LATTICE PATHS, q-MULTINOMIALS AND TWO VARIANTS OF THE ANDREWS-GORDON IDENTITIES

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Abstract. A few years ago Foda, Quano, Kirillov and Warnaar proposed and proved various finite analogs of the celebrated Andrews-Gordon identities. In this paper we use these polynomial identities along with the combinatorial techniques introduced in our recent paper to derive Garrett, Ismail, Stanton type formulas for two variants of the Andrews-Gordon identities.

1. Background and the first variant of the Andrews-Gordon identities

In 1961, Gordon [12] found a natural generalization of the Rogers-Ramanujan partition theorem.

Theorem 1. (*Gordon*) *For all $\nu \geq 1$, $0 \leq s \leq \nu$, the partitions of N of the frequency form $N = \sum_{j \geq 1} j f_j$ with $f_1 \leq s$ and $f_j + f_{j+1} \leq \nu$, $f_j \geq 0$ (for all $j \geq 1$) are equinumerous with the partitions of N into parts not congruent to 0 or $\pm(s+1)$ modulo $2\nu + 3$.*

Thirteen years later, Andrews [1] proposed and proved the following analytic counterpart to Gordon's theorem:

Theorem 2. (*Andrews*) *For all ν, s as in Theorem 1, and $|q| < 1$,*

$$\sum_{n_1, n_2, \ldots, n_\nu \geq 0} \frac{q^{N_1^2 + \ldots + N_\nu^2 + N_{s+1} + \cdots + N_\nu}}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_\nu}} = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j((2\nu+3)(j+1)-2(s+1))}{2}}$$

$$= \prod_{\substack{j \geq 1 \\ j \not\equiv 0, \pm(s+1) (\operatorname{mod} 2\nu+3)}} \frac{1}{1 - q^j}, \tag{1.1}$$

where

$$N_i = \begin{cases} n_i + n_{i+1} + \cdots + n_\nu, & \text{if } 1 \leq i \leq \nu, \\ 0, & \text{if } i = \nu + 1, \end{cases} \tag{1.2}$$

and

$$(a; q)_\infty = (a)_\infty = \prod_{j \geq 0} (1 - a q^j), \tag{1.3}$$

$$(a; q)_m = (a)_m = \prod_{j \geq 0} \frac{(1 - a q^j)}{(1 - a q^{j+m})}. \tag{1.4}$$

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We note that the last equality in (1.1) follows from Jacobi's triple product identity [14, (II.28)].
Subsequently, Bressoud [7] interpreted the l.h.s. of (1.1) in terms of weighted lattice paths. Bressoud path is made of three basic steps (see Fig. 1):

NE step:	from (i, j) to $(i+1, j+1)$,	
SE step:	from (i, j) to $(i+1, j-1)$,	only allowed if $j > 0$,
Horizontal step:	from $(i, 0)$ to $(i+1, 0)$,	only allowed along x-axis,

with $i \in \mathbb{Z}$ and $j \in \mathbb{Z}_{\geq 0}$.

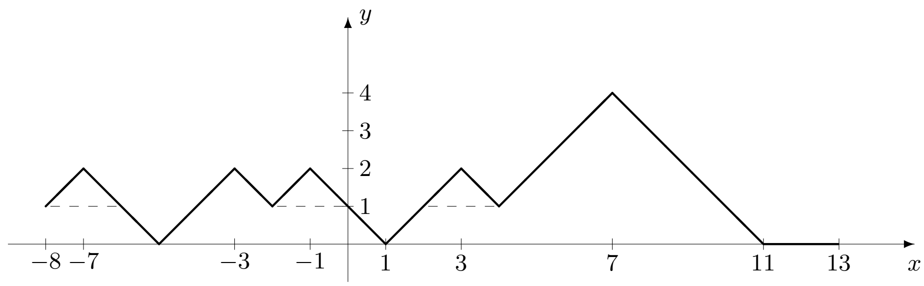

FIGURE 1. A Bressoud path starting at $(-8, 1)$ and ending at $(13, 0)$. Its weight is $-7 - 3 - 1 + 3 + 7 = -1$.

To calculate the weight of Bressoud path we define the peak of a path as a vertex preceded by the NE step and followed by the SE step. The height of a peak is its y-coordinate, the weight of a peak is its x-coordinate. The weight $w(p)$ of path p is defined as the sum of the weights of its peaks. In the above example (Fig. 1), the path has five peaks: $(-7, 2), (-3, 2), (-1, 2), (3, 2), (7, 4)$ and its weight is $-7-3-1+3+7 = -1$. The relative height of a peak (i, j) is the largest positive integer h, for which we can find two vertices on the path: $(i', j-h), (i'', j-h)$ such that $i' < i < i''$ and such that between these two vertices there are no peaks of height $> j$ and every peak of height $= j$ has weight $\geq i$. The peaks in the above example (Fig. 1) have relative heights $1, 2, 1, 1, 4$, respectively.

We can now state Bressoud's result [7].

Theorem 3. (*Bressoud*) *For ν, s as in Theorem 1 and $n_i \geq 0$, $1 \leq i \leq \nu$*

$$\frac{q^{N_1^2+\cdots+N_\nu^2+N_{s+1}+\cdots+N_\nu}}{(q)_{n_1}(q)_{n_2}\ldots(q)_{n_\nu}} = \lim_{L\to\infty} \sum_{p \in \mathbb{P}^\nu_{0,L}(\nu-s,0,\mathbf{n})} q^{w(p)}, \tag{1.5}$$

where

$$\mathbf{n} = (n_1, n_2, \ldots, n_\nu), \tag{1.6}$$

and $\mathbb{P}^\nu_{x_1,x_2}(y_1, y_2, \mathbf{n})$ denotes a collection of all Bressoud lattice paths that start at (x_1, y_1) and end at (x_2, y_2), which have no peaks higher than ν, and, in addition, the number of peaks of relative height j is n_j for $1 \leq j \leq \nu$.

Actually, it is straightforward to refine Bressoud's analysis to show that

$$q^{N_1^2+\cdots+N_\nu^2+N_{s+1}+\cdots+N_\nu} \prod_{i=1}^{\nu} \begin{bmatrix} n_i + L - 2\sum_{l=1}^{i} N_l - \alpha_{i,s} \\ n_i \end{bmatrix}_q = \sum_{p \in \mathbb{P}^\nu_{0,L}(\nu-s,0,\mathbf{n})} q^{w(p)}, \tag{1.7}$$

where

$$\alpha_{i,s} = \max(0, i-s), \tag{1.8}$$

and the q-binomial coefficients are defined as

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \begin{cases} \frac{(q)_{n+m}}{(q)_n(q)_m}, & \text{for } n,m \in \mathbb{Z}_{\geq 0}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.9}$$

This leads to

$$\sum_{\mathbf{n}} q^{N_1^2+\cdots+N_\nu^2+N_{s+1}+\cdots+N_\nu} \prod_{i=1}^{L} \begin{bmatrix} n_i + L - 2\sum_{l=1}^{i} N_l - \alpha_{i,s} \\ n_i \end{bmatrix}_q = C^\nu_{0,L}(\nu-s,0,q), \tag{1.10}$$

where for $0 \leq s,\ b \leq \nu$

$$C^\nu_{M,L}(s,b,q) = C^\nu_{M,L}(s,b) := \sum_{p \in \tilde{\mathbb{P}}^\nu_{M,L}(s,b)} q^{w(p)} \tag{1.11}$$

and

$$\tilde{\mathbb{P}}^\nu_{M,L}(s,b) = \sum_{\mathbf{n}} \mathbb{P}^\nu_{M,L}(s,b,\mathbf{n}). \tag{1.12}$$

Here and in the following, summation over $\mathbf{n}$ is over all non-negative integer tuples $\mathbf{n} = (n_1, \ldots, n_\nu)$.

Remark. We note that one can extract identity (1.7) from Lemma 4 in [9] with $a = s+1$, $k = \nu$, $m_i = N_i$, $n_i = L - \alpha_{i,s}$, if one recognizes that, in this case, the somewhat non-trivial conditions imposed on the peaks therein are equivalent to our concise statement that all paths end at $(L, 0)$.

Making use of (1.11), one easily derives the following recursion relations

$$C^\nu_{0,L}(s,\nu) = C^\nu_{0,L-1}(s,\nu-1), \tag{1.13}$$

$$\begin{aligned} C^\nu_{0,L}(s,b) &= C^\nu_{0,L-1}(s,b-1+\delta_{b,0}) + C^\nu_{0,L-1}(s,b+1) \\ &\quad + (q^{L-1}-1)C^\nu_{0,L-2}(s,b), \quad 0 \leq b < \nu, \end{aligned} \tag{1.14}$$

and verifies the initial conditions

$$C^\nu_{0,0}(s,b) = \delta_{s,b}, \tag{1.15}$$

where the Kronecker delta function $\delta_{i,j}$ is defined, as usual, as

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{1.16}$$

We note that formulas (1.13)–(1.15) specify the polynomials $C^\nu_{0,L}(s,b)$ uniquely. Next, for $L \equiv s+b \pmod 2$, we define polynomials $B^\nu_{s,b}(L,q)$ as

$$\begin{aligned} B^\nu_{s,b}(L,q) := B^\nu_{s,b}(L) := \sum_{j=-\infty}^{\infty} \Bigg\{ & q^{j((2j+1)(2\nu+3)-2s)} \begin{bmatrix} L \\ \frac{L+b-s}{2} + j(2\nu+3) \end{bmatrix}_q \\ & - q^{(2j+1)((2\nu+3)j+s)} \begin{bmatrix} L \\ \frac{L+b+s}{2} + j(2\nu+3) \end{bmatrix}_q \Bigg\}. \end{aligned} \tag{1.17}$$

Employing the standard q-binomial recurrences [14, (I.45)] one finds that

$$B^\nu_{s,b}(L) = B^\nu_{s,b-1}(L-1) + B^\nu_{s,b+1}(L-1) + (q^{L-1}-1)B^\nu_{s,b}(L-2). \tag{1.18}$$

It is not difficult to check that

$$B^{\nu}_{s,\nu+2}(L) = B^{\nu}_{2\nu+3-s,\nu+1}(L), \tag{1.19}$$

$$B^{\nu}_{s,1}(L) = B^{\nu}_{s,2}(L-1), \tag{1.20}$$

and for $1 \le s, b \le \nu + 1$

$$B^{\nu}_{s,b}(0) = \delta_{s,b}. \tag{1.21}$$

The formulas (1.13)–(1.15) and (1.18)–(1.21) imply that

$$C^{\nu}_{0,L}(s,b) = \begin{cases} B^{\nu}_{\nu+1-s,\nu+1-b}(L), & \text{if } L \equiv s+b (\operatorname{mod} 2), \\ B^{\nu}_{\nu+2+s,\nu+1-b}(L), & \text{otherwise}. \end{cases} \tag{1.22}$$

Indeed, both sides of (1.22) satisfy identical recurrences and initial conditions. Combining (1.10) and (1.22) we arrive at

Theorem 4. (*Foda, Quano and Kirillov*)

$$\sum_{\mathbf{n}} q^{N_1^2+\cdots+N_\nu^2+N_s+\cdots+N_\nu} \prod_{i=1}^{\nu} \begin{bmatrix} n_i + L - 2\sum_{l=1}^{i} N_l - \alpha_{i,s-1} \\ n_i \end{bmatrix}_q = \begin{cases} B^{\nu}_{s,\nu+1}(L), & \textit{if } L \not\equiv s+\nu (\operatorname{mod} 2), \\ B^{\nu}_{2\nu+3-s,\nu+1}(L), & \textit{otherwise.} \end{cases} \tag{1.23}$$

The above theorem was first proven in [11] and [15] in a somewhat different fashion. Using the following limiting formulas

$$\lim_{L\to\infty} \begin{bmatrix} L \\ n \end{bmatrix}_q = \frac{1}{(q)_n}, \tag{1.24}$$

$$\lim_{L\to\infty} B^{\nu}_{s,b}(L) = \lim_{L\to\infty} B^{\nu}_{2\nu+3-s,b}(L) = \prod_{\substack{j\ge 1 \\ j\not\equiv 0,\pm s (\operatorname{mod} 2\nu+3)}} \frac{1}{1-q^j}, \tag{1.25}$$

it is easy to check that in the limit $L \to \infty$, Theorem 4 reduces to Theorem 2. Recently, motivated by [4, 5, 13], we investigated in [6] the following multisums

$$\sum_{\mathbf{n}} \frac{q^{N_1^2+\cdots+N_\nu^2+N_i+\cdots+N_\nu-MN_1}}{(q)_{n_1}(q)_{n_2}\ldots(q)_{n_\nu}} \tag{1.26}$$

with $1 \le i \le \nu+1$, $M \in \mathbb{Z}_{\ge 0}$. The combinatorial analysis of (1.26) with $\nu = 1$ given in [6] can be upgraded to the more general case $\nu \ge 1$ as summarized in three steps below.

First step. We observe that

$$w(p') = w(p) - MN_1, \tag{1.27}$$

where the path $p' \in \mathbb{P}^{\nu}_{-M,L-M}(s,b,\mathbf{n})$ is obtained from the path $p \in \mathbb{P}^{\nu}_{0,L}(s,b,\mathbf{n})$ by moving p by M units to the left along the x-axis. Next, using (1.7) and (1.27) we derive that

$$C^{\nu}_{-M,L-M}(0,0,q) = \sum_{\mathbf{n}} \sum_{p\in\mathbb{P}^{\nu}_{-M,L-M}(0,0,\mathbf{n})} q^{w(p)} = \sum_{\mathbf{n}} \sum_{p\in\mathbb{P}^{\nu}_{0,L}(0,0,\mathbf{n})} q^{w(p)-MN_1}$$

$$= \sum_{\mathbf{n}} q^{N_1^2+\cdots+N_\nu^2-MN_1} \prod_{i=1}^{\nu} \begin{bmatrix} n_i + L - 2\sum_{l=1}^{i} N_l \\ n_i \end{bmatrix}_q. \tag{1.28}$$

Second step. For $0 \le M \le L$ every path $p \in \mathbb{P}^{\nu}_{-M,L-M}(0,0)$ consists of two pieces joined together at some point $(0,s)$ with $0 \le s \le \nu$. The first piece belongs to $\mathbb{P}^{\nu}_{-M,0}(0,s)$ and the second one to $\mathbb{P}^{\nu}_{0,L-M}(s,0)$. This observation is equivalent to

$$C^{\nu}_{-M,L-M}(0,0,q) = \sum_{s=0}^{\nu} C^{\nu}_{-M,0}(0,s,q)\; C^{\nu}_{0,L-M}(s,0,q). \tag{1.29}$$

Third step.

$$C^{\nu}_{-M,0}(0,s,q) = \sum_{p\in\mathbb{P}^{\nu}_{-M,0}(0,s)} q^{w(p)} = \sum_{p\in\mathbb{P}^{\nu}_{0,M}(s,0)} q^{-w(p)} = C^{\nu}_{0,M}(s,0,\frac{1}{q}). \tag{1.30}$$

Combining (1.28)–(1.30) one obtains

$$\sum_{\mathbf{n}} q^{N_1^2+\cdots+N_\nu^2-MN_1} \prod_{i=1}^{\nu} \begin{bmatrix} n_i + L - 2\sum_{l=1}^{i} N_l \\ n_i \end{bmatrix}_q = \sum_{s=0}^{\nu} C^{\nu}_{0,M}(s,0,\frac{1}{q})\; C^{\nu}_{0,L-M}(s,0,q). \tag{1.31}$$

Finally, letting L tend to infinity, we find with the aid of (1.22), (1.24), (1.25) and (1.31) our first variant of the Andrews-Gordon identities

$$\sum_{\mathbf{n}} \frac{q^{N_1^2+\cdots+N_\nu^2-MN_1}}{(q)_{n_1}(q)_{n_2}\dots(q)_{n_\nu}} = \sum_{\substack{s=1\\ M\not\equiv\nu+s (\mathrm{mod}\, 2)}}^{2\nu+2} \frac{B^{\nu}_{s,\nu+1}(M,\frac{1}{q})}{\prod_{j\ge1, j\not\equiv 0,\pm s (\mathrm{mod}\, 2\nu+3)}(1-q^j)}. \tag{1.32}$$

Formula (1.32) is a special case of (3.21) in [6] with $s = \nu+1$. The other cases there can be treated in a completely analogous manner.

Actually, neither the polynomial analogs (1.23) of (1.1), nor the path interpretation (1.10) of the Andrews-Gordon identities are unique. In particular, in [18] Warnaar considered the path space that is based on Gordon frequency conditions in Theorem 1, with an additional constraint that $f_j = 0$ for $j > L$. This led him to the new polynomial versions of the Andrews-Gordon identities. In the next section of this paper we will make essential use of Warnaar's analysis to investigate the following multisums

$$\sum_{\mathbf{n}} \frac{q^{N_1^2+\cdots+N_\nu^2+N_s+\cdots+N_\nu-M(N_1+N_2+\cdots+N_\nu)}}{(q)_{n_1}(q)_{n_2}\dots(q)_{n_\nu}}, \quad M \in \mathbb{Z}_{\ge 0}. \tag{1.33}$$

The rest of this article is organized as follows. In Section 2, we briefly discuss q-multinomial coefficients and Warnaar's terminating versions of the Andrews-Gordon identities. In Section 3, we review a particle interpretation of Gordon's frequency conditions given in [18]. In Section 4, by following easy steps being similar to three steps above, we derive our main formulas for (1.33) and their finite analogs. We conclude with a short description of prospects for future work opened by this investigation.

2. q-Multinomials and polynomial analogs of the

Andrews-Gordon identities

We start by recalling the binomial theorem

$$(1+x)^L = \sum_{a=0}^{L} \binom{L}{a} x^a, \tag{2.1}$$

where $\binom{L}{a}$ is the usual binomial coefficient.
By analogy, we introduce multinomial coefficients $\binom{L}{a}_\nu$ for $a = 0, 1, \ldots, \nu L$ as the coefficients in the expansion

$$(1+x+x^2+\cdots+x^\nu)^L = \sum_{a=0}^{\nu L} \binom{L}{a}_\nu x^a. \tag{2.2}$$

Multiple use of (2.1) yields an explicit sum representation

$$\binom{L}{a}_\nu = \sum_{j_1+\cdots+j_\nu=a} \binom{L}{j_\nu}\binom{j_\nu}{j_{\nu-1}}\cdots\binom{j_2}{j_1}. \tag{2.3}$$

Building on the work of Andrews [2], Schilling [16] and Warnaar [18] have introduced the following q-analogs of (2.3)

$$\begin{bmatrix} L \\ a \end{bmatrix}_\nu^p := \sum_{j_1+\cdots+j_\nu=a} q^{\sum_{l=2}^{\nu} j_{l-1}(L-j_l)-\sum_{l=1}^{p} j_l} \begin{bmatrix} L \\ j_\nu \end{bmatrix}_q \begin{bmatrix} j_\nu \\ j_{\nu-1} \end{bmatrix}_q \cdots \begin{bmatrix} j_2 \\ j_1 \end{bmatrix}_q \tag{2.4}$$

for $p = 0, 1, \ldots, \nu$.
We list some important properties of q-multinomials (2.4), which have been proven in [16] and [18].

Symmetries:

$$\begin{bmatrix} L \\ a \end{bmatrix}_\nu^p = q^{(\nu-p)L-a} \begin{bmatrix} L \\ \nu L - a \end{bmatrix}_\nu^{\nu-p} \tag{2.5}$$

and

$$\begin{bmatrix} L \\ a \end{bmatrix}_\nu^0 = \begin{bmatrix} L \\ \nu L - a \end{bmatrix}_\nu^0. \tag{2.6}$$

Recurrences:

$$\begin{bmatrix} L \\ a \end{bmatrix}_\nu^p = \sum_{m=0}^{\nu-p} q^{m(L-1)} \begin{bmatrix} L-1 \\ a-m \end{bmatrix}_\nu^m + \sum_{m=\nu-p+1}^{\nu} q^{L(\nu-p)-m} \begin{bmatrix} L-1 \\ a-m \end{bmatrix}_\nu^m. \tag{2.7}$$

***q*-Deformed tautologies:**

$$\begin{bmatrix} L \\ a \end{bmatrix}_\nu^p + q^L \begin{bmatrix} L \\ \nu L - a - p - 1 \end{bmatrix}_\nu^{p+1} = q^L \begin{bmatrix} L \\ a \end{bmatrix}_\nu^{p+1} + \begin{bmatrix} L \\ \nu L - a - p - 1 \end{bmatrix}_\nu^p \tag{2.8}$$

where $p = -1, 0, \ldots, \nu - 1$ and

$$\begin{bmatrix} L \\ a \end{bmatrix}_\nu^{-1} = 0. \tag{2.9}$$

Limiting behavior: For $\frac{\nu L}{2} - A = 0, 1, \ldots, \nu L$

$$\lim_{L\to\infty} \begin{bmatrix} L \\ \frac{\nu L}{2} - A \end{bmatrix}_\nu^p = \begin{cases} \frac{1}{(q)_\infty}, & \text{if } 0 \le p < \frac{\nu}{2}, \\ \frac{1+q^A}{(q)_\infty}, & \text{if } \nu \equiv 0 (\bmod 2) \text{ and } p = \frac{\nu}{2}, \\ \text{no limit}, & \text{if } \frac{\nu}{2} < p \le \nu. \end{cases} \tag{2.10}$$

Special values:

$$\begin{bmatrix} 0 \\ a \end{bmatrix}_\nu^p = \delta_{a,0}, \tag{2.11}$$

and

$$\begin{bmatrix} L \\ a \end{bmatrix}_\nu^p = 0, \quad \text{if } a < 0 \text{ or } a > \nu L. \tag{2.12}$$

Next, for each $\nu \in \mathbb{Z}_{>0}$ we consider the ordered sequences of integers $\{f_M, f_{M+1}, \ldots, f_{L-1}, f_L\}$ subject to Gordon's conditions

$$\begin{cases} f_j \geq 0, & M \leq j \leq L, \\ f_j + f_{j+1} \leq \nu, & M \leq j < L. \end{cases} \tag{2.13}$$

Each of these Gordon sequences can be represented graphically by a lattice path as illustrated by the example shown in Fig. 2.

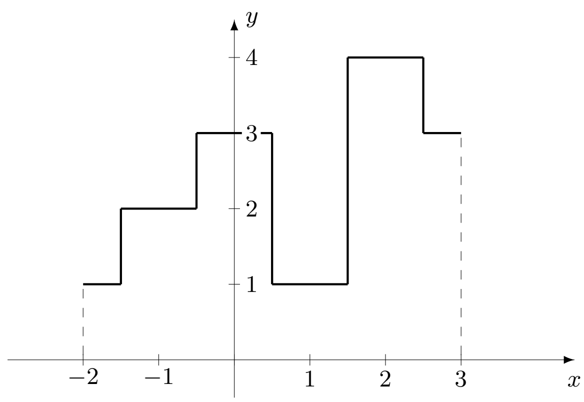

FIGURE 2. A lattice path representation of the sequence $\{f_{-2} = 1, f_{-1} = 2, f_0 = 3, f_1 = 1, f_2 = 4, f_3 = 3\}$. Here $M = -2$, $L = 3$, $\nu = 7$ and weight of the path $= -1 \cdot 2 + 0 \cdot 3 + 1 \cdot 1 + 2 \cdot 4 = 7$.

The weight $wg(p)$ of Gordon path p described above is

$$wg(p) = \sum_{j=M+1}^{L-1} j f_j. \tag{2.14}$$

For $0 \leq s,\ b \leq \nu,\ L, M \in \mathbb{Z}$ we perform a weighted path count with the help of the following polynomials

$$G^\nu_{M,L}(s, b, q) := \sum_{p \in \tilde{\mathbb{S}}^\nu_{M,L}(s,b)} q^{wg(p)}, \tag{2.15}$$

where $\tilde{\mathbb{S}}^\nu_{M,L}(s, b)$ denotes the space of all Gordon paths subject to (2.13) that start at (M, s) and end at (L, b).

In [18], Warnaar, building on the work of Andrews and Baxter [3], showed in a recursive fashion that for $0 \leq s,\ b \leq \nu$

$$G^\nu_{0,L}(\nu - s, b, q) = \begin{cases} W^\nu_{s+1,b}(L, q), & \text{if } b + s \equiv \nu(L + 1) (\operatorname{mod} 2), \\ W^\nu_{2\nu+2-s,b}(L, q), & \text{if } b + s \not\equiv \nu(L + 1) (\operatorname{mod} 2), \end{cases} \tag{2.16}$$

where

$$W^{\nu}_{s,b}(L,q) := \sum_{j=-\infty}^{\infty} \Bigg\{ q^{j((2j+1)(2\nu+3)-2s)} \begin{bmatrix} L \\ \frac{\nu(L+1)-s-b+1}{2} + (2\nu+3)j \end{bmatrix}^{b}_{\nu} - q^{(2j+1)((2\nu+3)j+s)} \begin{bmatrix} L \\ \frac{\nu(L+1)+s-b+1}{2} + (2\nu+3)j \end{bmatrix}^{b}_{\nu} \Bigg\}. \tag{2.17}$$

More specifically, it was proven in [18] that both sides of (2.16) satisfy the same recurrences

$$G^{\nu}_{0,L}(\nu-s,b,q) = \sum_{l=0}^{\nu-b} q^{(L-1)l} G^{\nu}_{0,L-1}(\nu-s,l,q) \tag{2.18}$$

and the same initial conditions

$$G^{\nu}_{0,0}(\nu-s,b,q) = \delta_{s+b,\nu}. \tag{2.19}$$

Also in [18], a particle interpretation of Gordon paths was given and, as a result, another representation for $G^{\nu}_{0,L}(s,b)$ was obtained; namely,

$$G^{\nu}_{0,L}(s,b) = F^{\nu}_{\nu-s,\nu-b}(L,q), \tag{2.20}$$

where $0 \le s,\ b \le \nu$, $L \ge 2$ and

$$F^{\nu}_{s,b}(L,q) := \sum_{\mathbf{n}} q^{N_1^2+\cdots+N_\nu^2+N_{s+1}+\cdots+N_\nu} \prod_{i=1}^{\nu} \begin{bmatrix} n_i + iL - 2\sum_{l=1}^{i} N_l - \alpha_{i,s} - \alpha_{i,b} \\ n_i \end{bmatrix}_q. \tag{2.21}$$

Comparing (2.16) and (2.20) we arrive at the polynomial identities

$$F^{\nu}_{s,b}(L,q) = \begin{cases} W^{\nu}_{s+1,\nu-b}(L,q), & \text{if } b+s \equiv \nu L (\operatorname{mod} 2), \\ W^{\nu}_{2\nu+2-s,\nu-b}(L,q), & \text{otherwise}, \end{cases} \tag{2.22}$$

which in the limit $L \to \infty$ reduce to the Andrews-Gordon identities (1.1). It is important to realize that while (1.23) and (2.22) are identical in the limit $L \to \infty$, these identities are substantially different for finite L.

Remark. It should be noted that (2.20) with $s = b = 0$ is a corollary of Bressoud's Lemma 3 in [9]. It appears that this Lemma, as stated, is true for $a = k+1$ only. For $a = 1, 2, \ldots, k$, the generating function $c_{a,k}(n, 2n, \ldots, kn; j)$ therein should be corrected to $c_{a,k}(n - \alpha_{1,a-1}, 2n - \alpha_{2,a-1}, \ldots, kn - \alpha_{k,a-1}; j)$.

3. Gordon paths and lattice particles

We begin our particle description of Gordon lattice paths by considering first paths in $\tilde{\mathbb{S}}^{\nu}_{M,L}(0,0)$. Following [18], we introduce a special kind of paths from which all other paths in $\tilde{\mathbb{S}}^{\nu}_{M,L}(0,0)$ can be constructed. These paths, termed the minimal paths in [18], are shown in Fig. 3.

In a minimal path, each column with a non-zero height t ($0 < t \le \nu$) is interpreted as a particle of charge t. Two adjacent particles in a minimal path are separated by a single empty column. In order to construct an arbitrary non-minimal path in $\tilde{\mathbb{S}}^{\nu}_{M,L}(0,0)$ out of one and only one minimal path, we need to introduce the rules of particle motion from left to right. Fig. 4 illustrates these rules in the simplest case of an isolated particle of charge t going from j to $j+1$. The complete transition requires t elementary moves.

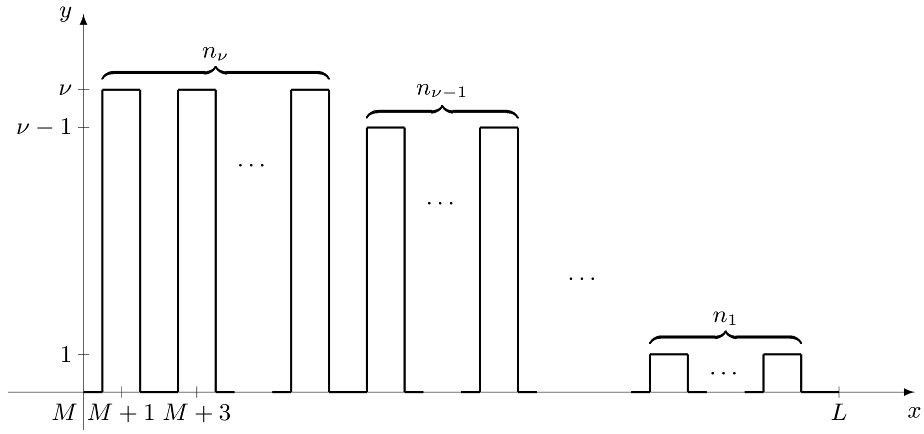

FIGURE 3. The minimal path in $\tilde{\mathbb{S}}^{\nu}_{M,L}(0,0)$ of particle content $\mathbf{n} = (n_1, n_2, \dots, n_\nu)$.

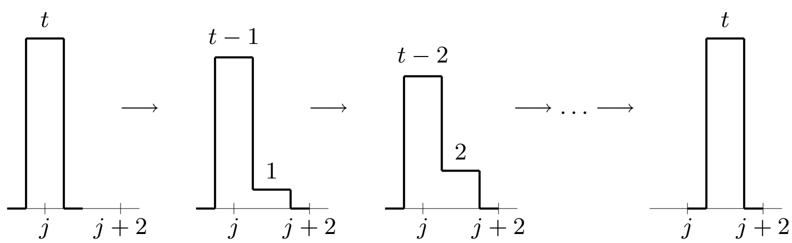

FIGURE 4. A particle of charge t in free motion from j to $j+1$.

Next, we consider the motion of a particle of charge t through a path configuration shown in Fig. 5 and Fig. 6(a).

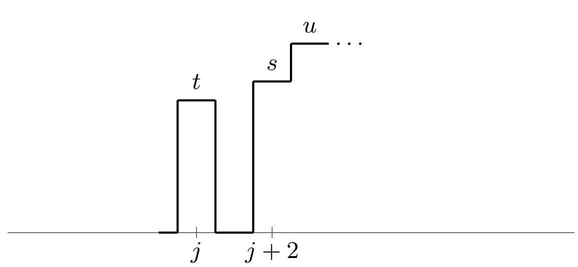

FIGURE 5. $t \leq s$

In case of the path configuration in Fig. 5, the particle of charge t can not make any further move to the right. In the case shown in Fig. 6(a), we can make $t - s$ elementary moves to obtain the new path configuration in Fig. 6(b). What happens next depends on the height of the column at $j+3$, as illustrated in Fig. 7 and Fig. 8. In case of the path configuration in Fig. 7, the particle of charge t can not move any further.

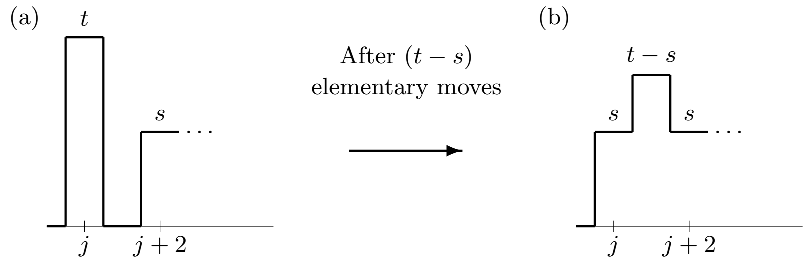

FIGURE 6. $t > s$

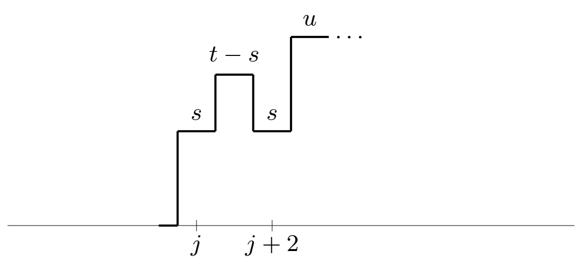

FIGURE 7. $u \geq t - s$

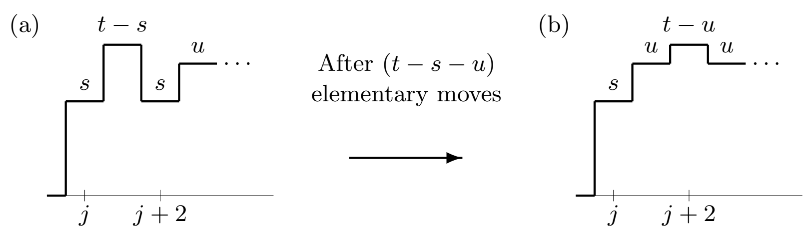

FIGURE 8. $u < t - s$

In the case shown in Fig. 8(a) we can make $t - u - s$ elementary moves to end up with the path configuration in Fig. 8(b). Ignoring the first column at j, we see that the last configuration is practically the same as the one in Fig. 6(b) with s replaced by u. It means that we can keep on moving the particle of charge t according to the rules in Fig. 4–Fig. 8.

Actually, it is easy to modify the above discussion in order to deal with the general boundary conditions $0 \leq s, b \leq \nu$. All we need to do is to refine the notion of a minimal path as in Fig. 9.

Two adjacent particles in Fig. 9 are separated by at most one empty column. Two half-columns at M and L are not interpreted as particles.

In [18], Warnaar proved that using rules of particle motion, one can obtain each non-minimal path from one and only one minimal path in a completely bijective fashion. The particle content $\mathbf{n}$ of $p \in \tilde{\mathbb{S}}^{\nu}_{M,L}(s,b)$ can be determined by reducing p to its minimal image $p_{\min}$.

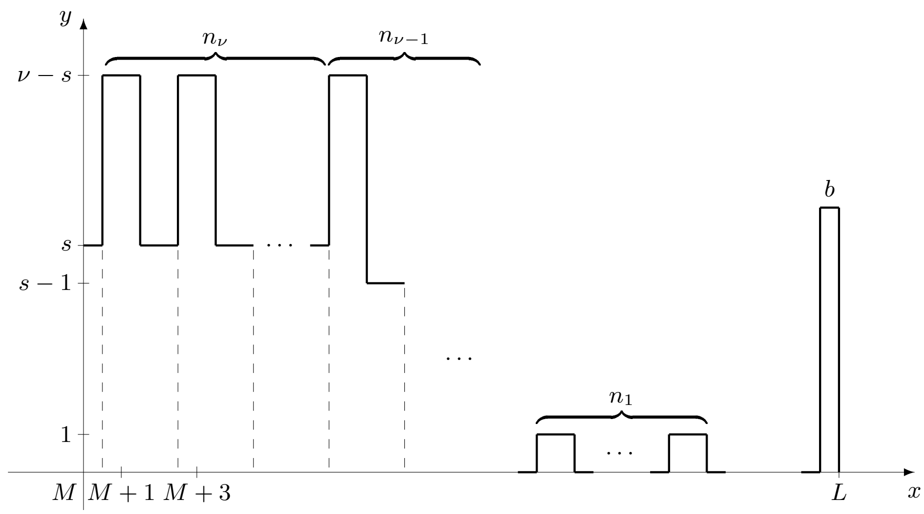

FIGURE 9. The minimal path in $\tilde{\mathbb{S}}^{\nu}_{M,L}(s,b)$ of particle content $\mathbf{n} = (n_1, n_2, \dots, n_\nu)$. The dashed lines separate different particles. All particles are arranged according to their charges in non-decreasing order from left to right.

Now, since the sum of heights $\sum_{j=M+1}^{L-1} f_j$ of the path $p \in \mathbb{S}^{\nu}_{M,L}(s,b,\mathbf{n})$ is the invariant of the motion we find by counting the heights of $p_{\min}$ that

$$\sum_{j=M-1}^{L-1} f_j = \sum_{j=1}^{\nu} j n_j = N_1 + N_2 + \cdots + N_\nu, \tag{3.1}$$

where $\mathbb{S}^{\nu}_{M,L}(s,b,\mathbf{n})$ denotes the space of all Gordon paths of the particle content $\mathbf{n}$, which start at (M,s) and end at (L,b).
Equation (3.1) implies that

$$wg(\tilde{p}) = wg(p) - M(N_1 + N_2 + \cdots + N_\nu), \tag{3.2}$$

where $\tilde{p} \in \mathbb{S}^{\nu}_{-M,L-M}(s,b,\mathbf{n})$ is obtained from $p \in \mathbb{S}^{\nu}_{0,L}(s,b,\mathbf{n})$ by moving p by M units to the left along the x-axis.
Formula (3.2) will play an important role in the sequel. We shall also require another result established in [18]:

Theorem 5. (*Warnaar*)

$$q^{N_1^2+\cdots+N_\nu^2+N_{s+1}+\cdots+N_\nu} \prod_{i=1}^{\nu} \begin{bmatrix} n_i + iL - 2\sum_{l=1}^{i} N_l - \alpha_{i,s} - \alpha_{i,b} \\ n_i \end{bmatrix}_q = \sum_{p \in \mathbb{S}^{\nu}_{0,L}(\nu-s,\nu-b,\mathbf{n})} q^{wg(p)}, \tag{3.3}$$

where $0 \le s,\ b \le \nu$ *and* $L \ge 2$.

Note that (2.20) is an immediate consequence of Theorem 5.

4. The second variant of the Andrews-Gordon identities

Having collected the necessary background information, we can derive the second variant of the Andrews-Gordon identities by following three easy steps very similar to those taken to derive our first variant in Section 1.

First step. We generalize (3.3) as

$$\sum_{p\in\mathbb{S}^{\nu}_{-M,L-M}(\nu-s,\nu-b,\mathbf{n})} q^{wg(p)} \overset{\text{by (3.2)}}{=} \sum_{p\in\mathbb{S}^{\nu}_{0,L}(\nu-s,\nu-b,\mathbf{n})} q^{wg(p)-M(N_1+\cdots+N_\nu)}$$
$$\overset{\text{by (3.3)}}{=} q^{N_1^2+\cdots+N_\nu^2+N_{s+1}+\cdots+N_\nu-M(N_1+\cdots+N_\nu)} \cdot\prod_{i=1}^{\nu}\begin{bmatrix} n_i+iL-2\sum_{l=1}^{i}N_l-\alpha_{i,s}-\alpha_{i,b}\\ n_i\end{bmatrix}_q. \tag{4.1}$$

Next, we sum over $\mathbf{n}$ to obtain

$$\sum_{\mathbf{n}} q^{N_1^2+\cdots+N_\nu^2+N_{s+1}+\cdots+N_\nu-M(N_1+\cdots+N_\nu)} \prod_{i=1}^{\nu}\begin{bmatrix} n_i+iL-2\sum_{l=1}^{i}N_l-\alpha_{i,s}-\alpha_{i,b}\\ n_i\end{bmatrix}_q$$
$$=\sum_{\mathbf{n}}\sum_{p\in\mathbb{S}^{\nu}_{-M,L-M}(\nu-s,\nu-b,\mathbf{n})} q^{wg(p)} = G^{\nu}_{-M,L-M}(\nu-s,\nu-b,q). \tag{4.2}$$

Second step. Here, the argument is exactly the same as the one used in deriving (1.29). We simply state the result

$$G^{\nu}_{-M,L-M}(\nu-s,\nu-b,q)=\sum_{s'=0}^{\nu} G^{\nu}_{-M,0}(\nu-s,s',q)\, G^{\nu}_{0,L-M}(s',\nu-b,q). \tag{4.3}$$

Third step. If the path $\tilde{p}\in\tilde{\mathbb{S}}^{\nu}_{-M,0}(s,b)$ is obtained from the path $p\in\tilde{\mathbb{S}}^{\nu}_{0,M}(b,s)$ by reflecting p across the y-axis, then

$$wg(\tilde{p})=-wg(p). \tag{4.4}$$

Hence,

$$G^{\nu}_{-M,0}(\nu-s,s',q)=\sum_{p\in\tilde{\mathbb{S}}^{\nu}_{-M,0}(\nu-s,s')} q^{wg(p)}$$
$$=\sum_{p\in\tilde{\mathbb{S}}^{\nu}_{0,M}(s',\nu-s)}\left(\frac{1}{q}\right)^{wg(p)} = G^{\nu}_{0,M}(s',\nu-s,\frac{1}{q}). \tag{4.5}$$

Next, combining (4.2), (4.3) and (4.5) we find that

$$\sum_{\mathbf{n}} q^{N_1^2+\cdots+N_\nu^2+N_{s+1}+\cdots+N_\nu-M(N_1+\cdots+N_\nu)} \prod_{i=1}^{\nu}\begin{bmatrix} n_i+iL-2\sum_{l=1}^{i}N_l-\alpha_{i,s}-\alpha_{i,b}\\ n_i\end{bmatrix}_q$$
$$=\sum_{s'=0}^{\nu} G^{\nu}_{0,M}(s',\nu-s,\frac{1}{q})\, G^{\nu}_{0,L-M}(s',\nu-b,q). \tag{4.6}$$

It follows from Theorem 1 that

$$\lim_{L\to\infty} G^{\nu}_{0,L}(s,b,q)=\frac{1}{\prod_{j\geq 1, j\not\equiv 0,\pm(\nu-s+1)(\mathrm{mod}\ 2\nu+3)}(1-q^j)}. \tag{4.7}$$

Letting L tend to infinity in (4.6), we obtain with the aid of (1.24) and (4.7)

$$\sum_{\mathbf{n}} \frac{q^{N_1^2+\cdots+N_\nu^2+N_{s+1}+\cdots+N_\nu-M(N_1+\cdots+N_\nu)}}{(q)_{n_1}(q)_{n_2}\dots(q)_{n_\nu}} = \sum_{s'=0}^{\nu} \frac{G^{\nu}_{0,M}(\nu-s',\nu-s,\frac{1}{q})}{\prod_{j\geq 1, j\not\equiv 0,\pm(s'+1) (\bmod 2\nu+3)}(1-q^j)}. \tag{4.8}$$

Finally, recalling (2.16) we arrive at the desired second variant

$$\sum_{\mathbf{n}} \frac{q^{N_1^2+\cdots+N_\nu^2+N_{s+1}+\cdots+N_\nu-M(N_1+\cdots+N_\nu)}}{(q)_{n_1}(q)_{n_2}\dots(q)_{n_\nu}} = \sum_{\substack{s'=1\\ s+s'\not\equiv \nu M (\bmod 2)}}^{2\nu+2} \frac{W^{\nu}_{s',\nu-s}(M,\frac{1}{q})}{\prod_{j\geq 1, j\not\equiv 0,\pm s' (\bmod 2\nu+3)}(1-q^j)}. \tag{4.9}$$

Setting $M=1$ in (4.9) we easily obtain

$$\sum_{\mathbf{n}} \frac{q^{N_1^2+\cdots+N_\nu^2-N_1-N_2-\cdots-N_s}}{(q)_{n_1}(q)_{n_2}\dots(q)_{n_\nu}} = \sum_{s'=\nu-s+1}^{\nu+1} \prod_{\substack{j\geq 1\\ j\not\equiv 0,\pm s' (\bmod 2\nu+3)}} \frac{1}{(1-q^j)}, \tag{4.10}$$

which is essentially the same (modulo misprint) as identity (3.3) in [8].

5. Further generalizations

Comparing (1.32) and (4.9) with $s=\nu$, the structural similarities of these two formulas become obvious. This resemblance strongly suggests that our two variants are special cases of a more general formula for multisums of the form

$$\sum_{\mathbf{n}} \frac{q^{N_1^2+\cdots+N_\nu^2-M_1N_1-\cdots-M_\nu N_\nu}}{(q)_{n_1}(q)_{n_2}\dots(q)_{n_\nu}}, \tag{5.1}$$

with $(M_1,M_2,\dots,M_\nu)\in\mathbb{Z}^\nu$.

The unifying formula requires generalized q-multinomials that depend on ν finitization parameters. Fortunately, such objects have already appeared in the literature [10, 17]:

$$\begin{bmatrix}\mathbf{L}\\ a\end{bmatrix}_\nu := \sum_{j_1+\cdots+j_\nu=a+\frac{1}{2}\sum_{i=1}^{\nu} iL_i} q^{\sum_{l=2}^{\nu} j_{l-1}(L_l+\cdots+L_\nu-j_l)} \cdot \begin{bmatrix}L_\nu\\ j_\nu\end{bmatrix}_q \begin{bmatrix}L_{\nu-1}+j_\nu\\ j_{\nu-1}\end{bmatrix}_q \cdots \begin{bmatrix}L_1+j_2\\ j_1\end{bmatrix}_q, \tag{5.2}$$

where

$$\mathbf{L}=(L_1,L_2,\dots,L_\nu)\in\mathbb{Z}^\nu, \tag{5.3}$$

and

$$a+\frac{1}{2}\sum_{i=1}^{\nu} iL_i\in\mathbb{Z}_{\geq 0}. \tag{5.4}$$

In [17], the polynomials (5.2) were termed q-supernomial coefficients.

We notice that

$$\begin{bmatrix}(L,0,\dots,0)\\ a\end{bmatrix}_\nu = \begin{bmatrix}L\\ a+\frac{L}{2}\end{bmatrix}_q, \tag{5.5}$$

and

$$\begin{bmatrix}(0,\dots,0,L)\\ a\end{bmatrix}_\nu = \begin{bmatrix}L\\ a+\frac{\nu L}{2}\end{bmatrix}_\nu^0. \tag{5.6}$$

Using (5.5), (5.6) along with (1.32) and (4.9) with $s=\nu$, it is easy to guess that

$$\sum_{\mathbf{n}} \frac{q^{N_1^2+\cdots+N_\nu^2-M_1N_1-\cdots-M_\nu N_\nu}}{(q)_{n_1}(q)_{n_2}\dots(q)_{n_\nu}} = \sum_{\substack{s=1\\ s+\nu+\sum_{i=1}^{\nu} M_i \text{ odd}}}^{2\nu+2} \frac{I^\nu_{s,\nu+1}(\tilde{\mathbf{M}},\frac{1}{q})}{\prod_{j\geq 1, j\not\equiv 0,\pm s (\bmod 2\nu+3)}(1-q^j)}, \tag{5.7}$$

where

$$\tilde{\mathbf{M}} = (M_1-M_2, M_2-M_3,\dots,M_{\nu-1}-M_\nu, M_\nu), \tag{5.8}$$

and

$$I^\nu_{s,b}(\mathbf{L},q) := \sum_{j=-\infty}^{\infty}\Bigg\{ q^{j((2j+1)(2\nu+3)-2s)} \begin{bmatrix}\mathbf{L}\\ \frac{b-s}{2}+(2\nu+3)j\end{bmatrix}_\nu - q^{(2j+1)((2\nu+3)j+s)} \begin{bmatrix}\mathbf{L}\\ \frac{b+s}{2}+(2\nu+3)j\end{bmatrix}_\nu \Bigg\}. \tag{5.9}$$

The remarkable formula (5.7) and its finite analogs will be the subject of our next paper.

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