

Examples of RISC Research

B. Buchberger.

Appendix of talk at Symposium "Mathematics and New Technologies: How to Learn? What to Teach?", Dec 10–11, 2003, Madrid.

■ Groebner Bases: Well Established in Research and Systems

BB's Groebner bases method is implemented in all current math software systems for all problems involving multivariate polynomials. For an overview see

B. Buchberger. Introduction to Groebner Bases.

In: Groebner Bases and Applications (B. Buchberger, F. Winkler, eds.), Cambridge University Press, 1998, pp.3–31.

$$F = \{x y^2 - 3 x y + 2, x^2 y - 5 x^2 - 2 x y + 3\}$$

$$\{2 - 3 x y + x y^2, 3 - 5 x^2 - 2 x y + x^2 y\}$$

GroebnerBasis[F]

$$\{-20 + 4 y + 15 y^2 - 14 y^3 + 3 y^4, 4 + 20 x - 10 y + y^2 + 3 y^3\}$$

$$\text{Solve}\{x y^2 - 3 x y + 2 == 0, x^2 y - 5 x^2 - 2 x y + 3 == 0\}$$

$$\left\{ \left\{ y \rightarrow \frac{1}{51} \left(-150 + 251 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} - \frac{55}{2} \left(\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} \right) - \frac{25}{2} \left(\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} \right)^{3/2} + 251 \sqrt{\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} \right. \right.$$

$$\left. \begin{aligned}
& \frac{1934}{125 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}}} \right) - \\
& 55 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} \\
& \sqrt{\left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} - \right.} \\
& \left. \frac{1934}{125 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}}} \right) - \\
& \frac{75}{2} \left(\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} \right) \\
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& \frac{55}{2} \left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} - \right. \\
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& \frac{75}{2} \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} \\
& \left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} - \right.
\end{aligned}$$

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& \frac{25}{2} \left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} - \right. \\
& \left. \frac{1934}{125 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}}} \right)^{3/2} \\
x \rightarrow & -\frac{4}{5} + \frac{1}{2} \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} + \\
& \frac{1}{2} \sqrt{\left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} - \right. \\
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\{y \rightarrow & \frac{1}{51} \left(-150 + 251 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} - \right. \\
& \frac{55}{2} \left(\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} \right) - \\
& \frac{25}{2} \left(\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} \right)^{3/2} - \\
& 251 \sqrt{\left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} - \right. \\
& \left. \frac{1934}{125 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}}} \right)} + \\
& \left. 55 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} \right)
\end{aligned}$$

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& \sqrt{\left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} - \right. \\
& \quad \left. \frac{1934}{125 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}}} \right) +} \\
& \frac{75}{2} \left(\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} \right) \\
& \sqrt{\left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} - \right. \\
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& \frac{55}{2} \left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} - \right. \\
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& \frac{75}{2} \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} \\
& \left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} - \right. \\
& \quad \left. \frac{1934}{125 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}}} \right) +} \\
& \frac{25}{2} \left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} - \right.
\end{aligned}$$

$$\left. \left. \left. \frac{1934}{125 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}}} \right)^{3/2} \right) \right\}$$

$$x \rightarrow -\frac{4}{5} + \frac{1}{2} \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} -$$

$$\frac{1}{2} \sqrt{\left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} - \right.$$

$$\left. \left. \left. \frac{1934}{125 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}}} \right)^{3/2} \right) \right\},$$

$$\left\{ y \rightarrow \frac{1}{51} \left(-150 - 251 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} - \right. \right.$$

$$\frac{55}{2} \left(\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} \right) +$$

$$\frac{25}{2} \left(\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} \right)^{3/2} +$$

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$$55 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}}$$

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& \frac{75}{2} \left(\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} \right) \\
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& x \rightarrow -\frac{4}{5} - \frac{1}{2} \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} +
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& \frac{1}{2} \sqrt{\left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} + \right. \\
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\{y \rightarrow \frac{1}{51} & \left[-150 - 251 \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} - \right. \\
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& \frac{25}{2} \left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} + \right. \\
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x \rightarrow & -\frac{4}{5} - \frac{1}{2} \sqrt{\frac{337}{75} + \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} + \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3}} - \\
& \frac{1}{2} \sqrt{\left(\frac{674}{75} - \frac{1}{30} (165727 - 408 \sqrt{21342})^{1/3} - \frac{1}{30} (165727 + 408 \sqrt{21342})^{1/3} + \right. \\
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\end{aligned}$$

■ Symbolic Summation: Well Established in Research

Zeilberger, Paule, ... method. Based Groebner bases for non-commutative polynomials play a role.

Example: Closed form for the following expression was an open problem for many years! The solution was given in P. Paule, *Computer-Solution of Problem 94-2*, SIAM REVIEW Vol.37 (1995), 105–106 using the Paule-Schorn automated conjecture generator / prover.

The method is not yet routinely available in math software systems.

Formula["SIAM series",

$$\sum_{k=1, \dots, n} \frac{(-1)^{k+1} (4k+1) (2k)!}{2^k (2k-1) (k+1)! 2^k k!}]$$

Simplify[Formula["SIAM series"], by → PauleSchorn-Telescope, built-in → Built-in["PauleSchorn"]]

If '-1 + n' is a natural number, then:

$$1 + (-1)^{-1+n} 2^{-2n} n!^{-1} (2n)! (1+n)!^{-1}$$

■ Automated Proving in Elementary Analysis: Under Study

The PCS prover (BB 2000): Experimental implementation in the frame of the *Theorema* system. See

B. Buchberger.

The PCS Prover in *Theorema*.

In: R. Moreno-Diaz, B. Buchberger, J.L. Freire (eds.), Proceedings of EUROCAST 2001 (8th International Conference on Computer Aided Systems Theory – Formal Methods and Tools for Computer Science), Feb. 19–23, 2001, Las Palmas de Gran Canaria, Lecture Notes in Computer Science 2178, 2201, Springer, Berlin – Heidelberg – New York, pp. 469–478

Definition["limit:", any[f, a],

$$\text{limit}[f, a] \iff \forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{n \geq N} |f[n] - a| < \epsilon]$$

Proposition["limit of sum", any[f, a, g, b],

$$(\text{limit}[f, a] \wedge \text{limit}[g, b]) \Rightarrow \text{limit}[f + g, a + b]$$

Definition["+":, any[f, g, x],

$$(f + g)[x] = f[x] + g[x]$$

Lemma["|+"], any[x, y, a, b, δ , ϵ],
 $((x + y) - (a + b)) < (\delta + \epsilon) \iff (|x - a| < \delta \wedge |y - b| < \epsilon)$

Lemma["max"], any[m, M1, M2],
 $m \geq \max[M1, M2] \Rightarrow (m \geq M1 \wedge m \geq M2)$

Theory["limit",

Definition["limit:"]

Definition["+:"]

Lemma["|+"]

Lemma["max"]

Prove[Proposition["limit of sum"], using \rightarrow Theory["limit"], by \rightarrow PCS]

By this *Theorema* call, the following proof is generated completely automatically by our PCS proving algorithm:

Prove:

(Proposition (limit of sum)) $\forall_{f,a,g,b} (\text{limit}[f, a] \wedge \text{limit}[g, b] \Rightarrow \text{limit}[f + g, a + b]),$

under the assumptions:

(Definition (limit:)) $\forall_{f,a} \left(\text{limit}[f, a] \Leftrightarrow \forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{n \geq N} (|f[n] - a| < \epsilon) \right),$

(Definition (+:)) $\forall_{f,g,x} ((f + g)[x] = f[x] + g[x]),$

(Lemma (+)) $\forall_{x,y,a,b,\delta,\epsilon} (|(x + y) - (a + b)| < \delta + \epsilon \Leftrightarrow (|x - a| < \delta \wedge |y - b| < \epsilon)),$

(Lemma (max)) $\forall_{m,M1,M2} (m \geq \max[M1, M2] \Rightarrow m \geq M1 \wedge m \geq M2).$

We assume

(1) $\text{limit}[f_0, a_0] \wedge \text{limit}[g_0, b_0],$

and show

(2) $\text{limit}[f_0 + g_0, a_0 + b_0].$

Formula (1.1), by (Definition (limit:)), implies:

(3) $\forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{n \geq N} (|f_0[n] - a_0| < \epsilon).$

By (3), we can take an appropriate Skolem function such that

(4) $\forall_{\epsilon > 0} \forall_{n \geq N_0[\epsilon]} (|f_0[n] - a_0| < \epsilon),$

Formula (1.2), by (Definition (limit:)), implies:

$$(5) \quad \forall_{\epsilon > 0} \exists_N \forall_{n \geq N} (|g_0[n] - b_0| < \epsilon).$$

By (5), we can take an appropriate Skolem function such that

$$(6) \quad \forall_{\epsilon > 0} \forall_{n \geq N_1[\epsilon]} (|g_0[n] - b_0| < \epsilon),$$

Formula (2), using (Definition (limit:)), is implied by:

$$(7) \quad \forall_{\epsilon > 0} \exists_N \forall_{n \geq N} (|(f_0 + g_0)[n] - (a_0 + b_0)| < \epsilon).$$

We assume

$$(8) \quad \epsilon_0 > 0,$$

and show

$$(9) \quad \exists_N \forall_{n \geq N} (|(f_0 + g_0)[n] - (a_0 + b_0)| < \epsilon_0).$$

We have to find N_2^* such that

$$(10) \quad \forall_n (n \geq N_2^* \Rightarrow |(f_0 + g_0)[n] - (a_0 + b_0)| < \epsilon_0).$$

Formula (10), using (Definition (+:)), is implied by:

$$(11) \quad \forall_n (n \geq N_2^* \Rightarrow |f_0[n] + g_0[n] - (a_0 + b_0)| < \epsilon_0).$$

Formula (11), using (Lemma (|+|)), is implied by:

$$(12) \quad \exists_{\delta, \epsilon} \forall_n (n \geq N_2^* \Rightarrow |f_0[n] - a_0| < \delta \wedge |g_0[n] - b_0| < \epsilon).$$

$\delta + \epsilon = \epsilon_0$

We have to find δ_0^* , ϵ_1^* , and N_2^* such that

$$(13) \quad (\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge_n (n \geq N_2^* \Rightarrow |f_0[n] - a_0| < \delta_0^* \wedge |g_0[n] - b_0| < \epsilon_1^*).$$

Formula (13), using (6), is implied by:

$$(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge_n (n \geq N_2^* \Rightarrow \epsilon_1^* > 0 \wedge n \geq N_1[\epsilon_1^*] \wedge |f_0[n] - a_0| < \delta_0^*),$$

which, using (4), is implied by:

$$(\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge_n (n \geq N_2^* \Rightarrow \delta_0^* > 0 \wedge \epsilon_1^* > 0 \wedge n \geq N_0[\delta_0^*] \wedge n \geq N_1[\epsilon_1^*]),$$

which, using (Lemma (max)), is implied by:

$$(14) \quad (\delta_0^* + \epsilon_1^* = \epsilon_0) \bigwedge_n (n \geq N_2^* \Rightarrow \delta_0^* > 0 \wedge \epsilon_1^* > 0 \wedge n \geq \max[N_0[\delta_0^*], N_1[\epsilon_1^*]]).$$

Formula (14) is implied by

$$(15) \quad (\delta_0^* + \epsilon_1^* = \epsilon_0) \wedge \delta_0^* > 0 \wedge \epsilon_1^* > 0 \wedge \bigwedge_n (n \geq N_2^* \Rightarrow n \geq \max[N_0[\delta_0^*], N_1[\epsilon_1^*]]).$$

Partially solving it, formula (15) is implied by

$$(16) \quad (\delta_0^* + \epsilon_1^* = \epsilon_0) \wedge \delta_0^* > 0 \wedge \epsilon_1^* > 0 \wedge (N_2^* = \max[N_0[\delta_0^*], N_1[\epsilon_1^*]]).$$

Now,

$$(\delta_0^* + \epsilon_1^* = \epsilon_0) \wedge \delta_0^* > 0 \wedge \epsilon_1^* > 0$$

can be solved for δ_0^* and ϵ_1^* by a call to Collins cad-method yielding a sample solution

$$\delta_0^* \leftarrow \frac{\epsilon_0}{2},$$

$$\epsilon_1^* \leftarrow \frac{\epsilon_0}{2}.$$

Furthermore, we can immediately solve

$$N_2^* = \max[N_0[\delta_0^*], N_1[\epsilon_1^*]]$$

for N_2^* by taking

$$N_2^* \leftarrow \max[N_0[\frac{\epsilon_0}{2}], N_1[\frac{\epsilon_0}{2}]].$$

Hence formula (16) is solved, and we are done. □

■ Automated Algorithm Invention by Failing Correctness Proofs: On the Horizon

Recent research (BB 2002) with some encouraging results, see

B. Buchberger.

Algorithm Invention and Verification by Lazy Thinking.

In: D. Petcu, V. Negru, D. Zaharie, T. Jebelean (eds), Proceedings of SYNASC 2003 (Symbolic and Numeric Algorithms for Scientific Computing, Timisoara, Romania, October 1–4, 2003), Mirton Publishing, ISBN 973–661–104–3pp. 2–26.

Give a problem specification, e.g. find 'sort' such that

$$\bigvee_{\text{is-tuple}[X]} \text{is-sorted-version}[X, \text{sorted}[X]]$$

where

$$\bigvee_{\text{is-tuple}[X]} \left(\text{is-sorted-version}[X, Y] \Leftrightarrow \left(\begin{array}{l} \text{is-tuple}[Y] \\ X \approx Y \\ \text{is-sorted}[Y] \end{array} \right) \right).$$

Press the button and get algorithms for the problem:

$$\forall_{\text{is-tuple}[X]} \left(\text{sorted}[X] = \begin{cases} \text{special}[X] & \leftarrow \text{is-trivial-tuple}[X] \\ \text{merged}[\text{sorted}[\text{left-split}[X]], \text{sorted}[\text{right-split}[X]]] & \leftarrow \text{otherwise} \end{cases} \right)$$

and

$$\text{sorted}[\langle \rangle] = c$$

$$\forall_x (\text{sorted}[\langle x \rangle] = d[x])$$

$$\forall_{x,y,z} (\text{sorted}[\langle x, y, \bar{z} \rangle] = i[x, \text{sorted}[\langle y, \bar{z} \rangle]])$$

and

...

Note after the conference: A major step forward was achieved by this method in Jan 2004, see

B. Buchberger.

Towards the Automated Synthesis of a Gröbner Basis Algorithm.

RACSAM (Rev. R. Acad. Cien. Serie A. Mat.), 10 pages, submitted.

In this paper I show that, with my above 2003 algorithm synthesis method, it is possible to automatically synthesize the Groebner bases algorithm illustrated in the first section above. This shows that the 2003 synthesis method is strong enough to synthesize also highly non-trivial algorithms.