Wavelets with Scale Dependent Properties

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Abstract— **In this paper we revisit the constitutive equations for coefficients of orthonormal wavelets. We construct wavelets that satisfy alternatives to the vanishing moments conditions giving orthonormal basis functions with scale dependent properties.**

I. Introduction

In this paper we construct filter coefficients of real, compactly supported, orthonormal wavelets with scale dependent properties.

Daubechies' construction of wavelets is based on the existence a *scaling function* ϕ, such that for $m \in \mathbf{Z}$ the functions $\phi_{m,k} := 2^{-m/2}\phi(2^{-m}x - k),\ \ k \in \mathbf{Z}$, are orthonormal with respect to $L^2(\mathbf{R})$. Moreover, ϕ is chosen in such a way that

$$V_m := \overline{\mathrm{span}\{\ \phi_{m,k},\ k \in \mathbf{Z}\ \}}$$

form a multiresolution analysis on $L^2(\mathbf{R})$, i.e.,

$$V_m \subset V_{m-1},\ \text{ for } m \in \mathbf{Z}\,,$$

with

$$\bigcap_{m\in\mathbf{Z}} V_m = \{0\} \text{ and } \overline{\bigcup_{m\in\mathbf{Z}} V_m} = L^2(\mathbf{R}).$$

The wavelet spaces W_m are the orthogonal complements of V_m in V_{m-1}, i.e.,

$$W_m := V_m^{\perp} \cap V_{m-1}.$$

For the *mother wavelet* ψ a function ψ is chosen such that $\psi_{m,k} := 2^{-m/2}\psi(2^{-m}x - k),\ \ k \in \mathbf{Z}$, form an orthonormal basis of W_m. Since $\phi = \phi_{0,0} \in V_0 \subset V_{-1}$, the scaling function ϕ must satisfy the *dilation equation*

$$\phi(x) = \sum_{k\in\mathbf{Z}} h_k \phi(2x-k), \qquad (1)$$

where the sequence $\{h_k\}$ is known as the *filter sequence* of the wavelet ψ. The filter coefficients have to satisfy certain conditions in order to guarantee that the scaling function ϕ and ψ satisfy various desired properties. These properties of wavelets and scaling functions will be reviewed in Section II. Daubechies' construction principle guarantees that the properties imposed on the mother wavelet ψ carry over to any $\psi_{m,k}$.

However, in practical applications (like in digital signal and image processing) one actually utilizes only a finite number of scales m. In data compression with wavelets of medical image data we experienced that five scales are sufficient to achieve high compression ratios.

This experience stimulated our work to design wavelets that have additional properties on an a–priori prescribed number of scales. As we show in Section III, Daubechies' construction principle leaves enough freedom to design such wavelets. In Section IV we present some examples of scale dependent wavelets. Finally we put our work in relation to recent work on rational spline wavelets.

II. A review on Daubechies' wavelets

Following Daubechies (see [1], [2]) the construction of orthonormal wavelet functions is reduced to the design of the corresponding filter sequence $\{h_k\}$ in (1). Moreover, one assumes that the mother wavelet ψ satisfies

$$\psi(x) = \sum_{k\in\mathbf{Z}} (-1)^k h_{1-k}\phi(2x-k)\ . \qquad (2)$$

In particular this choice guarantees that the wavelet ψ and the scaling function ϕ are orthogonal.

In orthogonal wavelet theory due to Daubechies (see e.g. [1], [2]) the desired properties on the scaling function and wavelets are:

1. For fixed integer $N \geq 1$ the scaling function ϕ has support in the interval $[1-N, N]$. This in particular holds when the filter coefficients satisfy

$$h_k = 0\,, \qquad (k < 1-N \text{ or } k > N). \qquad (3)$$

2. The existence of a scaling function ϕ satisfying (1) requires that

$$\sum_{k\in\mathbf{Z}} h_k = 2\,. \tag{4}$$

3. In order that the integer translates of the scaling function ϕ are orthonormal, i.e., $\int_{\mathbf{R}} \phi(x-l)\phi(x)dx = \delta_{0,l}$, the filter coefficients $\{h_k\}$ have to satisfy

$$\sum_{k\in\mathbf{Z}} h_k h_{k-2l} = 2\delta_{0,l}, \qquad (l = 0,\ldots,N-1). \tag{5}$$

4. The wavelet ψ is postulated to have N vanishing moments, i.e.,

$$\int_{\mathbf{R}} x^l \psi(x)dx = 0, \text{ for } l = 0,\ldots,N-1 \tag{6}$$

which requires the filter sequence to satisfy

$$\sum_{k\in\mathbf{Z}} (-1)^k h_{1-k} k^l = 0, \qquad (l = 0,\ldots,N-1). \tag{7}$$

III. Wavelets with scale dependent properties

In this section we are particularity interested in constructing wavelets that satisfy alternatives to the vanishing moments condition (7). Our motivation is to have more flexibility in adapting wavelets to practical needs.

All along this paper we restrict our attention to filter coefficients that satisfy the following general conditions:
1. the wavelet and the scaling function are compactly supported,
2. the scaling function is orthogonal to its integer translates on every scale.
To satisfy these properties we assume that the filter coefficients satisfy (3)-(5).

To derive alternatives to the vanishing moments condition (7) we redirect our attention to the connection between (6) and (7).

To this end we consider families $\{s_j : j \in I\}$ of functions on $\mathbf{R}$, I being a suitable set of indices which are othogonal with all wavelets on a fixed scale m. In other words, we want that for all j in I and all integers k,

$$\int_{\mathbf{R}} s_j(x)\psi_{m,k}dx = 0. \tag{8}$$

In order to achieve this goal we assume for all j in I the existence of functions $\{t_{j,k} : k \in I\}$ on $\mathbf{R}$ such that

$$s_j(x+y) = \sum_{k\in I} s_k(x) t_{j,k}(y). \tag{9}$$

The following computation will show that property (9) together with the conditions

$$\sum_{k\in\mathbf{Z}} (-1)^k h_{1-k}\, s_j(2^{m-1}k) = 0, \qquad (\text{for all } j \in I) \tag{10}$$

are sufficient to guarantee the orthogonality (8).

Namely, the left side of (8) equals

$$2^{-m/2}\int_{\mathbf{R}} s_j(x)\psi(2^{-m}x-k)dx = 2^{(m-2)/2}$$
$$\times \quad \sum_{l\in\mathbf{Z}} (-1)^l h_{1-l} \int_{\mathbf{R}} \phi(t) s_j\left(2^{m-1}(t+2k+l)\right)dt,$$

where we applied (2) and then substituted $x \to 2^{m-1}(t+2k+l)$. Next, invoking (9) with $x = 2^{m-1}l$ and $y = 2^{m-1}(t+2k)$ results in

$$2^{(m-2)/2}\sum_{i\in I}\left(\sum_{l\in\mathbf{Z}}(-1)^l h_{1-l} s_i(2^{m-1}l)\right)$$
$$\times \quad \int_{\mathbf{R}} \phi(t)\, t_{j,i}\left(2^{m-1}(t+2k)\right)dt,$$

which is 0 according to (10).

We conclude this section by making some obvious choices for $\{s_j\}$ and $\{t_{j,k}\}$; corresponding examples for computations of filter coefficients $\{h_k\}$ are presented in the next section. Alternative choices that guarantee (9) will be discussed in a forthcoming paper.

1. Sheffer Relations. If we restrict ourselves to polynomial sequences, there is the classical theory of *Sheffer sequences* $\{s_j : j \geq 0\}$ satisfying the Sheffer identity

$$s_j(x+y) = \sum_{k=0}^{j}\binom{j}{k} s_k(x) p_{j-k}(y), \qquad (j \geq 0) \tag{11}$$

where $\{p_j : j \geq 0\}$ is called an *associated* sequence. For further information see, for instance, Roman's book [3] which is devoted to Rota's view of umbral calculus.

In our context (11) is obtained from (9) by choosing $t_{j,k}(x) = \binom{j}{k} p_{j-k}(x)$. It is important to note that in order to be able to compute the filter coefficients $\{h_k\}$, it is necesssary to restrict the index set I to a *finite* subset of $\mathbf{N}$, for instance, to $\{0,1,\ldots,N-1\}$.

For the particular choice

$$s_j(x) = x^j \quad \text{and} \quad t_{j,k}(x) = \binom{j}{k} s_{j-k}(x)$$

the relations (9) and (11) are nothing but the binomial theorem. Additionally, taking $I = \{0,1,\ldots,N-1\}$ the conditions (10) turn into the Daubechies situation of (7), i.e., orthogonality holds on any scale m.

Finally we remark that special types of Sheffer sequences, namely *Appell* sequences, appear in other recent work [4, Remark 7] on wavelets, but in a different context of analysing orthonormal systems of multiwavelets.

2. Exponential Relations. The corresponding setting in full generality is as follows. Let $\{\omega_j : j \in I\}$ be a sequence of complex parameters. For all j, k in I define

$$s_j(x) = q^{\omega_j x} \quad \text{and} \quad t_{j,k}(x) = \delta_{j,k}\, s_j(x),$$

with $\delta_{j,k}$ being the Kronecker function and q being a fixed nonzero complex number. Then (9) turns into

$$q^{\omega_j(x+y)} = q^{\omega_j x} q^{\omega_j y};$$

additionally, (10) becomes

$$\sum_{k\in\mathbf{Z}} (-1)^k h_{1-k}\, q^{2^{m-1}\omega_j k} = 0 \qquad (\text{for all } j \in I). \tag{12}$$

In the following examples section we examine two special cases over the reals: *q-wavelets* where for all j in I we set $\omega_j = 1$ and q to a fixed positive real number; and *sin-wavelets* where we set $q = \exp(i)$, the complex exponential, and where we split (12) into two groups of equations over the reals by taking the real *and* the imaginery part of

$$\exp(i\, 2^{m-1}\omega_j\, k) = \cos(2^{m-1}\omega_j\, k) + i\, \sin(2^{m-1}\omega_j\, k).$$

IV. Examples

In this section we present several examples of wavelets with scale dependent properties. The resulting systems for $\{h_k\}$ are considered as algebraic equations which can be solved by the combined symbolic/numerical approach described in [5].

A. q-wavelets

Here we derive wavelet filter coefficients of orthonormal, compactly supported wavelet functions $\psi_{m,k}$ that are orthogonal to $s_j(x) = q^x$ on scales $m = 0, -1, \ldots, -(N-2)$. Since the s_j are independent of j this means, we can set $I = \{0\}$. For $N > 1$ the equations for the wavelet filter coefficients are

$$\begin{aligned} \sum_{k\in\mathbf{Z}} h_k &= 2 \\ \sum_{k\in\mathbf{Z}} h_k h_{k-2l} &= 2\delta_{0,l}, && (l = 0, \ldots, N-1) \\ \sum_{k\in\mathbf{Z}} (-1)^k h_{1-k} q^{2^{-(\overline{m}+1)}k} &= 0, && (\overline{m} = 0, \ldots, N-2)\ . \end{aligned} \tag{13}$$

For $N = 2$ the solution $h = (h_{-1}, h_0, h_1, h_2)$ of this system is

$$\begin{aligned}(h_{-1}, h_0, h_1, h_2) = \\ \left(\tfrac{q^{3/2}-q-\sqrt{q}Q}{q(-1+q)}, \tfrac{q-1-Q}{-1+q}, \tfrac{q^2-q^{3/2}+\sqrt{q}Q}{q(-1+q)}, \tfrac{Q}{-1+q}\right)\end{aligned} \tag{14}$$

where

$$\begin{aligned} Q \equiv Q_\pm &= \tfrac{q^2+q-q^{3/2}-\sqrt{q}}{2(q+1)} \\ &\pm \tfrac{\sqrt{q^4-5\,q^3+2\,q^{7/2}+4\,q^{5/2}-5\,q^2+2\,q^{3/2}+q}}{2(q+1)}. \end{aligned}$$

In Figures 1 and 2 we have plotted the scaling functions and the wavelets according to the filter coefficients (14). From such plottings one can see that for $q \to 1$ the scaling function and the wavelet converge to the Daubechies scaling function and to the wavelet, respectively. The proof of this observation is immediate since applying $q\,D_q$, where D_q is derivation with respect to q, $N-1$ times to the last equation in (13) and then setting $q = 1$ results in (7).

Moreover, from (14) it follows that for $q \to 0$ and $q \to \infty$ the coefficients of the q-wavelets according to Q_+ approach the coefficients of the Haar wavelet, which confirms the plots in Figure 1 and Figure 2.

In [7, Section 3] all cofficients h_k, $k \in \mathbf{Z}$ satisfying (3) – (5) (with $N = 2$) are calculated. It can be shown that the family of solutions can be parametrized by a single real parameter. Thus the q-wavelets for $N = 2$ form a subset of the coefficients obtained by Daubechies.

B. Variations of Daubechies' wavelets

In this subsection we investigate wavelet filter coefficients of orthonormal, compactly supported wavelet functions $\psi_{m,k}$ that have $K \geq 1$ vanishing moments and which, in addition, are orthogonal to $s_j(x) = q^x$ on scales $m = 0, -1, \ldots, -(N-K-1)$. As in the previous section we can again set $I = \{0\}$ since s_j is independent of j. Thus the equations for the filter coefficients are

$$\begin{aligned} \sum_{k\in\mathbf{Z}} h_k &= 2, \\ \sum_{k\in\mathbf{Z}} h_k h_{k-2l} &= 2\delta_{0,l}, && (l = 0, \ldots, N-1), \\ \sum_{k\in\mathbf{Z}} (-1)^k h_{1-k} q^{2^{-(\overline{m}+1)}k} &= 0, && (\overline{m} = 0, \ldots, N-K-1), \\ \sum_{k\in\mathbf{Z}} (-1)^k h_{1-k} k^l &= 0, && (l = 0, \ldots, K-1)\ . \end{aligned} \tag{15}$$

In Figure 3 we have plotted the associated scaling function and wavelet for $q \in \{2, 4, 16, 32\}$, $K = 2$, and $N = 3$.

C. sin-wavelets

Wavelet filter coefficients of orthonormal, compactly supported wavelet functions $\psi_{m,k}$ that are orthogonal on scale $m = 0$ to $\sin(2^{m-1}\omega_j x)$ and $\cos(2^{m-1}\omega_j x)$ for $j \in I = \{0, \ldots, K-1\}$; here $\omega_j \neq 0$ is a sequence of real numbers.

Setting $N = 2K+1$ the equations for such filter coefficients are

$$\begin{aligned} \sum_{k\in\mathbf{Z}} h_k &= 2\,, \\ \sum_{k\in\mathbf{Z}} h_k h_{k-2l} &= 2\delta_{0,l}, && (l = 0, \ldots, N-1)\,, \\ \sum_{k\in\mathbf{Z}} (-1)^k h_{1-k} \sin(\omega_j k/2)&, && (j = 0, \ldots, K-1)\ , \\ \sum_{k\in\mathbf{Z}} (-1)^k h_{1-k} \cos(\omega_j k/2)&, && (j = 0, \ldots, K-1)\ . \end{aligned} \tag{16}$$

Note that the last two sets of equations correspond to taking real and imaginary parts as explained in Section III.

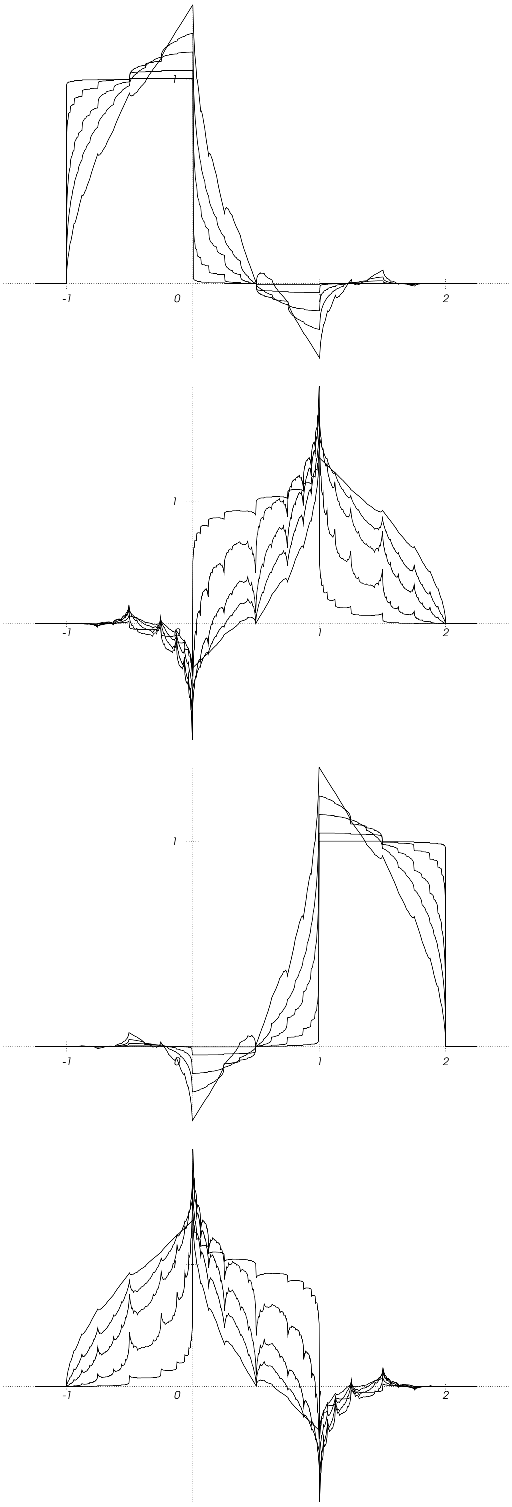

Fig. 1. q–scaling function for different values of q: $q \in (1, \infty)$ (rows 1 and 2), $q \in (0, 1)$ (rows 3 and 4); the two solutions for Q: Q_+ (rows 1 and 3), Q_- (rows 2 and 4).

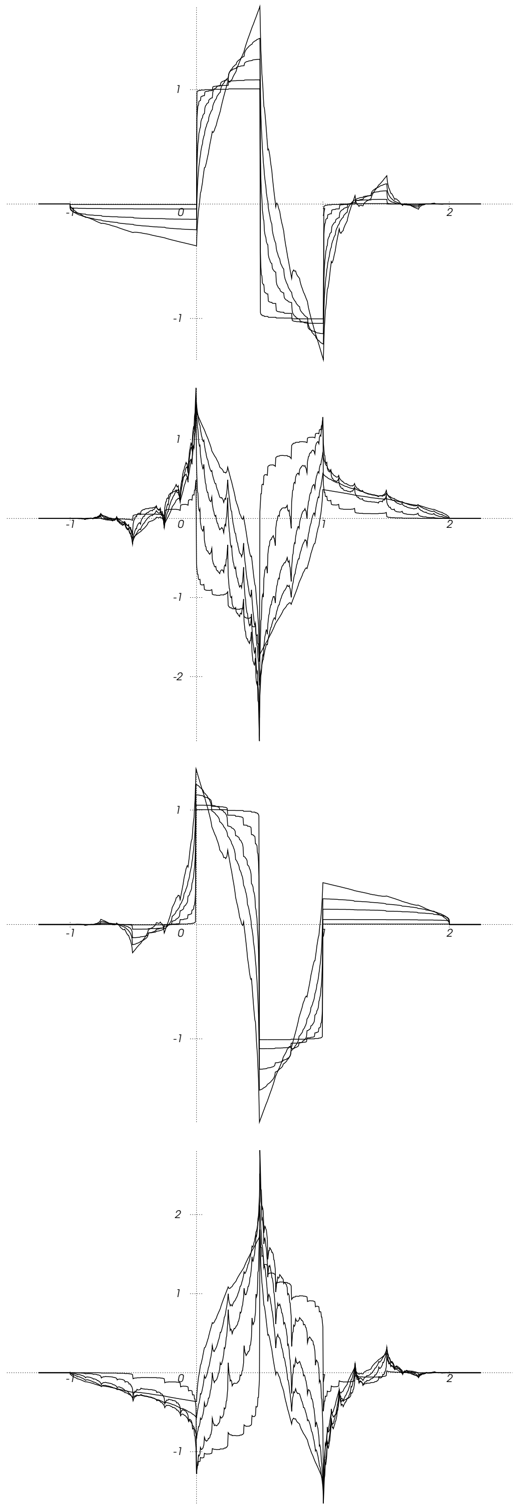

Fig. 2. q–wavelet for different values of q: $q \in (1, \infty)$ (rows 1 and 2), $q \in (0, 1)$ (rows 3 and 4); the two solutions for Q: Q_+ (rows 1 and 3), Q_- (rows 2 and 4).

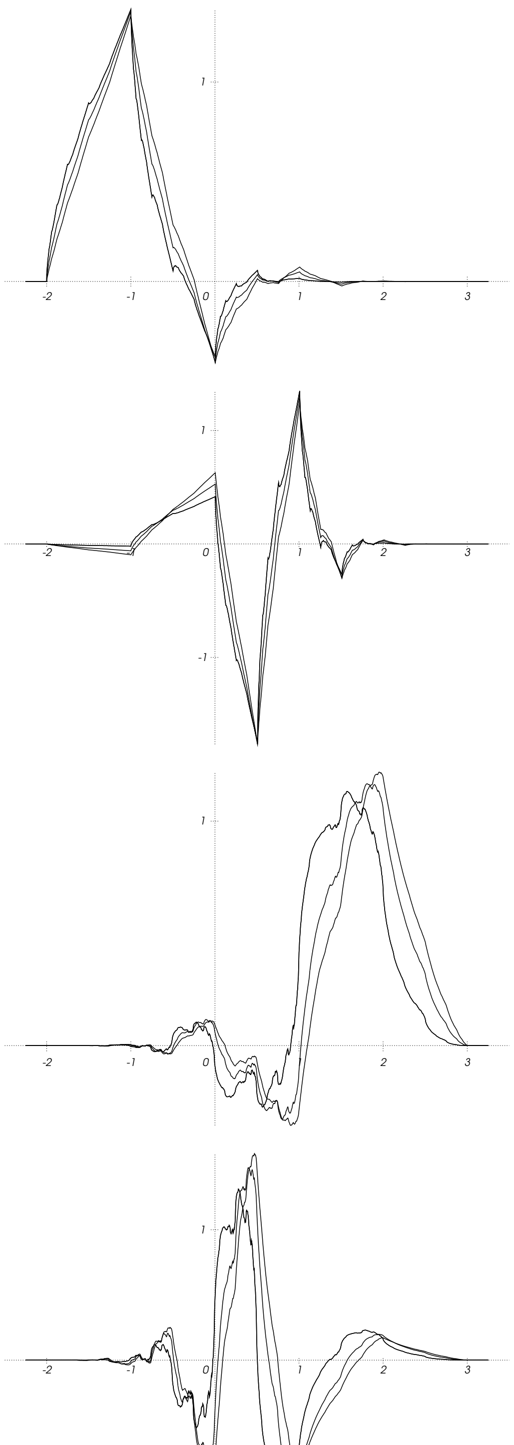

Fig. 3. Variation of Daubechies' Wavelets for $q \in \{2, 4, 16, 32\}$, $K = 2$, and $N = 3$. Scaling functions (rows 1,3) and wavelets (rows 2,4) for the two different solutions (rows 1,2 and 3,4).

In Figure 4 we have plotted the scaling function and wavelet for $N = 3$, $K = 1$, and $\omega_0 \in \{\pi/2, \pi/4, \pi/16\}$.

V. Discussion and related work

We have constructed orthogonal families of wavelet-functions with scale dependent properties. The constructed wavelet families and functions involve additional parameters: the parameter q for q-wavelets, the sequence of frequencies in the sin wavelets.

Striking is the obvious visual relationship between q-wavelets and fractional splines [8], [9], [10]: one may think of fractional splines of degree α as functions of the form

$$s^{\alpha}(x) = \sum_{k \in \mathbf{Z}} a_k (x - x_k)_+^{\alpha} ,$$

where x_k are the knots of the spline and $(\cdot)_+^{\alpha}$ denotes the one–sided power function. As becomes evident from the impressive graphics and the mathematical analysis in [10] (even more impressive graphics can be found at `http://bigwww.epfl.ch/art`) rational splines "interpolate´´ the common B-spline of integer order. In particular it interpolates between the Haar scaling function and B-splines of any order. The q-Wavelets in our paper reveal a similar interpolation behavior, where of course due to our construction we can only expect "interpolation´´ between the wavelets. A comparison with the work of Blu and Unser [8], [9], [10] immediately gives rise to the following open question. Smoothness of the fractional splines is equivalent by the (fractional) power α. We expect that also for q-wavelets smoothness can be directly linked to the parameter q. This question is also related to the existence of the scaling and wavelet functions.

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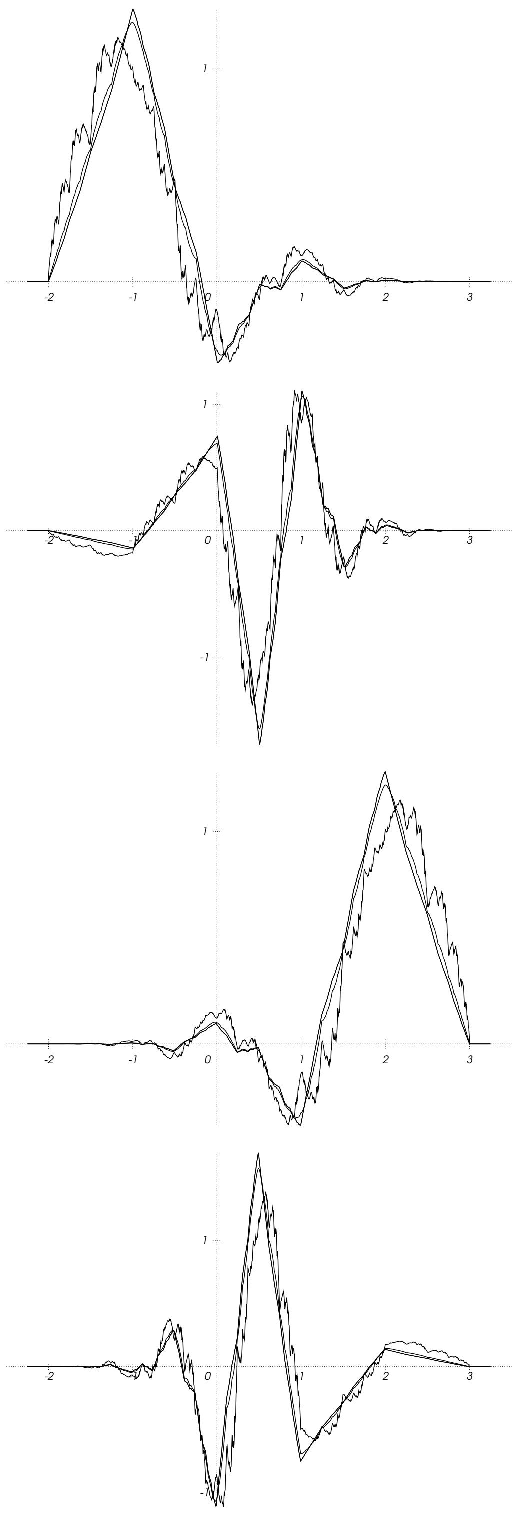

Fig. 4. sin-scaling functions (rows 1,3) and wavelets (rows 2,4) for $N = 3$, $K = 1$, and $\omega_0 \in \{\pi/2, \pi/4, \pi/16\}$; here the two different solutions (rows 1,2 and 3,4) are just mirrored versions.