

The proof generated completely automatically by the above call of the PCS algorithm is shown below:

Prove:

(Proposition (limit of sum)) $\forall_{f,a,g,b} (\text{limit}[f, a] \wedge \text{limit}[g, b] \Rightarrow \text{limit}[f + g, a + b])$,

under the assumptions:

(Definition (limit:)) $\forall_{f,a} \left(\text{limit}[f, a] \Leftrightarrow \forall_{\epsilon > 0} \exists_{N} \forall_{n \geq N} (|f[n] - a| < \epsilon) \right)$,

(Definition (+:)) $\forall_{f,g,x} ((f + g)[x] = f[x] + g[x])$,

(Lemma (|+|)) $\forall_{x,y,a,b,\delta,\epsilon} (|(x + y) - (a + b)| < \delta + \epsilon \Leftarrow (|x - a| < \delta \wedge |y - b| < \epsilon))$,

(Lemma (max)) $\forall_{m,M1,M2} (m \geq \max[M1, M2] \Rightarrow m \geq M1 \wedge m \geq M2)$.

We assume

(1) $\text{limit}[\mathrm{f}_0, \mathrm{a}_0] \wedge \text{limit}[\mathrm{g}_0, \mathrm{b}_0]$,

and show

(2) $\text{limit}[\mathrm{f}_0 + \mathrm{g}_0, \mathrm{a}_0 + \mathrm{b}_0]$.

Formula (1.1), by (Definition (limit:)), implies:

(3) $\forall_{\epsilon > 0} \exists_{N} \forall_{n \geq N} (|\mathrm{f}_0[n] - \mathrm{a}_0| < \epsilon)$.

By (3), we can take an appropriate Skolem function such that

(4) $\forall_{\epsilon > 0} \forall_{n \geq \mathrm{N}_0[\epsilon]} (|\mathrm{f}_0[n] - \mathrm{a}_0| < \epsilon)$,

Formula (1.2), by (Definition (limit:)), implies:

(5) $\forall_{\epsilon > 0} \exists_{N} \forall_{n \geq N} (|\mathrm{g}_0[n] - \mathrm{b}_0| < \epsilon)$.

By (5), we can take an appropriate Skolem function such that

(6) $\forall_{\epsilon > 0} \forall_{n \geq \mathrm{N}_1[\epsilon]} (|\mathrm{g}_0[n] - \mathrm{b}_0| < \epsilon)$,

Formula (2), using (Definition (limit:)), is implied by:

(7) $\underset{\substack{\epsilon \\ \epsilon>0}}{\forall} \underset{N}{\exists} \underset{\substack{n \\ n\geq N}}{\forall} (|(f_0 + g_0)[n] - (a_0 + b_0)| < \epsilon).$

We assume

(8) $\epsilon_0 > 0,$

and show

(9) $\underset{N}{\exists} \underset{\substack{n \\ n\geq N}}{\forall} (|(f_0 + g_0)[n] - (a_0 + b_0)| < \epsilon_0).$

We have to find N_2^* such that

(10) $\underset{n}{\forall} (n \geq N_2^* \Rightarrow |(f_0 + g_0)[n] - (a_0 + b_0)| < \epsilon_0).$

Formula (10), using (Definition (+:)), is implied by:

(11) $\underset{n}{\forall} (n \geq N_2^* \Rightarrow |(f_0[n] + g_0[n]) - (a_0 + b_0)| < \epsilon_0).$

Formula (11), using (Lemma (|+|)), is implied by:

(12) $\underset{\substack{\delta,\epsilon \\ \delta+\epsilon=\epsilon_0}}{\exists} \underset{n}{\forall} (n \geq N_2^* \Rightarrow |f_0[n] - a_0| < \delta \wedge |g_0[n] - b_0| < \epsilon).$

We have to find δ_0^*, ϵ_1^*, and N_2^* such that

(13) $(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge \underset{n}{\forall} (n \geq N_2^* \Rightarrow |f_0[n] - a_0| < \delta_0^* \wedge |g_0[n] - b_0| < \epsilon_1^*).$

Formula (13), using (6), is implied by:

$(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge \underset{n}{\forall} (n \geq N_2^* \Rightarrow \epsilon_1^* > 0 \wedge n \geq N_1[\epsilon_1^*] \wedge |f_0[n] - a_0| < \delta_0^*),$

which, using (4), is implied by:

$(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge \underset{n}{\forall} (n \geq N_2^* \Rightarrow \delta_0^* > 0 \wedge \epsilon_1^* > 0 \wedge n \geq N_0[\delta_0^*] \wedge n \geq N_1[\epsilon_1^*]),$

which, using (Lemma (max)), is implied by:

(14) $(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge \underset{n}{\forall} (n \geq N_2^* \Rightarrow \delta_0^* > 0 \wedge \epsilon_1^* > 0 \wedge n \geq \max[N_0[\delta_0^*], N_1[\epsilon_1^*]]).$

Formula (14) is implied by

(15) $(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge \delta_0^* > 0 \bigwedge \epsilon_1^* > 0 \bigwedge \underset{n}{\forall} (n \geq N_2^* \Rightarrow n \geq \max[N_0[\delta_0^*], N_1[\epsilon_1^*]]).$

Partially solving it, formula (15) is implied by

(16) $(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge \delta_0^* > 0 \bigwedge \epsilon_1^* > 0 \bigwedge (\mathrm{N}_2^* = \max[\mathrm{N}_0[\delta_0^*], \mathrm{N}_1[\epsilon_1^*]])$.

Now,

$$(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge \delta_0^* > 0 \bigwedge \epsilon_1^* > 0$$

can be solved for δ_0^* and ϵ_1^* by a call to Collins cad–method yielding a sample solution

$$\delta_0^* \leftarrow \frac{\epsilon_0}{2},$$

$$\epsilon_1^* \leftarrow \frac{\epsilon_0}{2}.$$

Furthermore, we can immediately solve

$$\mathrm{N}_2^* = \max[\mathrm{N}_0[\delta_0^*], \mathrm{N}_1[\epsilon_1^*]]$$

for N_2^* by taking

$$\mathrm{N}_2^* \leftarrow \max[\mathrm{N}_0[\frac{\epsilon_0}{2}], \mathrm{N}_1[\frac{\epsilon_0}{2}]].$$

Hence formula (16) is solved, and we are done.

□