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Some Questions Concerning Computer-Generated Proofs of a Binomial Double-Sum Identity

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Along two different proofs of a double-sum identity involving binomial coefficients this paper raises some questions of general character concerning computer-assisted treatment of the given double sum and of identities of similar type.

1. Introduction

In a lecture on his holonomic systems approach to special function identities at the 24^{th} Séminaire Lotharingien de Combinatoire (Liebfrauenberg, May 1990) D. Zeilberger asked for a "computer-generated" proof of the following identity:

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{j}^2\binom{4n-2i-2j}{2n-2j}=(2n+1)\binom{2n}{n}^2 \tag{1.1}$$

for all nonnegative integers n (Problem E3376 of *Amer. Math. Monthly* **97***, March 1990*, proposed by R. J. Blodgelt). Up to now not too much is known concerning structure and symbolic manipulation of binomial *multiple*-sum identities. Thus the interest in a "computer-generated" proof of (1.1) or, more general, of identities of similar type basically arises from the question whether similar algorithmic tools as those recently developed in the frame of Zeilberger's approach could be applied. If the answer is positive it is to expect that these methods will stimulate and assist further thorough investigations in this area.

In a more adequate frame that is closer to canonical form representation, the problem can be viewed as one concerning manipulation of *hypergeometric* multiple-sum identities. The great relevance of hypergeometric series to binomial coefficient identities was first pointed out by G. Andrews (1974), and by R. Askey. For further references see also Hayden and Lamagna (1986), Roy (1987), Graham, Knuth and Patashnik (1989), or Koornwinder (1991). Consequently, from hypergeometric theory point-of-view the *single*-sum case can be considered as well-studied, see for instance the books by Bailey (1935), Slater (1966), or Gasper and Rahman (1990). With respect to *algorithmic* treatment, recently a break-through has been achieved by D. Zeilberger in the frame of the holonomic systems approach (1990a). His "fast algorithm" (1990b), which is based on Gosper's

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summation algorithm (see Gosper (1978) or Graham, Knuth and Patashnik (1989)), provides an excellent algorithmic tool for finding and proving binomial single-sum identities (see also Zeilberger (1991a), (1991a), and the joint paper with H. Wilf (1990)). For instance, it succeeds in proving almost all identities listed by H. Gould (1972). Its wide range of applicability also is documented by manifold and interesting new applications it has found up to now (see for instance the corresp. references above, or, as another striking example, the application for enumerating totally symmetric, self-complementary plane partitions by G. Andrews(1991)).

Sometimes a multiple sum can be reduced to a single sum by applying some single-sum techniques iteratively, see for example eq. (3.1) in section 3. But it should be emphasized that this is not possible in general, at least not in an obvious way. The reason simply might be that no inner sum is representable in closed form, which, for instance, applies to the inner sum of the left hand side of (1.1). This e.g. is immediate by looking at the corresponding linear recurrences, both being of order 3 instead of order 1 in case of hypergeometric term evaluation. The recurrences, in i or n respectively, can be obtained automatically by applying Zeilberger's algorithm. For example, that one in i is obtained by the Zeilberger-package by P. Paule and M. Schorn (1993), written in Mathematica, as follows. (The package is available via email from *Peter.Paule@risc.uni-linz.ac.at.*) Let

$$SUM[i] = \sum_{j=0}^{n} \binom{i+j}{j}^2 \binom{4n-2i-2j}{2n-2j}, \tag{1.2}$$

then:

```
In[1]:= Zb[Binomial[i+j,j]^2 * Binomial[4n-2i-2j,2n-2j],j,i,3]

Out[1]= {-((1 + i) (-i + n) (-1 - 2 i + 2 n) SUM[i]) + (18 +

                2      3                       2         2        2
> 32 i + 22 i  + 6 i  + 11 n - 4 i n - 8 i  n + 30 n  + 20 i n )

                                            2                       2
> SUM[1 + i] - (2 + i) (27 + 23 i + 6 i  + 9 n - 4 i n + 18 n )

                                   2
> SUM[2 + i] + 2 (2 + i) (3 + i)  SUM[3 + i] == 0}
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In section 2 we present two different proofs of (1.1), both involving the application of computer algebra. Despite this fact both proofs still rely on ideas of a human as essential ingredients. Along a discussion of the concrete example (1.1), in section 3 we conclude by introducing some questions of general character concerning computer-assisted treatment of multiple sums to researchers interested in algorithmic problems in connection with symbolic manipulation of combinatorial formulae. One may expect that appropriate answers will provide substantial additional insight concerning the new developments in this subarea of symbolic computation initiated recently by D. Zeilberger.

2. The Proofs

We present two different proofs of (1.1). The human part of the first proof consists in using a somehow tricky generating function argument. The second proof is by observing that a generalization of the left hand side of (1.1) satisfies a simple structured recurrence relation. Here the part of the human consists in clever guessing on base of the data delivered by the computer. We want to remark that hypergeometric methods (see the references above) seemingly do not succeed applied in various standard ways.

Equipped with the Zeilberger-package by Paule and Schorn (1993), or that one written by Zeilberger (1991b), one has to do a - so far human - preprocessing step. One rewrites the double sum as a single sum, i.e. in a form, which is ready to serve as input for the proving-procedure. The left side of (1.1) is equal to the coefficient of $x^n y^n$ in

$$(1+x)^{2n}(1+y)^{2n}(1+xy)\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{i+j}{j}^2\frac{x^{i+k}y^{j+k}}{(1+x)^{i+j}(1+y)^{i+j}},$$

which is immediate from the binomial theorem and the Vandermonde-identity applied in the form

$$\binom{4n-2i-2j}{2n-2j} = \sum_{k=-n}^{n}\binom{2n-i-j}{n-i-k}\binom{2n-i-j}{n-j-k}$$
$$= \sum_{k=0}^{n}\{\binom{2n-i-j}{n-i-k}\binom{2n-i-j}{n-j-k} + \binom{2n-i-j}{n-1-i-k}\binom{2n-i-j}{n-1-j-k}\}.$$

Considering the triple sum without upper summation bounds, for any of the parameters i, j, k being greater than n, no contribution to the coefficient of $x^n y^n$ arises. Thus all sums can be taken as infinite ones. It is well known that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{i+j}{j}^2 u^i v^j = ((1-u-v)^2-4uv)^{-1/2}$$

(e.g. Riordan (1968), Sec. 4.4). Hence the left side of (1.1) turns out to be the coefficient of $x^n y^n$ in

$$\frac{(1+x)^{2n+1}(1+y)^{2n+1}(1+xy)}{(1-xy)^2},$$

which reads after expansion according to the binomial theorem as

$$\sum_{k=0}^{n}(2k+1)\binom{2n+1}{n-k}^2 = l(n). \tag{2.1}$$

Now, arriving at a *single* sum, the rest is done by Zeilberger's algorithm. After 3.25 CPU seconds (on an Apollo DN4500) one obtains with his Maple-program (1991b) (or with the Zeilberger-package by Paule and Schorn (1993) written in Mathematica) that $(-2n-1)F(n,k) = G(n,k) - G(n,k-1)$, with input $F(n,k) = (2k+1)\binom{2n+1}{n-k}^2$, and where $G(n,k) = \frac{(n-k)^2}{(2k+1)}F(n,k)$. It follows that

$$(-2n-1)l(n) \quad = \quad \sum_{k=0}^{n}(-2n-1)F(n,k) = G(n,n+1) - G(n,-1)$$

$$= \quad -(n+1)^2 \binom{2n+1}{n+1}^2,$$

which proves the identity in question. One should note that what is needed from Zeilberger's machinery in this special application is just the part played by Gosper's algorithm as observed by V. Strehl (1990).

Remark: The part of the computer in this proof certainly is simpler than the human one: identity (2.1) can be proved by any of the standard methods, either hypergeometric (see the references above) or those described in the books by Graham, Knuth and Patashnik (1989), or Wilf (1990).

The second proof is by proving more generally

$$\sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{i+j}{j}^2 \binom{m+n-2i-2j}{n-2j} = \frac{\left(\lfloor \frac{m+n+1}{2} \rfloor\right)! \left(\lfloor \frac{m+n+2}{2} \rfloor\right)!}{\left(\lfloor \frac{m}{2} \rfloor\right)! \left(\lfloor \frac{m+1}{2} \rfloor\right)! \left(\lfloor \frac{n}{2} \rfloor\right)! \left(\lfloor \frac{n+1}{2} \rfloor\right)!}, \tag{2.2}$$

where $\lfloor x \rfloor$ is the largest integer $\leq x$. Let's denote the left hand side as $L(m,n)$. This generalization of the double sum identity above can be obtained by an heuristic process using any computer algebra package. In this two-parameter form the resemblance to a certain representation of the Brock-Numbers (e.g. Riordan (1968), p. 145) is apparent. Now the proof follows immediately from the fact that each side satisfies the following initial conditions and "Brock-like" recurrence:

$$f(0,n) = f(n,0) = \lfloor n/2 \rfloor + 1, \tag{2.3}$$

$$f(m,n) - f(m-1,n) - f(m,n-1) = \begin{cases} \binom{\frac{m+n}{2}}{\frac{m}{2}}^2 & \text{if } m \text{ and } n \text{ are even} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

To prove that $L(m,n)$ satisfies (2.4) is easily done by applying $\binom{a+b}{a} = \binom{a+b-1}{a} + \binom{a+b-1}{a-1}$ and collecting together those terms which pairwise add up to zero. When both m and n are even the term corresponding to $(i,j) = (m/2, n/2)$ is the only one left. All other verifications are simple college algebra exercises.

Remark: We want to note that the technique applied in the first proof can be easily adapted to prove also the generalized double-sum evaluation. For that the reduction step to a single sum is done again by the Vandermonde-identity, now in the form:

$$\binom{m+n-2i-2j}{n-2j} = \sum_{k \geq 0} \Big\{ \binom{\lfloor \frac{m+n}{2} \rfloor - i - j}{\lfloor \frac{m}{2} \rfloor - i - k} \binom{\lfloor \frac{m+n+1}{2} \rfloor - i - j}{\lfloor \frac{n}{2} \rfloor - j - k} +$$
$$\binom{\lfloor \frac{m+n+1}{2} \rfloor - i - j}{\lfloor \frac{m+n+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor - 1 - i - k} \binom{\lfloor \frac{m+n}{2} \rfloor - i - j}{\lfloor \frac{m+n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor - 1 - j - k} \Big\}.$$

This equation holds in all cases except m even and n odd. But this is all one needs, because the evaluation of $L(m,n)$ for arbitrary m, n follows from the symmetry $L(m,n) = L(n,m)$.

3. Discussion and Open Questions

Several possible ways to come up with a computer-generated proof of a multiple-sum evaluation were opened by D. Zeilberger's "holonomic systems approach" (1990a). In the single-sum case, due to the possible use of Gosper's summation algorithm, the corresponding procedures work very efficiently. In the multiple-sum case, since nothing like

Gosper's algorithm is available, one has to introduce for instance some elimination procedure. Zeilberger (1990a) described a method ("Sylvester's dialytic elimination") which extends to the multiple-sum case. As indicated in the same paper there is another possibility, i.e. to do the elimination by using non-commutative Groebner bases methods. Zeilberger claimed that this method is far superior to "Sylvester's dialytic elimination". It is true that both methods can be applied successfully, for example in the case of a simpler double-sum evaluation problem:

$$\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}(-1)^{k_1+k_2}\binom{n_1}{k_1}\binom{n_2}{k_2}\binom{k_1+k_2}{k_1}=\delta(n_1,n_2), \tag{3.1}$$

where $\delta(n_1,n_2)$ is defined to be 1, for nonnegative integers $n_1=n_2$, and to be 0, otherwise.

Remark: This identity reflects the orthogonality relations for the case $\alpha=0$ of the Laguerre polynomials

$$\mathcal{L}_n{}^{(\alpha)}(x)=\sum_k\binom{n+\alpha}{n-k}\frac{n!}{k!}(-x)^k$$

with respect to the inner product

$$<p_1,p_2>=L\frac{1}{(1-D)^{\alpha+1}}p_1(x)p_2(x),$$

where L is the evaluation at the origin $(Lp)(x)=p(0)$, and D the differentiation operator.

We want to point out that we have chosen the double-sum evaluation (3.1) only for illustrating the *elimination* problem according an elementary instance. As already mentioned in the introduction chapter, it is not a *generic* example, because it can be solved by iterating single-sum techniques. For instance, applying the Zeilberger package by Paule and Schorn (1993) to

$$SUM[n_2]=\sum_{k_2=0}^{n_2}(-1)^{k_2}\binom{n_2}{k_2}\binom{k_1+k_2}{k_1} \tag{3.2}$$

one gets back a first-order linear recurrence:

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In[2]:= Zb[(-1)^k2 * Binomial[n2,k2] * Binomial[k1+k2,k1],k2,n2,1]

Out[2]= {(-k1 + n2) SUM[n2] + (-1 - n2) SUM[1 + n2] == 0}
```

Consequently, the inner sum (3.2) of the left side of (3.1) has closed form representation $\binom{n_2-k_1-1}{n_2}$. This reduces the original double sum to

$$S(n_1,n_2)=\sum_{k_1=0}^{n_1}(-1)^{k_1}\binom{n_1}{k_1}\binom{n_2-k_1-1}{n_2}. \tag{3.3}$$

By running Zeilberger's algorithm again, one obtains the recurrence

$$(n_1-n_2)S(n_1,n_2)=0, \tag{3.4}$$

i.e. $S(n_1,n_2)=0$ if $n_1\neq n_2$. Finally, if $n_1=n_2$, Zeilberger's algorithm delivers the recurrence

$$3S(n_1,n_1)-3S(n_1+1,n_1+1)=0 \tag{3.5}$$

whose solution $S(n_1, n_1) = 1$ is determined uniquely by the boundary value $S(0, 0) = 1$.

Applying *elimination* techniques for proving (3.1) leads to the following observations. As shown by Paule (1990), here "Sylvester's dialytic elimination" is equivalent to the computation of the determinant of a 10 by 10 matrix in order to derive the desired annihilating operator. Concerning the use of non-commutative Groebner bases methods, Apel (1990) (see also Apel and Lassner (1988)) succeeded to obtain the same operator by using a program written for the algorithmic treatment of more general types of non-commutative algebras. For further references see e.g. Galligo (1985), Mora (1986), Kandri-Rody and Weispfenning (1990), or Takayama (1992).

But in the case of the double sum (2.2) in question the situation seems to be somewhat different, despite the fact that $L(m, n)$ is solution of a simple structured difference equation (see (2.3), (2.4)). Here, using "Sylvester's dialytic elimination" leads to a tremendous increase in the size of the corresponding square-matrix, a fact which suggests that this method is of almost no practical use in multiple-sum problems of more complicated structure. But also the application of non-commutative Groebner bases methods seems to meet several serious difficulties (Apel (1990)). From these observations the following questions arise:

> Is it possible to evaluate $L(m, n)$ (or $L(2n, 2n)$, resp.) by following Zeilberger's "holonomic systems approach" using non-commutative Groebner bases methods?
>
> Is it possible to evaluate $L(m, n)$ (or $L(2n, 2n)$, resp.) by following Zeilberger's "holonomic systems approach" using any elimination method?

Finally we raise an additional question, which is motivated by the strategy of the first proof presented above:

> Is it possible to provide any algorithmic device for reducing multiple sums to single sums?

Only recently, Wilf and Zeilberger (1992) showed that every 'proper-hypergeometric' multisum/integral identity, or q-identity, with a fixed number of summations and/or integration signs, indeed possesses a computer-constructable proof. Despite proceeding along the lines of the 'holonomic' paper of Zeilberger (1990a), the method, by which the authors succeed to prove a variety of interesting examples, does not involve elimination in the operator algebra.

In connection with the third question, Paule (1992) found a way to apply Zeilberger's "fast algorithm" not only for proving (1.1), but also for *finding* the closed form evaluation starting out given only the double sum on the left hand side.

We also want to point out that meanwhile sketches of our proofs of the Monthly Problem (1.1) appeared, see Problem E3376 Solution, *Amer. Math. Monthly* **99** (1992), 63-65. There, a number of additional issues are referred to, also indicating that it is a problem of wider interest.

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