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Note

A Note on Bailey's Lemma

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Combining the q-binomial theorem in the version of J. Cigler (Monatsh. Math. **88**, 87–105 (1979)) with the ε-technique of L. J. Rogers (G. E. Andrews, *Math. Chronicle*, **11**, 1–15, 1982) a short operator proof of a significant special case of Bailey's Lemma is given. © 1987 Academic Press, Inc.

1. Introduction

We consider the following highly instructive special case of Bailey's Lemma

$$\sum_{k=0}^{n} \frac{a_k x^k}{(qx)_{n+k}(q)_{n-k}} = \sum_{j=0}^{n} \frac{q^{j^2} x^j}{(q)_{n-j}} \sum_{k=0}^{j} \frac{a_k q^{-k^2}}{(qx)_{j+k}(q)_{j-k}}, \tag{1}$$

using the standard notation [1]: $(a)_m = (a;q)_m = (1-a)(1-qa)\cdots(1-q^{m-1}a)$, $(a)_\infty = (a;q)_\infty = (1-a)(1-qa)(1-q^2a)\cdots = \lim_{m\to\infty}(a)_m$ and for an integer n: $(a)_n = (a;q)_n = (a)_\infty/(q^n a)_\infty$, where q is a real number with $q \neq 0$ and $|q| < 1$.

To obtain (1) from Bailey's Lemma [4; (3.1)] take $\rho_1 = q^{-s}$, $\rho_2 = q^{-t}$, simplify and send t to infinity. Substituting $\alpha_k = q^{-k^2} a_k$ yields (1).

Besides its special importance within the scope of the theory of basic hypergeometric functions transform (1) is extremely useful in handling identities of the Rogers–Ramanujan type.

Namely, the power of (1) lies in the fact that the second sum of the right-hand side of (1) is of the same form as the sum on its left-hand side. Thus we may iterate (1) substituting the whole formula (modified by taking $a_k q^{-k^2}$ instead of $a_k x^k$) in the place of the second sum of the right-hand

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side as often as we want, in order to reduce the initial sum on the left to a well-known one, as in many cases to one of the finite forms of the q-binomial theorem, e.g.,

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k x^k q^{(1/2)k^2}}{(q)_{j+k}(q)_{j-k}} = \frac{(x^{-1}q^{1/2})_j(xq^{1/2})_j}{(q)_{2j}}. \tag{2}$$

Many applications of this principle can be found in [7], where the following representations of (1)—notice that all sums are finite—

$$\sum_{k=-\infty}^{\infty} \frac{c_k}{(q)_{n+k}(q)_{n-k}} = \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_{n-j}} \sum_{k=-\infty}^{\infty} \frac{c_k q^{-k^2}}{(q)_{j+k}(q)_{j-k}} \tag{3}$$

($x=1$, $a_0=c_0$ and $a_k=c_k+c_{-k}$ for $k \geqslant 1$ in (1); the special case $c_k = x^k q^{ck^2}$ was first stated by Bressoud in [5]) and

$$\sum_{k=-\infty}^{\infty} \frac{c_k}{(q)_{n+k}(q)_{n+1-k}} = \sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q)_{n-j}} \sum_{k=-\infty}^{\infty} \frac{c_k q^{-k^2+k}}{(q)_{j+k}(q)_{j+1-k}} \tag{4}$$

($x=q$, $a_k=(q^{-k}/1-q)(c_{k+1}+c_{-k})$ for $k \geqslant 0$ in (1)) are used, in combination with their limiting forms $n \to \infty$, to give simple "iteration-proofs" of partition identities like the Rogers–Ramanujan identities, the Rogers–Selberg identities (mod 7), the Göllnitz–Gordon identities (mod 8) and some of their multiple-series generalizations (cf. [1]).

Remark. In [3] G. Andrews has brought this idea to its full generality, using the form of Bailey's Lemma from Section 4 of [4] and introducing the notion of Bailey chains.

In the following section we present a new proof of transform (1), in order to give some insight into its structure from the operator-point of view.

2. Proof of (1)

Let R denote the set of all power series in the variable x over the reals. On R we define the following linear operators: the multiplication operator $(\mathbf{x}f)(x)=xf(x)$, the ε-operator $(\varepsilon f)(x)=f(qx)$ and its inverse $(\varepsilon^{-1}f)(x)=f(q^{-1}x)$. The following properties are easy to check:

$$\varepsilon(fg)=(\varepsilon f)(\varepsilon g), \tag{5}$$

$$(\varepsilon^{-1}+\mathbf{x})(qx)_\infty=(qx)_\infty \tag{6}$$

and

$$\mathbf{x}\varepsilon^{-1}f(x)=q\varepsilon^{-1}\mathbf{x}f(x) \tag{7}$$

for all $f(x), g(x) \in R$.

We shall use the following version of the q-binomial theorem (cf., Cigler [6]): For linear operators A, B on R with $BA=qAB$ the following formula holds ($n=0, 1, 2,\ldots$)

$$(A+B)^n=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} A^k B^{n-k}, \tag{8}$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$, the Gaussian polynomial, is defined to be zero for $k<0$ or $k>n$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]=(q)_n/(q)_k(q)_{n-k}$ for $0\leqslant k\leqslant n$.

Its proof is an easy induction exercise using the recurrence formula $\left[\begin{smallmatrix} n+1 \\ k \end{smallmatrix}\right]=q^k\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]+\left[\begin{smallmatrix} n \\ k-1 \end{smallmatrix}\right]$.

Now we are ready to prove Bailey's transform (1): Using the ε-operator the left-hand side of (1) can be rewritten as

$$\sum_{k=0}^{n}\frac{a_k x^k}{(qx)_{n+k}(q)_{n-k}}=\frac{1}{(qx)_\infty}\left(\sum_{k=0}^{n}\frac{a_k x^k}{(q)_{n-k}}\varepsilon^{n+k}(qx)_\infty\right)$$

$$=\frac{1}{(qx)_\infty}\left(\sum_{k=0}^{n}\frac{a_k x^k}{(q)_{n-k}}\varepsilon^{n+k}(\varepsilon^{-1}+\mathbf{x})^{n-k}(qx)_\infty\right)$$

(by the fixpoint-property (6))

$$=\frac{1}{(qx)_\infty}\left(\sum_{k=0}^{n}\frac{a_k x^k}{(q)_{n-k}}\varepsilon^{n+k}\sum_{j=0}^{n-k}\begin{bmatrix} n-k \\ j \end{bmatrix}(\varepsilon^{-1})^{n-k-j}\mathbf{x}^j(qx)_\infty\right)$$

(using (7) together with the q-binomial theorem (8))

$$=\frac{1}{(qx)_\infty}\sum_{k=0}^{n}\frac{a_k x^k}{(q)_{n-k}}\sum_{j=0}^{n-k}\begin{bmatrix} n-k \\ j \end{bmatrix}q^{j(j+2k)}x^j(q^{j+2k+1}x)_\infty$$

(by (5))

$$=\sum_{k=0}^{n}\frac{a_k x^k}{(q)_{n-k}}\sum_{j=k}^{n}\begin{bmatrix} n-k \\ j-k \end{bmatrix}q^{j^2-k^2}\frac{x^{j-k}}{(qx)_{j+k}}$$

$$=\sum_{j=0}^{n}\frac{q^{j^2}x^j}{(q)_{n-j}}\sum_{k=0}^{j}\frac{a_k q^{-k^2}}{(qx)_{j+k}(q)_{j-k}},$$

as desired.

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