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A Remark on a Lemma of Ingleton and Piff and the Construction of Bijections

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Abstract. It is shown that the Linkage Lemma of Ingleton and Piff [5; Lemma 3.1] serves as an useful tool for the construction of combinatorial bijections. For example, one of its direct applications yields the Garsia-Milne Involution Principle [3].

INTRODUCTION

In their paper on gammoids and transversal matroids Ingleton and Piff made extensive use of their Linkage Lemma [5; Lemma 3.1]. To mention one graph theoretic example, they established a duality between the graph theorems of Menger and König. In Welsh [7] a simple proof of Menger's theorem can be found. This proof just needs the Linkage Lemma and Hall's theorem.

Apart from these various applications of the Linkage Lemma, the goal of the present note is to point out an algorithmic aspect of its applicability: its importance as an useful tool for the construction of combinatorial bijections.

As a paradigmatic example we shall show that it implies in a straightforward manner the Involution Principle of Garsia and Milne. This also sheds some additional light on the mechanism of this principle. In [3] Garsia and Milne ingeniously used this principle to give the first bijective proof of the famous Rogers-Ramanujan identities. For further applications of their powerful technique see e.g. Bressoud-Zeilberger [1],[2] or Remmel [6].

Finally we shall also present Gordon's Complementary Bijection Principle [4] as a direct consequence of the Linkage Lemma.

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THE LINKAGE LEMMA AND THE CONSTRUCTION OF BIJECTIONS

Let $G = (V, E)$ denote a graph with vertex set V and edge set E (loops and multiple edges are allowed). In this note all sets are supposed to be finite.

A path from one vertex v_0 of G to another vertex v_k of G is a sequence $P = [v_0, v_1, \dots, v_k]$ of distinct vertices of G such that $k \geq 1$ and $\{v_{i-1}, v_i\}$ in E , for each $i \in \{1, 2, \dots, k\}$.

For each $v \in V$ we denote by $[v]$ the trivial path from v to v . Two paths are disjoint if their vertex sets are disjoint. Two subsets X, Y of V are said to be linked in G , if for some bijection $\phi: X \rightarrow Y$ we can find disjoint paths $\{P_x \mid x \in X\}$ such that P_x is a path from x to $\phi(x)$.

For $Z \subseteq V$ we define $Z^* = \{v \in V \mid \{z, v\} \in E \text{ for some } z \in Z\} \cup Z$ and $st(Z) = \{\{z\}^* \mid z \in Z\}$.

Now we are ready to state the Linkage Lemma due to Ingleton and Piff [5; Lemma 3.1]. We present it in an undirected version (cf. Welsh [7; Chapt. 13]):

THE LINKAGE LEMMA.

For a graph $G = (V, E)$ and subsets X, Y of V the following statements are equivalent:

- (i) X and Y are linked in G .
- (ii) $V \setminus X$ is a transversal of $st(V \setminus Y)$.
- (iii) $V \setminus Y$ is a transversal of $st(V \setminus X)$.

Proof. The equivalence of (ii) and (iii) is trivial.

To conclude (ii) from (i), according to Hall's theorem (cf. e.g. Welsh [7]) we just have to check for each $Z \subseteq V \setminus Y: |Z^* \setminus X| \geq |Z|$.

Suppose, X and Y are linked by the disjoint paths $\{P_x \mid x \in X\}$. Trivially, $x \in Z \setminus X$ implies $x \in Z^* \setminus X$. On the other hand, following the path P_x for each $x \in X \cap Z$ we meet a vertex $x(Z) \in P_x$ such that $x(Z) \in Z^* \setminus (X \cup Z)$. All these $\{x(Z) \mid x \in X\}$ are distinct, so Hall's condition is satisfied.

To show (i) from (ii), the desired linkage is constructed by the following algorithm. Condition (ii) is equivalent to the existence of a bijection $\varphi: V \setminus Y \rightarrow V \setminus X$ such that $\varphi(v) \in \{v\}^*$ for all $v \in V \setminus Y$. Now we have to link X and Y by disjoint paths $\{P_x \mid x \in X\}$. For all $x \in X \cap Y$ we define $P_x = [x]$. Suppose $x \in X \setminus Y$, then if $\varphi(x) \in Y$ define $P_x = [x, \varphi(x)]$, else

check $\varphi^2(x)$. If $\varphi^2(x) \in Y$ define $P_x = [x, \varphi(x), \varphi^2(x)]$, else check $\varphi^3(x)$ a.s.o.. This procedure will terminate with $\varphi^s(x) \in Y$ for some $s = s(x) \geq 1$. For if not, we would obtain an infinite sequence $[x, \varphi(x), \varphi^2(x), \dots]$ of distinct vertices ($\varphi^i(x) = \varphi^j(x)$ where $i < j$ implies $x = \varphi^{j-i}(x) \in V \setminus X$ contrary to $x \in X \setminus Y$), a contradiction to the finiteness of $V \setminus (X \cup Y)$. Clearly, all paths $\{P_x \mid x \in X\}$ constructed this way are disjoint and yield the desired linking of X and Y .

REMARK.

(a) The step "(i) \Rightarrow (ii)" can be done without Hall's theorem by the following construction (cf. Welsh [7]):

Define the function $\varphi: V \setminus Y \rightarrow V \setminus X$ by

$$\varphi(v) = \begin{cases} v', & \text{if } v' \text{ is the right hand neighbour of } v \text{ in } P_x \text{ for some } x \in X \\ v, & \text{else } \varphi. \end{cases}$$

It is easy to see that φ is well defined and injective, since the paths P_x, x in X are disjoint. In addition we have, for all $v \in V \setminus Y$, that $\varphi(v) \in \{v\}^*$, thus $V \setminus X$ is a transversal of $st(V \setminus Y)$.

(b) The Linkage Lemma can be extended to infinite X and Y . In that case we have to replace the finiteness condition on V by providing $V \setminus (X \cup Y)$ to be finite. It is easy to see that the Linkage Lemma still remains valid.

The construction of the disjoint linking paths $\{P_x \mid x \in X\}$ of X and Y in the proof yields the following corollary as a by-product:

COROLLARY.

Let $G = (V, E)$ be a graph, $X, Y \subseteq V$ and $\varphi: V \setminus Y \rightarrow V \setminus X$ a bijection such that $\varphi(v) \in \{v\}^*$ for all $v \in V \setminus Y$. Then the function

$$\phi: X \rightarrow Y; x \mapsto \varphi^{s(x)}(x)$$

is a bijection, where $s(x)$ is the smallest non-negative integer such that $\varphi^{s(x)}(x) \in Y$.

In this form the Linkage Lemma turns out to be a welcome advice for the construction of bijections. The technique lies in defining an appropriate graph together with a suitable bijection φ possessing the transversal property.

So the Linkage Lemma of Ingleton and Piff may well serve as an unifying concept of constructing bijections. We give some examples to illustrate that.

First we will show that the setting of Garsia and Milne's Involution Principle fits perfectly into the frame of the Linkage Lemma.

THE GARSIA-MILNE INVOLUTION PRINCIPLE

Let $C = C^+ \cup C^-$ be the disjoint union of finite sets C^+ and C^- . Let α and β be the involutions on C with fixed point sets C_α and C_β , respectively, such that $C_\alpha, C_\beta \subseteq C^+$ and $\gamma(C^+ \setminus C_\gamma) = C^-$ for each $\gamma \in \{\alpha, \beta\}$; i.e. α and β are sign-reversing outside their fixed point sets. Clearly $|C_\alpha| = |C_\beta|$, and we meet the problem to construct a bijection $\phi: C_\alpha \rightarrow C_\beta$ out of α and β . This is immediately solved by our corollary:

We define a suitable graph $G = (V, E)$ and $X, Y \subseteq V$ by setting $V = C^+, E = \{\{v, v'\} \subseteq C^+ \mid v' = \alpha\beta(v)\}$ and $X = C_\alpha, Y = C_\beta$. The function $\varphi = \alpha\beta: C^+ \setminus C_\beta \rightarrow C^+ \setminus C_\alpha$ is a bijection such that $\varphi(v) \in \{v\}^*$ for all $v \in C^+ \setminus C_\beta$. Thus the corollary gives the desired bijection by

$$\phi: C_\alpha \rightarrow C_\beta; v \mapsto (\alpha\beta)^{s(v)}(v),$$

where $s(v)$ is the smallest non-negative integer such that $(\alpha\beta)^{s(v)}(v) \in C_\beta$.

REMARK.

The more general situation (cf. Remmel [6]) of two finite disjoint set partitions $A = A^+ \cup A^-, B = B^+ \cup B^-$ together with a sign-preserving bijection $f: A \rightarrow B$ (i.e. $f(A^+) = B^+, f(A^-) = B^-$) and corresponding bijections α on A, β on B , which are sign-reversing outside their fixed point sets $A_\alpha \subseteq A^+$ and $B_\alpha \subseteq B^+$, respectively, can be easily played back to the situation above (which remains valid if we replace "involution" by "bijection"):

Define $\beta' = f^{-1}\beta f: A \rightarrow A$, which is a bijection. Clearly, the fixed point set of β' is $A'_\beta = f^{-1}(B_\beta) \subseteq A^+$ and β' again is sign-reversing outside A'_β . Hence the corollary yields the bijection $\phi': A_\alpha \rightarrow A'_\beta, v \mapsto (\alpha\beta')^{s(v)}(v)$. Composing ϕ' and f we obtain $\phi = f \circ \phi': A_\alpha \rightarrow B_\beta$ as the desired bijection.

Finally we would like to point out that Gordon's Complementary Bijection Principle [4], which is essentially a reformulation of the Garsia-Milne Involution Principle without using involutions or signed elements, again can be derived directly from the Linkage Lemma:

THE COMPLEMENTARY BIJECTION PRINCIPLE OF GORDON

Given finite sets $A, B, A_1 \subseteq A, B_1 \subseteq B$ and two bijections $f: A \rightarrow B$, and $g: A \setminus A_1 \rightarrow B \setminus B_1$, the problem is to construct a bijection $h: A_1 \rightarrow B_1$ out of f and g .

To execute that by our corollary, the appropriate graph $G = (V, E)$ and the corresponding φ again are easy to find.

We set $V = A, E = \{\{v, v'\} \subseteq A \mid v' = g^{-1}f(v)\}, X = A_1$ and $Y = f^{-1}(B_1)$. Clearly, the bijection $\varphi = g^{-1}f: A \setminus f^{-1}(B_1) \rightarrow A \setminus A_1$ has the desired transversal property, i.e. $\varphi(v) \in \{v\}^*$ for each $v \in A \setminus f^{-1}(B_1)$. The corollary now yields the bijection $\phi: A_1 \rightarrow f^{-1}(B_1), v \mapsto (g^{-1}f)^{s(v)}(v)$, where $s(v)$ is the smallest non-negative integer such that $(g^{-1}f)^{s(v)}(v) \in f^{-1}(B_1)$. Hence $h = f \circ \phi$ is the desired bijection, which corresponds exactly to that one given by Gordon in [4].

REFERENCES

- [1] D.M. Bressoud and D. Zeilberger, A short Rogers- Ramanujan bijection, *Discrete Math.*, 38 (1982), 313-315.
- [2] D.M. Bressoud and D. Zeilberger, A bijection proof of Euler's recurrence for the number of partitions of an integer, *Amer. Math. Monthly*, 92 (1985), 54-55.
- [3] A.M. Garsia and S.C. Milne, A Rogers-Ramanujan bijection, *J. Comb. The.*, Ser. A, 31 (1981), 289-339.
- [4] B. Gordon, Sieve equivalence and explicit bijections, *J. Comb. Th.*, Ser. A, 34 (1983), 90-93.
- [5] A.W. Ingleton and M.J. Piff, Gammoids and transversal matroids, *J. Comb. Th.*, Ser. B, 15 (1973), 51-68.
- [6] J. Remmel, Bijective proofs of some classical partition identities, *J. Comb. The.*, Ser. A, 33 (1982), 273-286.
- [7] D.J.A. Welsh, "Matroid Theory", Academic Press, London, 1976.

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