VARIANTS OF THE ANDREWS-GORDON IDENTITIES

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Abstract. The object of this paper is to propose and prove a new generalization of the Andrews-Gordon Identities, extending a recent result of Garrett, Ismail and Stanton. We also give a combinatorial discussion of the finite form of their result which appeared in the work of Andrews, Knopfmacher, and Paule.

1. Introduction

The celebrated Rogers-Ramanujan identities are given analytically as follows

$$\sum_{t\geq 0}\frac{q^{t^2+at}}{(q)_t}=\frac{1}{(q)_\infty}\sum_{j=-\infty}^{\infty}\left\{q^{j(10j+1+2a)}-q^{(2j+1)(5j+2-a)}\right\}$$
$$=\frac{1}{(q^{1+a},q^5)_\infty(q^{4-a},q^5)_\infty},\tag{1.1}$$

where $a=0,1$ and the q-shifted factorials $(z;q)_t$ are defined as usual as

$$(z;q)_t=(z)_t=\begin{cases}\prod_{j=0}^{t-1}(1-zq^j), & \text{if } t\in\mathbb{Z}_{>0},\\ 1, & \text{if } t=0.\end{cases}\tag{1.2}$$

It is well known that these identities have polynomial analogs. In particular, building on the work of Schur and MacMahon, Andrews [1] has shown that for $L\in\mathbb{Z}_{\geq 0}$

$$\sum_{t\geq 0}q^{t^2}\begin{bmatrix}L-t\\ t\end{bmatrix}_q=e_L(q)\tag{1.3}$$

and

$$\sum_{t\geq 0}q^{t^2+t}\begin{bmatrix}L-t-1\\ t\end{bmatrix}_q=d_L(q),\tag{1.4}$$

where

$$e_L(q)=\sum_{j=-\infty}^{\infty}\left\{q^{j(10j+1)}\begin{bmatrix}L\\ \lfloor\frac{L}{2}\rfloor-5j\end{bmatrix}_q-q^{(2j+1)(5j+2)}\begin{bmatrix}L\\ \lfloor\frac{L-4}{2}\rfloor-5j\end{bmatrix}_q\right\}\tag{1.5}$$

and

$$d_L(q)=\sum_{j=-\infty}^{\infty}\left\{q^{j(10j+3)}\begin{bmatrix}L\\ \lfloor\frac{L-1}{2}\rfloor-5j\end{bmatrix}_q-q^{(2j+1)(5j+1)}\begin{bmatrix}L\\ \lfloor\frac{L-3}{2}\rfloor-5j\end{bmatrix}_q\right\}.\tag{1.6}$$

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As usual, $\lfloor x \rfloor$ denotes the integer part of x and q-binomial coefficients are defined as follows

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \begin{cases} \frac{(q^{n+1})_m}{(q)_m}, & \text{if } m \in \mathbb{Z}_{\geq 0}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.7}$$

Both polynomial sequences (e_L) and (d_L) satisfy the recurrence

$$c_L(q) = c_{L-1}(q) + q^{L-1} c_{L-2}(q), \quad L \geq 2. \tag{1.8}$$

The above equation along with the initial conditions

$$d_0(q) = 0, \quad e_0(q) = e_1(q) = d_1(q) = 1 \tag{1.9}$$

specifies these sequences uniquely. Moreover, one can read (1.8) backward to define (e_L), (d_L) for negative subindices; i.e., for $L \geq 1$,

$$\begin{cases} e_{-L}(q) = (-1)^L q^{\binom{L}{2}} d_{L-1}(\frac{1}{q}), \\ d_{-L}(q) = (-1)^{L+1} q^{\binom{L}{2}} e_{L-1}(\frac{1}{q}). \end{cases} \tag{1.10}$$

Despite the long history of the Rogers-Ramanujan identities, the following variants found by Garrett et al. [14]

$$\sum_{t \geq 0} \frac{q^{t^2+mt}}{(q)_t} = \frac{(-1)^m q^{-\binom{m}{2}} d_{m-1}(q)}{(q,q^5)_\infty (q^4,q^5)_\infty} + \frac{(-1)^{m+1} q^{-\binom{m}{2}} e_{m-1}(q)}{(q^2,q^5)_\infty (q^3,q^5)_\infty}, \qquad m \geq 0 \tag{1.11}$$

appeared to be new, even though closely related results were derived before in [10] and [3].

Actually, (1.11) can be extended to negative m with the aid of (1.10) as

$$\sum_{t \geq 0} \frac{q^{t^2-Mt}}{(q)_t} = \frac{e_M(\frac{1}{q})}{(q,q^5)_\infty (q^4,q^5)_\infty} + \frac{d_M(\frac{1}{q})}{(q^2,q^5)_\infty (q^3,q^5)_\infty} \tag{1.12}$$

with $M \in \mathbb{Z}_{\geq 0}$. The authors of [14] gave two proofs of (1.11). In the first proof they evaluated a certain integral involving q-Hermite polynomials in two different ways and equated the results. Their second proof made essential use of Schur's involution. A very different approach was taken by Andrews et al. in [5], where identity (1.12) appeared as a limiting case of the much stronger identity

$$\begin{aligned} \sum_{t \geq 0} q^{t^2+mt} \begin{bmatrix} L-t \\ t \end{bmatrix}_q &= (-1)^m q^{-\binom{m}{2}} d_{m-1}(q) e_{L+m}(q) \\ &+ (-1)^{m+1} q^{-\binom{m}{2}} e_{m-1}(q) d_{L+m}(q), \quad (L, m \geq 0), \end{aligned} \tag{1.13}$$

which was proven recursively. It is trivial to verify that in the limit $L \to \infty$ (1.13) turns into (1.11). It was pointed out in [5] that (1.13) may be viewed as a q-analog of a famous Euler-Cassini's identity for Fibonacci numbers. We remark that a new approach to identities of q-Euler-Cassini type has been given in [11].

Once again, we may employ (1.10) to extend (1.13) to negative m. The result is

$$\sum_{t \geq 0} q^{t^2-Mt} \begin{bmatrix} L-t \\ t \end{bmatrix}_q = e_M(\frac{1}{q}) e_{L-M}(q) + d_M(\frac{1}{q}) d_{L-M}(q), \quad M \geq 0. \tag{1.14}$$

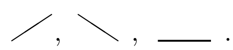
Remarkably, this reformulation of (1.13) enables us to reduce it to (1.3), (1.4) in an elementary combinatorial fashion. This is done in Section 2. In Section 3, we briefly discuss a polynomial version of the Andrews-Gordon identities and then, move on to our main results (3.19)–(3.21): variants of the Andrews-Gordon identities, which are straightforward multisum generalizations of (1.14). In Section 4, some problems for further investigation motivated by this work are indicated. Finally, certain technical details pertaining to the recurrences for multisums are relegated to the Appendix.

2. Combinatorial analysis of (1.14)

We start by recalling a well-known fact.

Lemma. *For $L \geq 2t \geq 0$, $q^{t^2}\left[\begin{smallmatrix} L-t \\ t \end{smallmatrix}\right]_q$ is the generating function for partitions into exactly t parts with difference at least 2 between parts, such that each part $< L$.*

We wish to describe this generating function in "path" language. To this end we define an admissible sequence of integers Σ as an ordered sequence $(\sigma_i, \sigma_{i+1}, \ldots, \sigma_{f-1}, \sigma_f)$ such that $\sigma_l \in \{0, 1\}$ for $i \leq l \leq f$ and $\sigma_j \sigma_{j+1} = 0$ for $i \leq j \leq f-1$. Given Σ we can construct an admissible lattice path $P(\Sigma)$ by connecting points $(j; \sigma_j)$ and $(j+1; \sigma_{j+1})$ by the straight line segments. Thus, any admissible path is made out of three basic segments:



Note that a horizontal segment is always of height 0. On such a path we distinguish points $(j; \sigma_j = 1)$ with $i \neq j$ and $i \neq f$, which we call peaks. Clearly, the distance between two peaks is at least 2, as can be seen from Figure 1.

Let us denote the space of all admissible paths $P(\sigma_i = s, \sigma_{i+1}, \ldots, \sigma_f = b)$ with exactly t peaks and fixed end points $(i; s), (f; b)$ as $\mathbf{P}^t_{s,b}(i, f)$. For a given path $\in \mathbf{P}^t_{0,0}(0, L)$ we can identify the corresponding j-coordinates of its peaks with parts of partitions described in the Lemma (see Figure 1).

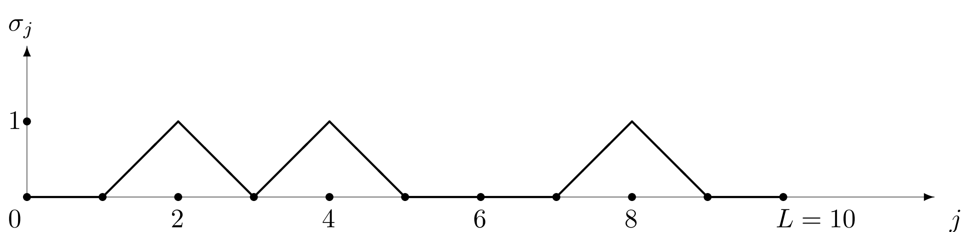


Figure 1. Path representation of the partition $14 = 2 + 4 + 8$ into 3 parts, each $< L = 10$.

Hence, we can reformulate this Lemma as

$$q^{t^2}\begin{bmatrix} L-t \\ t \end{bmatrix}_q = \sum_{\mathbf{P}^t_{0,0}(0,L)} q^{\sum_{j=1}^{L-1} j\sigma_j}, \tag{2.1}$$

where the symbol $\Sigma_{\mathbf{P}^t_{s,b}(i,f)}$ denotes the sum over all admissible paths $\in \mathbf{P}^t_{s,b}(i, f)$. The advantage of the path interpretation is that while partitions always have non-negative parts (by definition), j-coordinates of peaks in general may take on negative values.

If we move $\mathbf{P}^t_{0,0}(0,L)$ to the left by M units, then the resulting path space $\mathbf{P}^t_{0,0}(-M,L-M)$ can be used to prove that

$$q^{t^2-Mt}\begin{bmatrix} L-t \\ t \end{bmatrix}_q = \sum_{\mathbf{P}^t_{0,0}(-M,L-M)} q^{\sum_{j=1-M}^{L-M-1} j\sigma_j}. \tag{2.2}$$

Indeed, using (2.1) we have

$$\begin{aligned}\sum_{\mathbf{P}^t_{0,0}(-M,L-M)} q^{\sum_{j=1-M}^{L-M-1} j\sigma_j} &= \sum_{\mathbf{P}^t_{0,0}(0,L)} q^{\sum_{j=1}^{L-1}(j-M)\sigma_j} = q^{-Mt}\sum_{\mathbf{P}^t_{0,0}(0,L)} q^{\sum_{j=1}^{L-1} j\sigma_j} \\ &= q^{t^2-Mt}\begin{bmatrix} L-t \\ t \end{bmatrix}_q. \end{aligned} \tag{2.3}$$

And so,

$$\sum_{t\geq 0} q^{t^2-Mt}\begin{bmatrix} L-t \\ t \end{bmatrix}_q = \sum_{\mathbf{P}_{0,0}(-M,L-M)} q^{\sum_{j=1-M}^{L-M-1} j\sigma_j}, \tag{2.4}$$

where $\mathbf{P}_{s,b}(i,f)$ is defined the same way as $\mathbf{P}^t_{s,b}(i,f)$, except that we no longer require that the number of peaks is exactly t.

More generally, one can easily show that for $s,b\in\{0,1\}$

$$f_{s,b}(L,M,q) = C_{s,b}(-M,L-M,q) \tag{2.5}$$

with

$$f_{s,b}(L,M,q) := \sum_{t\geq 0} q^{t^2+st-Mt}\begin{bmatrix} L-t-s-b \\ t \end{bmatrix}_q \tag{2.6}$$

and

$$C_{s,b}(i,f,q) := \sum_{\mathbf{P}_{s,b}(i,f)} q^{\sum_{j=i+1}^{f-1} j\sigma_j}. \tag{2.7}$$

Next, for $0\leq M\leq L$ every admissible path $\in \mathbf{P}_{s,b}(-M,L-M)$ consists of two pieces joined together at point $(0; s'=0,1)$. The first piece belongs to $\mathbf{P}_{s,s'}(-M,0)$ and the second one to $\mathbf{P}_{s',b}(0,L-M)$. This observation is equivalent to

$$C_{s,b}(-M,L-M,q) = \sum_{s'=0}^{1} C_{s,s'}(-M,0,q)C_{s',b}(0,L-M,q). \tag{2.8}$$

Now, because

$$\sum_{\mathbf{P}_{s,s'}(-M,0)} q^{\sum_{j=1-M}^{-1} j\sigma_j} = \sum_{\mathbf{P}_{s',s}(0,M)} (\frac{1}{q})^{\sum_{j=1}^{M-1} j\sigma_j} \tag{2.9}$$

we infer that

$$C_{s,s'}(-M,0,q) = C_{s',s}(0,M,\frac{1}{q}). \tag{2.10}$$

Next, combining (2.5), (2.8) and (2.10), we arrive at

$$f_{s,b}(L,M,q) = \sum_{s'=0}^{1} f_{s',s}(M,0,\frac{1}{q})f_{s',b}(L-M,0,q). \tag{2.11}$$

The desired formula (1.14) is an easy consequence of (2.11) with $s = b = 0$ and the Rogers-Ramanujan identities (1.3) and (1.4), which we restate again as

$$f_{s,0}(L, 0, q) = \sum_{t\geq 0} q^{t^2+st} \begin{bmatrix} L-t-s \\ t \end{bmatrix}_q = \begin{cases} e_L(q), & \text{if } s = 0, \\ d_L(q), & \text{if } s = 1. \end{cases} \tag{2.12}$$

Remark: If we set $s = 1, b = 0, q = 1$ in (2.11), we immediately derive the following well known identity for the Fibonacci numbers $\mathrm{Fi}(L)$:

$$\mathrm{Fi}(L) = \mathrm{Fi}(M)\mathrm{Fi}(L - M + 1) + \mathrm{Fi}(M - 1)\mathrm{Fi}(L - M). \tag{2.13}$$

If we perform the substitution $M \to -M$ and use, according to (1.10),

$$\mathrm{Fi}(-M) = (-1)^{M+1}\mathrm{Fi}(M) \tag{2.14}$$

in (2.13), we obtain

$$(-1)^M \mathrm{Fi}(L) = \mathrm{Fi}(M + 1)\mathrm{Fi}(L + M) - \mathrm{Fi}(M)\mathrm{Fi}(L + M + 1), \tag{2.15}$$

which is a specialization of the Euler-Cassini formula. We would like to point out that (2.13) is "minus sign" free. As a result, the combinatorial proof of (2.13) given here is very different from that of Werman and Zeilberger [18]. Namely, their proof of (2.15) given in [18] made essential use of involution technique.

We conclude this section by pointing out that our analysis can be trivially extended to show that

$$\begin{aligned} f_{s,b}(L, M, q) &= \sum_{s'=0}^{1} f_{s,s'}(M + x, M, q)\, q^{s'x} f_{s',b}(L - M - x, -x, q) \\ &= \sum_{s'=0}^{1} q^{s'x} f_{s',s}(M + x, x, \frac{1}{q})\, f_{s',b}(L - M - x, -x). \end{aligned} \tag{2.16}$$

Note that (2.11) is (2.16) with $x = 0$.

3. Variants of the Andrews-Gordon Identities

For $\nu \in \mathbb{Z}_{>0}$, the analytical generalizations of the Rogers-Ramanujan identities known as Andrews-Gordon identities [2] can be stated as

$$\begin{aligned} &\sum_{n_1,n_2,\ldots,n_\nu} \frac{q^{(N_1^2+N_2^2+\cdots+N_\nu^2)+(N_s+N_{s+1}+\cdots+N_\nu)}}{(q)_{n_1}(q)_{n_2}\ldots(q)_{n_\nu}} \\ &\quad = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} \left\{ q^{2(2\nu+3)j^2+j(2\nu+3-2s)} - q^{(2j+1)((2\nu+3)j+s)} \right\} \\ &\quad = \frac{1}{\prod_{n\not\equiv 0,\pm s \pmod{2\nu+3}}(1-q^n)}, \quad s = 1, 2, \ldots, \nu + 1 \end{aligned} \tag{3.1}$$

with

$$N_i = \begin{cases} n_i + n_{i+1} + \cdots + n_\nu, & \text{if } 1 \leq i \leq \nu, \\ 0, & \text{if } i = \nu + 1. \end{cases} \tag{3.2}$$

Here and throughout, we adopt the convention that in the product

$$\prod_{n\not\equiv 0,\pm s \pmod{2\nu+3}} (1-q^n)$$

n takes on positive integer values not congruent to $0, \pm s$ modulo $2\nu+3$. Clearly, if $\nu=1$, (3.1) reduces to (1.1). As in the case of the Rogers-Ramanujan identities, the identities (3.1) have polynomial analogs. To describe these polynomial versions we need to introduce polynomials $\tilde{F}_{s,b}(L,q)$ defined for $0\le s,b\le\nu$ as follows

$$\tilde{F}_{s,b}(L,q) := \sum_{\mathbf{n}} q^{(N_1^2+\cdots+N_\nu^2)+(N_{s+1}+\cdots+N_\nu)} \begin{bmatrix}\mathbf{n}+\mathbf{m}\\ \mathbf{n}\end{bmatrix}_q \tag{3.3}$$

with

$$\mathbf{n}=(n_1,n_2,\ldots,n_\nu), \quad \mathbf{m}=(m_1,m_2,\ldots,m_\nu) \tag{3.4}$$

and

$$\begin{bmatrix}\mathbf{n}+\mathbf{m}\\ \mathbf{n}\end{bmatrix}_q = \prod_{i=1}^{\nu} \begin{bmatrix}n_i+m_i\\ n_i\end{bmatrix}_q, \tag{3.5}$$

where

$$m_i = L-2(N_1+N_2+\cdots+N_i)-\chi(i>s)(i-s)-\chi(i>b)(i-b) \tag{3.6}$$

and

$$\chi(i>a)=\begin{cases}1, & \text{if } i>a,\\ 0, & \text{if } i\le a.\end{cases} \tag{3.7}$$

Next, for $L\equiv s+b \pmod 2$ and $1\le s,b\le \nu+1$, we define polynomials $B_{s,b}(L,q)$ as

$$\begin{aligned}B_{s,b}(L,q) := \sum_{j=-\infty}^{\infty}\Bigg\{ & q^{2(2\nu+3)j^2+j(2\nu+3-2s)} \begin{bmatrix}L\\ \frac{L+s-b}{2}-j(2\nu+3)\end{bmatrix}_q \\ & - q^{(2j+1)((2\nu+3)j+s)} \begin{bmatrix}L\\ \frac{L-s-b}{2}-j(2\nu+3)\end{bmatrix}_q \Bigg\}.\end{aligned} \tag{3.8}$$

Equipped with these definitions we are in the position to state the polynomial analogs of (3.1), namely

$$\tilde{F}_{s,b}(L,q)=\begin{cases}B_{s+1,b+1}(L,q), & \text{if } L\equiv s+b \pmod 2,\\ B_{(2\nu+3)-(s+1),b+1}(L,q), & \text{if } L\not\equiv s+b \pmod 2,\end{cases} \tag{3.9}$$

with $0\le s,b\le\nu$.

For $b=\nu$, formulas (3.9) first appeared in the works of Foda, Quano [13] and Kirillov [15], for other values of b, these formulas were derived in [8]. It is important to keep in mind that in case $s\ne\nu$ and $b\ne\nu$, the summands in (3.3) may be non-zero in value even if $n_\nu=-1$.

To prove (3.9) the authors of [8] showed that both sides of (3.9) satisfy identical recurrences for $1\le b\le\nu$,

$$\begin{cases}\tilde{F}_{s,0}(L,q)=\tilde{F}_{s,1}(L-1,q),\\ \tilde{F}_{s,b}(L,q)=\tilde{F}_{s,b-1}(L-1,q)+\tilde{F}_{s,b+1-\delta_{b,\nu}}(L-1,q)+(q^{L-1}-1)\tilde{F}_{s,b}(L-2,q)\end{cases} \tag{3.10}$$

and the initial conditions

$$\tilde{F}_{s,b}(0,q)=\delta_{s,b} \tag{3.11}$$

where the Kronecker delta function $\delta_{i,j}$ is defined as usual as

$$\delta_{i,j}=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{otherwise.}\end{cases} \tag{3.12}$$

We would like to emphasize that there is more than one way to finitize the Andrews-Gordon identities. In particular, Warnaar [17] found polynomial versions of (3.1) involving q-multinomial coefficients.

Now we would like to alter the Andrews-Gordon identities in the spirit of Garrett et al. [14]. However, because we have more than one summation variable, it is not immediately clear how to accomplish this. Our guiding principle is that additional linear terms should modify the recurrences (3.10) in a minimal way by simple shifts, since this is precisely what happened in case of Rogers-Ramanujan identities. More specifically, we would like to have polynomials $F_{s,b}(L, M, q)$ satisfying the following relations for $1 \le b \le \nu, \;\; 0 \le s \le \nu$

$$\begin{cases} F_{s,0}(L, M, q) = F_{s,1}(L-1, M, q) \\ F_{s,b}(L, M, q) = F_{s,b-1}(L-1, M, q) + F_{s,b+1-\delta_{b,\nu}}(L-1, M, q) \\ \qquad\qquad\qquad + (q^{L-M-1} - 1)F_{s,b}(L-2, M, q). \end{cases} \tag{3.13}$$

The above requirement leads us to define

$$F_{s,b}(L, M, q) := \sum_{\mathbf{n}} q^{(N_1^2+N_2^2+\cdots+N_\nu^2)+(N_{s+1}+\cdots+N_\nu)-MN_1} \begin{bmatrix} \mathbf{n}+\mathbf{m} \\ \mathbf{n} \end{bmatrix}_q \tag{3.14}$$

with $s, b \in \{0, 1, \ldots, \nu\}$ and the rest of notations the same as in (3.3). In the Appendix we will prove that these polynomials indeed satisfy the recurrences (3.13).

Now, since $F_{s,b}(L, M, q)$ and $F_{s,b}(L-M, 0, q) = \tilde{F}_{s,b}(L-M, q)$ satisfy the same recursion relations we can write

$$F_{s,b}(L, M, q) = \sum_{s'=0}^{\nu} A_{s,s'}(M, q)\tilde{F}_{s',b}(L-M, q). \tag{3.15}$$

The connection coefficients $A_{s,s'}(M, q)$ can be easily determined from the boundary conditions

$$F_{s,b}(L, L, q) = \sum_{s'=0}^{\nu} A_{s,s'}(L, q)\tilde{F}_{s',b}(0, q) \overset{\text{by (3.11)}}{=} \sum_{s'=0}^{\nu} A_{s,s'}(L, q)\delta_{s',b} = A_{s,b}(L, q). \tag{3.16}$$

Making use of

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_{\frac{1}{q}} = q^{-nm} \begin{bmatrix} n+m \\ n \end{bmatrix}_q \tag{3.17}$$

one can easily verify that

$$F_{s,b}(L, L, q) = F_{b,s}(L, 0, \frac{1}{q}) = \tilde{F}_{b,s}(L, \frac{1}{q}). \tag{3.18}$$

Hence,

$$F_{s,b}(L, M, q) = \sum_{s'=0}^{\nu} \tilde{F}_{s',s}(M, \frac{1}{q})\tilde{F}_{s',b}(L-M, q), \tag{3.19}$$

which is a perfect analog of formula (2.11). Recalling (3.9), we can rewrite (3.19) as

$$F_{s,b}(L,M,q) = \sum_{\substack{s'=0 \\ s+s'\equiv M \ (\mathrm{mod}\ 2)}}^{\nu} B_{s'+1,s+1}(M,\frac{1}{q})\tilde{F}_{s',b}(L-M,q) + \sum_{\substack{s'=0 \\ s+s'\not\equiv M \ (\mathrm{mod}\ 2)}}^{\nu} B_{(2\nu+3)-(s'+1),s+1}(M,\frac{1}{q})\tilde{F}_{s',b}(L-M,q). \quad (3.20)$$

In the limit $L\to\infty$, (3.20) gives

$$\sum_{\mathbf{n}} \frac{q^{N_1^2+N_2^2+\cdots+N_\nu^2+(N_s+\cdots+N_\nu)-MN_1}}{(q)_{n_1}(q)_{n_2}\ldots(q)_{n_\nu}} = \sum_{\substack{s'=1 \\ s+s'\equiv M \ (\mathrm{mod}\ 2)}}^{\nu+1} \frac{B_{s',s}(M,\frac{1}{q})}{\prod_{n\not\equiv 0,\pm s' \ (\mathrm{mod}\ 2\nu+3)}(1-q^n)} + \sum_{\substack{s'=1 \\ s+s'\not\equiv M \ (\mathrm{mod}\ 2)}}^{\nu+1} \frac{B_{2\nu+3-s',s}(M,\frac{1}{q})}{\prod_{n\not\equiv 0,\pm s' \ (\mathrm{mod}\ 2\nu+3)}(1-q^n)}, \quad (3.21)$$

where we used the limiting formulas

$$\lim_{L\to\infty} \tilde{F}_{s-1,b}(L,q) = \frac{1}{\prod_{n\not\equiv 0,\pm s \ (\mathrm{mod}\ 2\nu+3)}(1-q^n)}, \quad (3.22)$$

which follow from

$$\lim_{L\to\infty} \begin{bmatrix} L \\ n \end{bmatrix}_q = \frac{1}{(q)_n} \quad (3.23)$$

and (3.1). It is easy to check that in case $\nu=1, s=2$ (3.21) reduces to (1.12).

4. Concluding Remarks

The interested reader may wonder if the combinatorial analysis given in Section 2 can be upgraded to explain the formulas (3.19). The answer to this question is affirmative. However, for $\nu>1$ the path interpretation of the $F_{s,b}(L,M,q)$ polynomials is much more involved than that of the $f_{s,b}(L,M,q)$ polynomials considered in Section 2. Here, one should deal with peaks of different heights [9] and in addition with certain boundary defects. We plan to come back to the combinatorial derivation of (3.19) in our future work. Here, we confine ourselves to remark that the introduction of an additional linear term $-MN_1$ in (3.21) amounts to the shift to the left by M units of Bressoud's path described in [9].

However, with respect to peaks with different heights we want to mention that these pop-up also in connection with another polynomial version of an identity of Garrett-Ismail-Stanton type. Namely, for integers $L,m\geq 0$ one has

$$q^m(q;q^2)_m \sum_{t\geq 0} \begin{bmatrix} L \\ 2t+1 \end{bmatrix}_q q^{2t^2+2(m+1)t} = S_m(q)T_{L+m}(q) - T_m(q)S_{L+m}(q), \quad (4.1)$$

where

$$S_m(q) = \sum_{t\geq 0} q^{2t^2} \begin{bmatrix} L \\ 2t \end{bmatrix}_q$$

and

$$T_m(q) = \sum_{t\geq 0} q^{2t^2+2t} \begin{bmatrix} L \\ 2t+1 \end{bmatrix}_q$$

are the Andrews-Santos polynomials discussed in [4]. Identity (4.1) arose in work of Andrews et al. [6] and yields, in the limit $L \to \infty$, a combination of Slater's identities (38) and (39) from [16].

The methods of Sections 1 and 2 above can be applied and lead to the following generalization which extends (4.1) also to negative integers: For $L \geq 0$ and arbitrary integer M,

$$\sum_{t\geq 0} \begin{bmatrix} L \\ 2t+1 \end{bmatrix}_q q^{2t^2-2Mt} = q^{M+1} S_{M+1}(\frac{1}{q}) T_{L-M-1}(q) + q^M T_{M+1}(\frac{1}{q}) S_{L-M-1}(q). \tag{4.2}$$

Here we understand that for negative indices, i.e., for $m \geq 0$, one has

$$S_{-m}(q) = (-1)^m \frac{q^{m^2}}{(q;q^2)_m} S_m(\frac{1}{q})$$

and

$$T_{-m}(q) = (-1)^{m+1} \frac{q^{m^2-1}}{(q;q^2)_m} T_m(\frac{1}{q}).$$

In the limit $L \to \infty$ one obtains from (4.2) another new identity of Garrett-Ismail-Stanton type. The proof and the underlying combinatorics will be presented in a forthcoming paper.

Our variants of the Andrews-Gordon identities were determined by the polynomial versions (3.9) and a requirement that the introduction of additional linear terms should modify the recursion relations (3.10) by trivial shifts as in (3.13). We intend to use more general polynomial versions of Andrews-Gordon identities containing ν finitization parameters to investigate the most general multisum

$$\sum_{\mathbf{n}} \frac{q^{N_1^2+\cdots+N_\nu^2-M_1N_1-M_2N_2-MN_\nu}}{(q)_{n_1}(q)_{n_2}\ldots(q)_{n_\nu}}. \tag{4.3}$$

Finally, we would like to mention that many new generalizations of Rogers-Ramanujan identities were introduced in [7] and proven in [8] and [12]. Techniques developed in this paper are adequate to produce and prove variants of all these identities.

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5. Appendix

Here we will prove the recurrences (3.13). To this end we need to define vectors $\mathbf{e}_i$ and $\mathbf{E}_{a,b}$ as

$$(\mathbf{e}_i)_j = \begin{cases} 1, & \text{if } i=j \text{ and } 1\le i\le \nu, \\ 0, & \text{otherwise,} \end{cases} \tag{5.1}$$

and

$$\mathbf{E}_{a,b} = \sum_{i=a}^{b} \mathbf{e}_i. \tag{5.2}$$

The first recurrence in (3.13) is trivial. To prove the second relation in (3.13), we expand $F_{s,b}(L,M,q)$ in a telescopic fashion as

$$\begin{aligned} F_{s,b}(L,M,q) &= \sum_{\mathbf{n}} q^{\Phi_s(\mathbf{N},M)} \begin{bmatrix} \mathbf{n}+\mathbf{m}-\mathbf{E}_{1,b} \\ \mathbf{n} \end{bmatrix}_q \\ &\quad + \sum_{\mathbf{n}} q^{\Phi_s(\mathbf{N},M)} \begin{bmatrix} \mathbf{n}+\mathbf{m}-\mathbf{E}_{1,b} \\ \mathbf{n}-\mathbf{e}_b \end{bmatrix}_q q^{m_b} \end{aligned}$$

$$+\sum_{\mathbf{n}} q^{\Phi_s(\mathbf{N},M)} \begin{bmatrix} \mathbf{n}+\mathbf{m}-\mathbf{E}_{1,b-1} \\ \mathbf{n}-\mathbf{e}_{b-1} \end{bmatrix}_q q^{m_{b-1}}$$
$$+\sum_{\mathbf{n}} q^{\Phi_s(\mathbf{N},M)} \begin{bmatrix} \mathbf{n}+\mathbf{m}-\mathbf{E}_{1,b-2} \\ \mathbf{n}-\mathbf{e}_{b-2} \end{bmatrix}_q q^{m_{b-2}}$$
$$\cdots$$
$$+\sum_{\mathbf{n}} q^{\phi_s(\mathbf{N},M)} \begin{bmatrix} \mathbf{n}+\mathbf{m}-\mathbf{E}_{1,1} \\ \mathbf{n}-\mathbf{e}_{1} \end{bmatrix}_q q^{m_1}, \tag{5.3}$$

where $\mathbf{N} = (N_1, N_2, \ldots, N_\nu)$ and

$$\Phi_s(\mathbf{N}, M) = (N_1^2 + N_1^2 + \cdots + N_\nu^2) + (N_{s+1} + \cdots + N_\nu) - MN_1, \tag{5.4}$$

with the rest of notations the same as in (3.3).

It is important to remember that the vectors $\mathbf{m}$ in (5.3) are actually functions of $\mathbf{N}$ and L as can be seen from (3.6). To make sure that (5.3) is a correct expansion of $F_{s,b}(L, M, q)$ we merge the first and second sum in (5.3) into a single sum using

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \begin{bmatrix} n+m-1 \\ n \end{bmatrix}_q + q^m \begin{bmatrix} n-1+m \\ n-1 \end{bmatrix}_q. \tag{5.5}$$

This single sum, in turn, can be merged with the third sum in (5.3). This process can be repeated until all sums are merged together to yield $F_{s,b}(L, M, q)$. It is trivial to recognize the first sum in (5.3) as $F_{s,b+1-\delta_{b,\nu}}(L-1, M, q)$. With regard to the last sum in (5.3), we perform the change $n_1 \to n_1 + 1$ to recognize it as $q^{L-M-1} F_{s,b}(L-2, M, q)$. So, it follows that

$$F_{s,b}(L, M, q) - F_{s,b+1-\delta_{b,\nu}}(L-1, M, q) - q^{L-M-1} F_{s,b}(L-2, M, q)$$
$$= \sum_{i=2}^{b} \sum_{\mathbf{n}} q^{\Phi_s(\mathbf{N},M)+m_i} \begin{bmatrix} \mathbf{n}+\mathbf{m}-\mathbf{E}_{1,i} \\ \mathbf{n}-\mathbf{e}_{i} \end{bmatrix}_q. \tag{5.6}$$

If $b = 1$, then the rhs of (5.6) is just zero, so

$$F_{s,1}(L, M, q) = F_{s,2}(L-1, M, q) + q^{L-M-1} F_{s,1}(L-2, M, q). \tag{5.7}$$

Combining (5.7) with the first recurrence in (3.13)

$$F_{s,0}(L-1, M, q) = F_{s,1}(L-2, M, q) \tag{5.8}$$

we obtain

$$F_{s,1}(L, M, q) = F_{s,0}(L-1, M, q) + F_{s,2}(L-1, M, q) + (q^{L-M-1} - 1) F_{s,1}(L-2, M, q), \tag{5.9}$$

as desired.

If $b \neq 0, 1$, we perform an "i" dependent change of the summation variables in (5.6)

$$\mathbf{n} \to \mathbf{n} + (\mathbf{e}_i - \mathbf{e}_{i-1}) - (\mathbf{e}_b - \mathbf{e}_{b-1}) \tag{5.10}$$

to obtain

$$F_{s,b}(L, M, q) - F_{s,b+1-\delta_{b,\nu}}(L-1, M, q) - q^{L-M-1} F_{s,b}(L-2, M, q)$$
$$= \sum_{i=1}^{b-1} \sum_{\mathbf{n}} q^{\Phi_s(\mathbf{N},M)+m_i+m_b-m_{b-1}} \begin{bmatrix} \mathbf{n}+\mathbf{m}-\mathbf{E}_{1,b-1}+\mathbf{e}_{b-1}-\mathbf{e}_b-\mathbf{E}_{i,b-1} \\ \mathbf{n}+\mathbf{e}_{b-1}-\mathbf{e}_b-\mathbf{e}_i \end{bmatrix}_q. \tag{5.11}$$

Next, we rewrite the polynomial $F_{s,b-1}(L-1,M,q)$ in terms of the same $\mathbf{n}$ vectors as in (5.11)

$$F_{s,b-1}(L-1,M,q)=\sum_{\mathbf{n}} q^{\Phi_s(\mathbf{N},M)+m_b-m_{b-1}+1}\begin{bmatrix}\mathbf{n}+\mathbf{m}-\mathbf{E}_{1,b-1}+\mathbf{e}_{b-1}-\mathbf{e}_b\\ \mathbf{n}+\mathbf{e}_{b-1}-\mathbf{e}_b\end{bmatrix}_q \tag{5.12}$$

and then expand it in a telescopic fashion to get

$$\begin{aligned}&F_{s,b-1}(L-1,M,q)\\ &\quad=\sum_{\mathbf{n}} q^{\Phi_s(\mathbf{N},M)+m_b-m_{b-1}+1}\begin{bmatrix}\mathbf{n}+\mathbf{m}-\mathbf{E}_{1,b-1}+\mathbf{e}_{b-1}-\mathbf{e}_b-\mathbf{E}_{1,b-1}\\ \mathbf{n}+\mathbf{e}_{b-1}-\mathbf{e}_b\end{bmatrix}_q\\ &\quad+\sum_{i=1}^{b-1}\sum_{\mathbf{n}} q^{\Phi_s(\mathbf{N},M)+m_i+m_b-m_{b-1}}\begin{bmatrix}\mathbf{n}+\mathbf{m}-\mathbf{E}_{1,b-1}+\mathbf{e}_{b-1}-\mathbf{e}_b-\mathbf{E}_{i,b-1}\\ \mathbf{n}+\mathbf{e}_{b-1}-\mathbf{e}_b-\mathbf{e}_i\end{bmatrix}_q.\end{aligned} \tag{5.13}$$

Once again, performing the change $\mathbf{n}\to\mathbf{n}-\mathbf{e}_{b-1}+\mathbf{e}_b$ in the first sum on the rhs of (5.13) we recognize it as $F_{s,b}(L-2,M,q)$. Hence,

$$\begin{aligned}&F_{s,b-1}(L-1,M,q)-F_{s,b}(L-2,M,q)\\ &\quad=\sum_{i=1}^{b-1}\sum_{\mathbf{n}} q^{\Phi_s(\mathbf{N},M)+m_i+m_b-m_{b-1}}\begin{bmatrix}\mathbf{n}+\mathbf{m}+\mathbf{e}_{b-1}-\mathbf{e}_b-\mathbf{E}_{i,b-1}\\ \mathbf{n}+\mathbf{e}_{b-1}-\mathbf{e}_b-\mathbf{e}_i\end{bmatrix}_q.\end{aligned} \tag{5.14}$$

Comparing the rhs of (5.11) and (5.14) we immediately infer that

$$\begin{aligned}F_{s,b}(L,M,q)&=F_{s,b-1}(L-1,M,q)+F_{s,b+1-\delta_{b,\nu}}(L-1,M,q)\\ &\quad+(q^{L-M-1}-1)F_{s,b}(L-2,M,q),\end{aligned} \tag{5.15}$$

as desired.

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