Computing Limits of Sequences

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1 Introduction

Automated asymptotic methods only recently became subject of symbolic computation. Contributions like [3, 12, 5, 10, 13] discuss a variety of aspects concerning the asymptotic analysis of continuous, real-valued functions. First approaches employed *generalized series expansion* [5, 3], later methods are based on the theory of *Hardy fields* [6].

So far, it seems that more emphasis was laid on the treatment of continuous functions, and although some mathematical fundaments are already available [1, 2], algorithmic approaches for the discrete case seem rare.

Our poster focuses on limit computation of *sequences* in $\mathbb{Q}$, more specifically, of sequences that can be defined by $\Pi\Sigma$-expressions [7]. We will present a rather simple approach which avoids the use of heavy theory but whose first applications already give promising fine results.

2 A discrete Analogon to l'Hospital's Rule

As is pointed out, e.g., in [4], the difference operator Δ, defined by $\Delta a_n := a_{n+1} - a_n$ can be viewed as a discrete analogon to the differential operator $D\colon f \mapsto f'$ of the continuous world. For instance, the discrete product rule

$$\Delta(a_n b_n) := (\Delta a_n) b_n + a_{n+1} \Delta b_n$$

closely resembles the product rule for differentiation.

Analogously, the summation operator Σ, defined by $\Sigma a_n := \sum_{k=1}^{n} a_k$, corresponds to indefinite integration in the continuous setting.

This correspondence between discrete and continuous operators has been exploited in algorithms for difference equations and indefinite summation (see, e.g., [7, 11]). Following this spirit, the following theorem may be regarded as a discrete analogon to l'Hospital's rule.

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Theorem (Discrete L'Hospital's Rule) Let $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ be real sequences, both tending to infinity as $n \to \infty$ and suppose that (b_n) is asymptotically strictly monotonous. Then,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\Delta a_n}{\Delta b_n},$$

provided that the limit on the right-hand side exists. □

The theorem follows immediately from statement I.1.70 in [9] or II §8, ex. 5 in [8] (see also Lemma 8.7 of [2]).

As an example,

$$\lim_{n\to\infty} \frac{\binom{2n}{n}}{\sum\limits_{k=1}^{n} \binom{2k}{k}} = \lim_{n\to\infty} \frac{\binom{2(n+1)}{n+1} - \binom{2n}{n}}{\binom{2(n+1)}{n+1}} = \lim_{n\to\infty} \left(1 - \frac{n+1}{2(2n+1)}\right) = \tfrac{3}{4}.$$

Note that this limit is found neither by Maple nor by Mathematica because standard limit computation techniques cannot deal with the summation sign in the denominator.

3 Towards an Algorithm

We consider terms that are built from rational functions, indefinite sums, and indefinite products over an indeterminate n. Such expressions are called $\Pi\Sigma$-expressions. A $\Pi\Sigma$-expression is called *admissible,* if the scopes of all occuring product quantifiers are asymptotically postive. It can be shown that every admissible term is asymptotically monotonous, so we need not worry about a decision procedure for monotonicity.

Examples for admissible terms include expressions involving exponentials a^n $(a \geq 0)$, factorials $n!$, binomial coefficients $\binom{n}{k}$, etc. while terms like $(-1)^n$ are excluded.

We illustrate our method with by example. In order to compute, for instance, $\lim\limits_{n\to\infty} \frac{\mathrm{H}_n \sum_{k=1}^{n} \mathrm{H}_k}{n\mathrm{H}_{2n}^2}$, where $\mathrm{H}_n := \sum\limits_{k=1}^{n} \frac{1}{k}$ denotes the nth harmonic number, we first apply l'Hospital's rule obtaining

$$\lim_{n\to\infty} \frac{p_n}{q_n} \tag{1}$$

with

$$\begin{aligned} p_n = {} & 4n^3\mathrm{H}_n\mathrm{H}_{2n} + 4n^2 \sum_{k=1}^{n} \mathrm{H}_k + 10n^2\mathrm{H}_n\mathrm{H}_{2n} + 7n \sum_{k=1}^{n} \mathrm{H}_k + 4n^2\mathrm{H}_n \\ & + 4n^2\mathrm{H}_{2n} + 8n\mathrm{H}_n\mathrm{H}_{2n} + 7n\mathrm{H}_n + 3\sum_{k=1}^{n} \mathrm{H}_k + 4n + 6n\mathrm{H}_{2n} \\ & + 2\mathrm{H}_n\mathrm{H}_{2n} + 3\mathrm{H}_n + 2\mathrm{H}_{2n} + 3 \quad \text{and} \\ q_n = {} & 4n^3\mathrm{H}_n^2 + 8n^3\mathrm{H}_n + 10n^2\mathrm{H}_n^2 + 20n^2\mathrm{H}_n + 4n^2 + 8n\mathrm{H}_n^2 + 16n\mathrm{H}_n \\ & + 6n + 2\mathrm{H}_n^2 + 4\mathrm{H}_n + 2 \end{aligned}$$

after expanding products and clearing double denominators. Subsequent applications of l'Hospital's rule would lead to more and more complicated expressions and would most likely never come to a conclusion.

Instead, we determine in the next step a *dominant term* of both numerator and denominator and consider only the quotient of these "leading terms" — the other terms are negligible asymptotically. This is known from calculus exercises like

$$\lim_{n\to\infty} \frac{5n^3 + 3n^2 + 4}{3n^3 + 4n - 5} = \tfrac{5}{3},$$

which we now extend from univariate to multivariate rational functions. Taking the point of view of $\Pi\Sigma$-fields [7], we may regard the expressions p_n and q_n of (1) as polynomials in the ring

$$R = \mathbb{Q}[n, \mathrm{H}_n, \mathrm{H}_{2n}, \sum_{k=1}^{n} \mathrm{H}_k].$$

This view regards n, H_n, etc. no longer as sequences but as pure algebraic variables. For monomials m_1, m_2 in R corresponding to admissible sequences $m_n^{(1)}, m_n^{(2)}$, respectively, let

$$m_1 \prec m_2 \quad :\Longleftrightarrow \quad \lim_{n\to\infty} \frac{m_n^{(1)}}{m_n^{(2)}} = 0.$$

This defines a strict partial ordering $\prec$ on the monomials of R.

Continuing the example of (1), we determine the leading term of both p_n and q_n with respect to $\prec$ by calling the algorithm recursively. In general, since $\prec$ is only a partial ordering, a unique greatest summand need not exists. This case is more complicated to deal with, but in the situation of (1) the leading terms are unique, and the problem reduces to

$$\lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} \frac{4n^3 \mathrm{H}_n \mathrm{H}_{2n}}{4n^3 \mathrm{H}_n^2} = \lim_{n\to\infty} \frac{\mathrm{H}_{2n}}{\mathrm{H}_n} \stackrel{\text{l'H}}{=} \lim_{n\to\infty} \frac{4n+3}{4n+2} = 1.$$

Putting things together, we obtain

$$\lim_{n\to\infty} \frac{\mathrm{H}_n \sum_{k=1}^{n} \mathrm{H}_k}{n \mathrm{H}_{2n}^2} = 1.$$

4 Open Problems

Although a Maple implementation of the method outlined in the previous section works quite nice (see Section 5 for examples), some questions remain open and will be subject of further investigations.

Currently, we deal with monotonicity by restricting the admissible sequences to a class where everything is asymptotically monotonous, and we employ a set of ad-hoc criterions for checking that numerator and denominator go to infinity. Naturally, future work will focus on enlarging the class of admissible terms and on a more systematic treatment for deciding unboundedness.

The method described in Section 3 is a semi-decision procedure in the sense that every computed limit is correct but the method may fail or it may run forever. Refined versions should focus on the treatment of sequences which currently cause a failure of the method, and an a priory criteria for termination would be interesting.

5 Some Examples

For giving a flavor of the problems that can be treated by our method, we provide some examples, none of which can be found by the standard general purpose CA-systems.

- $\lim\limits_{n\to\infty} \dfrac{\sum_{k=1}^{n}\left(k^2 \sum_{i=1}^{k} \frac{2^i}{i}\right)}{2^n n} = 4,$
- $\lim\limits_{n\to\infty} \dfrac{\sum_{k=1}^{n} \binom{3k}{k}\binom{5k}{k}}{\binom{3n}{n}\binom{5n}{n}} = \frac{84375}{83351},$
- $\lim\limits_{n\to\infty} \dfrac{\sum_{k=1}^{n} \frac{2^k}{k^2}\binom{2k}{k}}{\left(\sum_{k=1}^{n} \frac{2^k}{k^2}\right) \sum_{k=1}^{n} \binom{2k}{k}} = \frac{3}{7},$
- $\lim\limits_{n\to\infty} \dfrac{(\mathrm{H}_n)^3 \sum_{k=1}^{n} \frac{1}{\mathrm{H}_k}}{n \sum_{k=1}^{n} \frac{\mathrm{H}_k}{k}} = 2$
- $\lim\limits_{n\to\infty} \dfrac{n^3 \left(\sum_{k=1}^{n} \frac{2^k}{k^2}\right)^2}{2^n \sum_{k=1}^{n} \frac{2^k}{k}} = 2,$
- $\lim\limits_{n\to\infty} \dfrac{\mathrm{H}_n \sum_{k=1}^{n} \frac{2^k}{k}}{n \sum_{k=1}^{n} \frac{2^k \mathrm{H}_k}{k^2}} = 1,$
- $\lim\limits_{n\to\infty} \dfrac{\sum_{k=1}^{2n} \frac{2^k k!}{k^2}}{\left(\sum_{k=1}^{n} \frac{4^k}{k^2}\right) \sum_{k=1}^{2n} k!} = \frac{3}{16},$
- $\lim\limits_{n\to\infty} \dfrac{\sum_{k=1}^{2n} \frac{(\mathrm{H}_k)^2}{k}}{(\mathrm{H}_n)^3} = \frac{1}{3}.$

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