A Classical Hypergeometric Proof of an Important Transformation Formula Found by J.-B. Baillon and R.E. Bruck

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Abstract

We give a classical hypergeometric proof of a crucial transformation formula arising in work of J.-B. Baillon and R.E. Bruck on asymptotic regularity.

1 The Problem

In order to derive a quantitative form of the Ishikawa-Edelstein-O'Brian asymptotic regularity theorem, J.-B. Baillon and R.E. Bruck [1] needed to verify the hypergeometric identity

$$ {}_2F_1\left(\begin{matrix} 1/2, -m \\ 2 \end{matrix}; 4x(1-x)\right) = \tag{1} $$

$$ (m+1)(1-x)x^{2m-1}{}_2F_1\left(\begin{matrix} -m, -m \\ 2 \end{matrix}; (\frac{1-x}{x})^2\right) + $$

$$(2x-1)x^{2m-1}{}_2F_1\left(\begin{matrix}-m,-m\\1\end{matrix};(\frac{1-x}{x})^2\right).$$

Using Zeilberger's algorithm [5], J.-B. Baillon and R.E. Bruck gave a computer proof of this identity which is the key to the integral representation ([1], (2.1)) of their main theorem.

In this note we show how (1) can be proved by classical hypergeometric machinery, hence solving open problem 9.10 posed by J.-B. Baillon and R.E. Bruck [1].

All what we need is,

$$\frac{abz}{(c-1)c}{}_2F_1\left(\begin{matrix}a+1,b+1\\c+1\end{matrix};z\right)={}_2F_1\left(\begin{matrix}a,b\\c-1\end{matrix};z\right)-{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right)\tag{2}$$

and

$${}_2F_1\left(\begin{matrix}a+1,b\\c+1\end{matrix};z\right)=\frac{a-c}{a}{}_2F_1\left(\begin{matrix}a,b\\c+1\end{matrix};z\right)+\frac{c}{a}{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right),\tag{3}$$

together with

$${}_2F_1\left(\begin{matrix}b,a\\2b\end{matrix};\frac{4z}{(1+z)^2}\right)=(1+z)^{2a}{}_2F_1\left(\begin{matrix}a,a-b+\frac{1}{2}\\b+\frac{1}{2}\end{matrix};z^2\right).\tag{4}$$

The relations (2), (3) are called "contiguous relations". They can be easily verified. (For instance, (3) is a special case of Exercise 5.25 in [2].) The quadratic transformation (4) is due to Gauss. For a proof see, for instance, the book of Rainville [4].

2 The Proof

Using (2) the left hand side of (1) is equal to

$$B_m(x):=m(1-x)x{}_2F_1\left(\begin{matrix}\frac{3}{2},1-m\\3\end{matrix};4x(1-x)\right)+{}_2F_1\left(\begin{matrix}\frac{1}{2},-m\\1\end{matrix};4x(1-x)\right).$$

Now the key step is to apply (4) to each of the summands of $B_m(x)$ with $z:=(1-x)/x$ and $b=3/2$, respectively $b=1/2$,

$$B_m(x)\quad=\quad m(1-x)x^{2m-1}{}_2F_1\left(\begin{matrix}1-m,-m\\2\end{matrix};(\frac{1-x}{x})^2\right)+$$

$$x^{2m}{}_2F_1\left(\begin{matrix}-m,-m\\1\end{matrix};(\frac{1-x}{x})^2\right).$$

In order to arrive at the right hand side of (1) one only has to apply (3) to the first summand series.

Remark: We want to note that Krattenthaler's Mathematica package `hyp.m` [3] was used to come up with this proof.

References

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