

SYMBOLIC SUMMATION WITH SINGLE-NESTED SUM EXTENSIONS (EXTENDED VERSION)

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ABSTRACT. We present a streamlined and refined version of Karr’s summation algorithm. Karr’s original approach constructively decides the telescoping problem in $\Pi\Sigma$ -fields, a very general class of difference fields that can describe rational terms of arbitrarily nested indefinite sums and products. More generally, our new algorithm can decide constructively if there exists a so called single-nested $\Pi\Sigma$ -extension over a given $\Pi\Sigma$ -field in which the telescoping problem for f can be solved in terms that are not more nested than f itself. This allows to eliminate an indefinite sum over f by expressing it in terms of additional sums that are not more nested than f . Moreover, our refined algorithm contributes to definite summation: it can decide constructively if the creative telescoping problem for a fixed order can be solved in single-nested Σ^* -extensions that are less nested than the definite sum itself.

1. INTRODUCTION

Let (\mathbb{F}, σ) be a difference field, i.e., a field¹ \mathbb{F} together with a field automorphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$, and let \mathbb{K} be its constant field, i.e., $\mathbb{K} = \text{const}_\sigma \mathbb{F} := \{k \in \mathbb{F} \mid \sigma(k) = k\}$. Then Problem *PFLDE* plays an important role in symbolic summation.

Problem *PFLDE*: Solving **P**arameterized **F**irst Order **L**inear **D**ifference **E**quations

- Given $a_1, a_2 \in \mathbb{F}^*$ and $(f_1, \dots, f_n) \in \mathbb{F}^n$.
 - Find all $g \in \mathbb{F}$ and $(c_1, \dots, c_n) \in \mathbb{K}^n$ with $a_1 \sigma(g) + a_2 g = \sum_{i=1}^n c_i f_i$.
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For instance, if one takes the field of rational functions $\mathbb{F} = \mathbb{K}(k)$ with the shift $\sigma(k) = k + 1$ and specializes to $n = 1$, $a_1 = 1$ and $a_2 = -1$, one considers the telescoping problem for a rational function $f_1 = f'(k) \in \mathbb{K}(k)$. Moreover, if $\mathbb{K} = \mathbb{K}'(m)$ and $f_i = f'(m + i - 1, k) \in \mathbb{K}'(m)(k)$ for $1 \leq i \leq n$, one formulates the creative telescoping problem [14] of order $n - 1$ for definite rational sums.

More generally, $\Pi\Sigma$ -fields, introduced in [6, 7], are difference fields (\mathbb{F}, σ) with constant field \mathbb{K} where $\mathbb{F} := \mathbb{K}(t_1) \dots (t_e)$ is a rational function field and the application of σ on the t_i ’s is recursively defined over $1 \leq i \leq e$ with $\sigma(t_i) = \alpha_i t_i + \beta_i$ for $\alpha_i, \beta_i \in \mathbb{K}(t_1) \dots (t_{i-1})$; we omitted some technical conditions given in Section 2. Note that $\Pi\Sigma$ -fields enable to describe a huge class of sequences, like hypergeometric terms, as shown in [13], or most d’Alembertian solutions [1, 9], a subclass of Liouvillian solutions [5] of linear recurrences. More generally, $\Pi\Sigma$ -fields allow to describe rational terms consisting of arbitrarily nested indefinite sums and products. We want to emphasize that the nested depth of these sums and products gives a

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¹Throughout this paper all fields will have characteristic 0.

measure of the complexity of expressions. This can be carried over to $\Pi\Sigma$ -fields by introducing the *depth* of t_i as the number of recursive definition steps that are needed to describe the application of σ on t_i ; for more details see Section 2. Moreover, the depth of $f \in \mathbb{F}$ is the maximum depth of the t_i 's that occur in f , and the depth of (\mathbb{F}, σ) is the maximum depth of all the t_i .

The main result in [6] is an algorithm that solves Problem *PFLDE* and therefore the telescoping and creative telescoping problem for a given $\Pi\Sigma$ -field (\mathbb{F}, σ) where the constant field \mathbb{K} is σ -computable. This means that **(1)** for any $k \in \mathbb{K}$ one can decide if $k \in \mathbb{Z}$, **(2)** polynomials in $\mathbb{K}[t_1, \dots, t_n]$ can be factored over \mathbb{K} , and **(3)** one knows how to compute a basis of $\{(n_1, \dots, n_k) \in \mathbb{Z}^k \mid c_1^{n_1} \dots c_k^{n_k} = 1\}$ for $(c_1, \dots, c_k) \in \mathbb{K}^k$ which is a submodule of \mathbb{Z}^k over \mathbb{Z} . For instance, any rational function field $\mathbb{K} = \mathbb{A}(x_1, \dots, x_r)$ over an algebraic number field \mathbb{A} is σ -computable; see [13].

In this paper we will present a streamlined and simplified version of Karr's original algorithm [6] for Problem *PFLDE* using Bronstein's denominator bound [2] and results from [6, 12, 10, 11]. Afterwards we will extend this approach to an algorithm that can solve

Problem *RS*: Refined Summation

- Given a $\Pi\Sigma$ -field (\mathbb{F}, σ) with depth d , constant field \mathbb{K} and $(f_1, \dots, f_n) \in \mathbb{F}^n$.
 - *Decide* constructively if there are $(0, \dots, 0) \neq (c_1, \dots, c_n) \in \mathbb{K}^n$ and $g \in \mathbb{F}(x_1) \dots (x_e)$ for $\sigma(g) - g = \sum_{i=1}^n c_i f_i$ in an extended $\Pi\Sigma$ -field $(\mathbb{F}(x_1) \dots (x_e), \sigma)$ with depth d and $\sigma(x_i) = \alpha_i x_i + \beta_i$ where $\alpha_i, \beta_i \in \mathbb{F}$.
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Suppose we fail to find a solution g with $\sigma(g) - g = f$ in a given $\Pi\Sigma$ -field (\mathbb{F}, σ) with depth d and $f \in \mathbb{F}^*$ with depth d , but there exists such an extended $\Pi\Sigma$ -field $(\mathbb{F}(x_1) \dots (x_e), \sigma)$ and a solution g with depth d for $\sigma(g) - g = f$. Then our new algorithm can compute such an extension with such a solution g . As a side result we will show that it suffices to restrict to the sum case, i.e., $\sigma(x_i) - x_i \in \mathbb{F}$. In some sense our results shed new constructive light on Karr's Fundamental Theorem [6].

For instance, in Karr's approach [6] one can find the right hand side in (1) only by setting up manually the corresponding $\Pi\Sigma$ -field in terms of the harmonic numbers $H_n := \sum_{i=1}^n \frac{1}{i}$ and the generalized versions $H_n^{(r)} := \sum_{i=1}^n \frac{1}{i^r}$, $r > 1$, whereas with our new algorithm the underlying $\Pi\Sigma$ -field is constructed completely automatically. Additional examples are

$$\begin{aligned} \sum_{k=1}^n \frac{1}{k} \sum_{j=1}^k \frac{1}{j} \sum_{i=1}^j \frac{1}{i} &= \frac{1}{6} [H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}], \\ \sum_{k=1}^n \frac{1}{k} \sum_{i=1}^k \frac{1}{H_i} &= -n + H_n \sum_{i=1}^n \frac{1}{H_i} + \sum_{i=1}^n \frac{1}{iH_i}, \\ \sum_{k=0}^a \left(\sum_{i=0}^k \binom{n}{i} \right)^2 &= (n-a) \binom{n}{a} \sum_{i=0}^a \binom{n}{i} - \frac{n-2a-2}{2} \left(\sum_{i=0}^a \binom{n}{i} \right)^2 - \frac{n}{2} \sum_{i=1}^n \binom{n}{i}^2. \end{aligned} \tag{1}$$

Our new approach also refines creative telescoping: we might find a recurrence of smaller order by introducing additional sums with depths smaller than the definite sum.

All these algorithms have been implemented in form of the summation package *Sigma* in the computer algebra system Mathematica. The wide applicability of this new approach is illustrated for instance in [9, 8, 4].

2. REFINED SUMMATION IN $\Pi\Sigma$ -FIELDS

First we introduce some notations and definitions. Let (\mathbb{F}, σ) be a difference field with $\mathbb{K} = \text{const}_\sigma \mathbb{F}$, $\mathbf{a} = (a_1, a_2) \in \mathbb{F}^2$ and $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. For any $\mathbf{h} = (h_1, \dots, h_n) \in \mathbb{F}^n$ and $p \in \mathbb{F}$ we write $\mathbf{f} \mathbf{h} := \sum_{i=1}^n f_i h_i$, $\sigma(\mathbf{h}) := (\sigma(h_1), \dots, \sigma(h_n))$, and $\mathbf{h} p := (h_1 p, \dots, h_n p)$. We define $\mathbf{0}_n := (0, \dots, 0) \in \mathbb{K}^n$, and write $\mathbf{0} = \mathbf{0}_n$ if it is clear from the context. We call \mathbf{a} *homogeneous over* \mathbb{F} if $a_1 a_2 \neq 0$ and $a_1 \sigma(g) + a_2 g = 0$ for some $g \in \mathbb{F}^*$.

Now let \mathbb{V} be a subspace of \mathbb{F} over \mathbb{K} and suppose that $\mathbf{a} \neq \mathbf{0}$. Then we define the *solution space* $V(\mathbf{a}, \mathbf{f}, \mathbb{V})$ as the subspace $\{(c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{V} \mid a_1 \sigma(g) + a_2 g = \sum_{i=1}^n c_i f_i\}$ of the vector space $\mathbb{K}^n \times \mathbb{F}$ over \mathbb{K} . By difference field theory [3], the dimension is at most $n + 1$; see also [9, 10]. Therefore Problem *PFLDE* is equivalent to find a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$.

A difference field (\mathbb{E}, σ') is a *difference field extension* of (\mathbb{F}, σ) if \mathbb{F} is a subfield of \mathbb{E} and $\sigma'(g) = \sigma(g)$ for $g \in \mathbb{F}$; note that from now σ and σ' are not distinguished anymore.

A difference field extension $(\mathbb{F}(t), \sigma)$ of (\mathbb{F}, σ) is a Π - (resp. Σ^* -) *extension* if $\mathbb{F}(t)$ is a rational function field, $\sigma(t) = at$ ($\sigma(t) = t + a$ resp.) for some $a \in \mathbb{F}^*$ and $\text{const}_\sigma \mathbb{F}(t) = \text{const}_\sigma \mathbb{F}$. A difference field extension $(\mathbb{F}(t), \sigma)$ of (\mathbb{F}, σ) is a Σ -*extension* if $\mathbb{F}(t)$ is a rational function field, $\sigma(t) = \alpha t + \beta$ for some $\alpha, \beta \in \mathbb{F}^*$, $\text{const}_\sigma \mathbb{F}(t) = \text{const}_\sigma \mathbb{F}$, and the following two properties hold: (1) there does not exist a $g \in \mathbb{F}(t) \setminus \mathbb{F}$ with $\sigma(g)/g \in \mathbb{F}$, and (2) if there is a $g \in \mathbb{F}^*$ and $n \neq 0$ with $\sigma(g)/g = \alpha^n$ then there is also a $g \in \mathbb{F}^*$ with $\sigma(g)/g = \alpha$. Note that any Σ^* -extension is also a Σ -extension; for more details see [6, 7, 2, 9, 13]. A $\Pi\Sigma$ -*extension* is either a Π - or a Σ -extension. A difference field extension $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ of (\mathbb{F}, σ) is a (*nested*) $\Sigma^*/\Pi\Sigma$ -*extension* if $(\mathbb{F}(t_1) \dots (t_i), \sigma)$ is a $\Sigma^*/\Pi\Sigma$ -extension of $(\mathbb{F}(t_1) \dots (t_{i-1}), \sigma)$ for all $1 \leq i \leq e$; for $i = 0$ we define $\mathbb{F}(t_1) \dots (t_{i-1}) = \mathbb{F}$. Note that $e = 0$ gives the trivial extension.

For $\mathbb{H} \subseteq \mathbb{F}$, a $\Pi\Sigma$ -extension $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ of (\mathbb{F}, σ) is *single-nested over* \mathbb{H} , or in short *over* \mathbb{H} , if $\sigma(t_i) = \alpha_i t_i + \beta_i$ with $\alpha_i, \beta_i \in \mathbb{H}$ for all $1 \leq i \leq e$. A $\Pi\Sigma$ -extension of (\mathbb{F}, σ) is called *single-nested*, if it is single-nested over \mathbb{F} .

Finally, a $\Pi\Sigma$ -*field* (\mathbb{F}, σ) *over* \mathbb{K} is a $\Pi\Sigma$ -extension of (\mathbb{K}, σ) with $\text{const}_\sigma \mathbb{K} = \mathbb{K}$, i.e., $\text{const}_\sigma \mathbb{F} = \mathbb{K}$.

In [6] alternative definitions of $\Pi\Sigma$ -extensions are introduced that allow to decide constructively if an extension $(\mathbb{F}(t), \sigma)$ of (\mathbb{F}, σ) is a $\Pi\Sigma$ -extension under the assumption that (\mathbb{F}, σ) is a $\Pi\Sigma$ -field over a σ -computable \mathbb{K} . For instance, for Σ^* -extensions there is the following result given in [7, Theorem 2.3] or [9, Corollary 2.2.4].

Theorem 1. *Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of (\mathbb{F}, σ) . Then this is a Σ^* -extension iff $\sigma(t) = t + \beta$, $t \notin \mathbb{F}$, $\beta \in \mathbb{F}$, and there is no $g \in \mathbb{F}$ with $\sigma(g) - g = \beta$.*

In particular, this result states that indefinite summation and building up Σ^* -extensions are closely related. Namely, if one fails to find a $g \in \mathbb{F}$ with $\sigma(g) - g = \beta \in \mathbb{F}$, i.e., one cannot solve the telescoping problem in \mathbb{F} , one can adjoin the solution t with $\sigma(t) + t = \beta$ to \mathbb{F} in form of the Σ^* -extension $(\mathbb{F}(t), \sigma)$ of (\mathbb{F}, σ) .

Our refined simplification strategy for a given sum is as follows: If we fail to solve the telescoping problem, we do not adjoin immediately the sum in form of a Σ^* -extension, but we first try to find an appropriate $\Pi\Sigma$ -extension in which the sum can be formulated less nested.

These ideas can be clarified further with the depth-function. Let $\mathbb{F} = \mathbb{K}(t_1, \dots, t_e)$ be a function field over \mathbb{K} . Then for $g = \frac{g_1}{g_2} \in \mathbb{F}^*$ with $g_i \in \mathbb{K}[t_1, \dots, t_e]$ and $\gcd_{\mathbb{K}[t_1, \dots, t_e]}(g_1, g_2) = 1$ we define the support of g , in short $\text{supp}_{\mathbb{F}}(g)$, as those t_i that occur in g_1 or g_2 . Then for a $\Pi\Sigma$ -field (\mathbb{F}, σ) over \mathbb{K} with $\mathbb{F} := \mathbb{K}(t_1) \dots (t_e)$ and $\sigma(t_i) = \alpha_i t_i + \beta_i$ for $\alpha_i, \beta_i \in \mathbb{K}(t_1) \dots (t_{i-1})$, the depth-function $\text{depth} : \mathbb{F} \rightarrow \mathbb{N}_0$ is defined recursively as follows. For any $g \in \mathbb{K}$ set $\text{depth}(g) = 0$. If the depth-function is defined for $(\mathbb{K}(t_1) \dots (t_{i-1}), \sigma)$ with $i > 1$, we define $\text{depth}(t_i) = \max(\text{depth}(\alpha_i), \text{depth}(\beta_i)) + 1$ and for $g \in \mathbb{K}(t_1) \dots (t_i)$ we define $\text{depth}(g) = \max(\{\text{depth}(x) \mid x \in \text{supp}_{\mathbb{K}(t_1, \dots, t_i)}(g)\} \cup \{0\})$. The depth of (\mathbb{F}, σ) , in short $\text{depth}(\mathbb{F})$, is the maximal depth of all elements in \mathbb{F} , i.e., $\text{depth}(\mathbb{F})$ is equal to $\max(0, \text{depth}(t_1), \dots, \text{depth}(t_e))$. We say that a $\Pi\Sigma/\Sigma^*$ -extension $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ of a $\Pi\Sigma$ -field (\mathbb{F}, σ) has maximal depth d if $\text{depth}(t_i) \leq d$ for all $1 \leq i \leq e$.

Now we can reformulate Problem *RS* as follows. *Given a $\Pi\Sigma$ -field (\mathbb{F}, σ) with depth d and $\mathbf{f} \in \mathbb{F}^n$. Decide constructively if there is a single-nested $\Pi\Sigma$ -extension (\mathbb{E}, σ) of (\mathbb{F}, σ) with maximal depth d , $g \in \mathbb{E}$ and $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}^n$ such that $\sigma(g) - g = \mathbf{c} \mathbf{f}$.*

Example 1. Denote the left side in (1) with $S_n^{(3)}$ and define $S_n^{(1)} := \sum_{i=1}^n \frac{1}{i}$ and $S_n^{(2)} := \sum_{j=1}^n S_j^{(1)}/j$. In the straightforward summation approach one applies usual telescoping which results in the $\Pi\Sigma$ -field $(\mathbb{Q}(t_1)(t_2)(t_3)(t_4), \sigma)$ over \mathbb{Q} with $\sigma(t_1) = t_1 + 1$, $\sigma(t_2) = t_2 + \frac{1}{t_1+1}$, $\sigma(t_3) = t_3 + \sigma(\frac{t_2}{t_1})$ and $\sigma(t_4) = t_4 + \sigma(\frac{t_3}{t_1})$, i.e., there is no $g \in \mathbb{Q}(t_1)$ with $\sigma(g) - g = \frac{1}{t_1+1}$ and no $g \in \mathbb{Q}(t_1) \dots (t_r)$ with $\sigma(g) - g = \sigma(\frac{t_r}{t_1})$ for $r = 2, 3$. Then t_r represents $S_n^{(r-1)}$ with $\text{depth}(t_r) = r$ for $r = 2, 3, 4$, and $\text{depth}(\mathbb{Q}(t_1) \dots (t_4)) = 4$. But with our refined summation approach we obtain the following improvement starting from the $\Pi\Sigma$ -field (\mathbb{F}, σ) with $\mathbb{F} := \mathbb{Q}(t_1)(t_2)$. We find the Σ^* -extension $(\mathbb{F}(s), \sigma)$ of (\mathbb{F}, σ) with $\sigma(s) = s + \frac{1}{(t_1+1)^2}$ with the solution $g := \frac{t_2^2+s}{2}$ for $\sigma(g) - g = \sigma(\frac{t_2}{t_1})$ that represents the sum $S_n^{(2)}$. Moreover, we find the Σ^* -extension $(\mathbb{F}(s)(s'), \sigma)$ of $(\mathbb{F}(s), \sigma)$ with $\sigma(s') = s' + \frac{1}{(t_1+1)^3}$ and the solution $g' = \frac{1}{6}(t_2^3 + 3t_2 s + 2s')$ for $\sigma(g') - g' = \sigma(g/t_1)$. Then $S_n^{(3)}$ is represented by g' with $\text{depth}(g') = 2$ which gives the right hand side of identity (1).

Besides refined indefinite summation, we obtain a generalized version of creative telescoping in $\Pi\Sigma$ -fields. Suppose that the sequences $f'(m+i-1, k)$ can be represented with $f_i \in \mathbb{F}$ for $i \geq 1$ in a $\Pi\Sigma$ -field (\mathbb{F}, σ) over $\mathbb{K}(m)$ with $\text{depth}(f_i) = d$. Moreover assume that we do not find a $g \in \mathbb{F}$ and $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}(m)^n$ with $\sigma(g) - g = \mathbf{c} \mathbf{f}$ for $\mathbf{f} = (f_1, \dots, f_n)$. Then the usual strategy is to increase n , i.e., the order of the possibly resulting creative telescoping recurrence. But if we find a solution for Problem *RS*, we derive a recurrence of order $n-1$ in terms of sum extensions with maximal depth d .

Summarizing, for telescoping and creative telescoping we are interested in finding a single-nested $\Pi\Sigma$ -extension in which a nontrivial linear combination of (f_1, \dots, f_n) in the solution space exists. More generally, we will ask for those extensions that will give us additional linear combinations. To make this more precise, we define for any $\mathbb{A} \subseteq \mathbb{F}^{n+1}$ the set $\Pi_n(\mathbb{A}) := \{(a_1, \dots, a_n) \mid (a_1, \dots, a_n, a_{n+1}) \in \mathbb{A}\}$.

Definition 1. Let (\mathbb{E}, σ) be a $\Pi\Sigma$ -field over \mathbb{K} with depth d , $1 \leq \delta \leq d+1$, and $\mathbf{f} \in \mathbb{E}^n$. We call a $\Pi\Sigma$ -extension (\mathbb{G}, σ) of (\mathbb{E}, σ) single-nested δ -complete for \mathbf{f} if for all single-nested $\Pi\Sigma$ -extensions (\mathbb{H}, σ) of (\mathbb{E}, σ) with maximal depth δ we have

$$\Pi_n(\mathbb{V}((1, -1), \mathbf{f}, \mathbb{H})) \subseteq \Pi_n(\mathbb{V}((1, -1), \mathbf{f}, \mathbb{G})). \quad (2)$$

In this paper we solve the following problem. *Given a $\Pi\Sigma$ -field (\mathbb{E}, σ) over a σ -computable \mathbb{K} with depth d , $\mathbf{f} \in \mathbb{E}^n$ and $\delta \in \mathbb{N}$ with $1 \leq \delta \leq d + 1$; compute a single-nested Σ^* -extension (\mathbb{G}, σ) of (\mathbb{E}, σ) with maximal depth δ which is single-nested δ -complete for \mathbf{f} , and compute a basis of $V((1, -1), \mathbf{f}, \mathbb{G})$. Note that Problem *RS* for single-nested $\Pi\Sigma$ -extension is contained in this problem by setting $\delta := d$.*

3. A MORE GENERAL PROBLEM

In order to treat the problem stated in the previous paragraph, we solve the more general problem to find an \mathbb{F} -complete extension of (\mathbb{E}, σ) for \mathbf{f} defined in

Definition 2. *Let (\mathbb{E}, σ) be a $\Pi\Sigma$ -extension of (\mathbb{F}, σ) and $\mathbf{f} \in \mathbb{E}^n$. We call a $\Pi\Sigma$ -extension (\mathbb{G}, σ) of (\mathbb{E}, σ) single-nested \mathbb{F} -complete for \mathbf{f} , or in short \mathbb{F} -complete for \mathbf{f} , if (2) holds for all $\Pi\Sigma$ -extensions (\mathbb{H}, σ) of (\mathbb{E}, σ) over \mathbb{F} .*

The following lemma is crucial in order to show in Theorem 2 that there exists a Σ^* -extension of (\mathbb{E}, σ) over \mathbb{F} which is \mathbb{F} -complete for \mathbf{f} . This means that it suffices to restrict to Σ^* -extensions. Moreover this lemma is needed to prove Theorem 6 which gives us the essential idea how one can compute such \mathbb{F} -complete extensions.

Lemma 1. *Let (\mathbb{E}, σ) be a $\Pi\Sigma$ -extension of (\mathbb{F}, σ) and $f \in \mathbb{E}^*$. If there exists a single-nested $\Pi\Sigma$ -extension (\mathbb{G}, σ) of (\mathbb{E}, σ) over \mathbb{F} with a $g \in \mathbb{G} \setminus \mathbb{E}$ such that $\sigma(g) - g = f$ then there exists a Σ^* -extension $(\mathbb{E}(s), \sigma)$ of (\mathbb{E}, σ) over \mathbb{F} with a $w \in \mathbb{E}$ such that $\sigma(s + w) - (s + w) = f$.*

Proof. Let (\mathbb{G}, σ) be a $\Pi\Sigma$ -extension of (\mathbb{E}, σ) over \mathbb{F} , i.e., $\mathbb{G} = \mathbb{E}(t_1) \dots (t_e)$ with $\sigma(t_i) = \alpha_i t_i + \beta_i$ and $\alpha_i, \beta_i \in \mathbb{F}$, and suppose that there is a $g \in \mathbb{G} \setminus \mathbb{E}$ with $\sigma(g) - g = f$. Then by Karr's Fundamental Theorem [6, Theorem 24], see also [7, Section 4], it follows that $g = \sum_{i=0}^e c_i t_i + w$ for some $w \in \mathbb{E}$ and $c_i \in \mathbb{K}$, where $c_i = 0$ if $\sigma(t_i) - t_i \notin \mathbb{F}$. In particular, $\mathbf{0} \neq (c_1, \dots, c_e)$, since $g \notin \mathbb{E}$. Now let $\mathbb{E}(s)$ be a rational function field and suppose that the difference field extension $(\mathbb{E}(s), \sigma)$ of (\mathbb{E}, σ) with $\sigma(s) - s = \sum_{i=1}^e c_i (\sigma(t_i) - t_i) =: \beta \in \mathbb{F}$ is not a Σ^* -extension. Then by Theorem 1 we can take a $g' \in \mathbb{E}$ with $\sigma(g') - g' = \beta$. Let j be maximal such that $c_j \neq 0$. Then we have $\sigma(v) - v = \sigma(t_j) - t_j \in \mathbb{F}$ for $v := \frac{1}{c_j}(g' - \sum_{i=1}^{j-1} c_i t_i) \in \mathbb{E}(t_1) \dots (t_{j-1})$, and thus $(\mathbb{E}(t_1) \dots (t_{j-1})(t_j), \sigma)$ is not a Σ^* -extension of $(\mathbb{E}(t_1) \dots (t_{j-1}), \sigma)$ by Theorem 1, a contradiction. Hence $(\mathbb{E}(s), \sigma)$ is a Σ^* -extension of (\mathbb{E}, σ) over \mathbb{F} , and $\sigma(s + w) - (s + w) = \sum_{i=1}^e c_i (\sigma(t_i) - t_i) + \sigma(w) - w = \sigma(g) - g = f$. \square

Observe that Lemma 1 follows immediately by Theorem 1 if one restricts to the special case $\mathbb{E} = \mathbb{F}$. For the case $\mathbb{F} \subsetneq \mathbb{E}$, in which we are actually interested, we needed Karr's Fundamental Theorem [6].

Theorem 2. *Let (\mathbb{E}, σ) be a $\Pi\Sigma$ -extension of (\mathbb{F}, σ) and $\mathbf{f} \in \mathbb{E}^n$. Then there is a Σ^* -extension of (\mathbb{E}, σ) over \mathbb{F} which is \mathbb{F} -complete for \mathbf{f} .*

Proof. Let (\mathbb{G}, σ) be a Σ^* -extension of (\mathbb{E}, σ) over \mathbb{F} which is not \mathbb{F} -complete for \mathbf{f} . Then we can take a $\mathbf{c} \in \mathbb{K}^n$ such that $\sigma(g) - g = \mathbf{c}\mathbf{f} \in \mathbb{E}$ has a solution in some $\Pi\Sigma$ -extension of (\mathbb{E}, σ) over \mathbb{F} , but no solution in \mathbb{E} . Then by Lemma 1 it follows that there is a Σ^* -extension $(\mathbb{E}(s), \sigma)$ of (\mathbb{E}, σ) over \mathbb{F} with $\sigma(s + w) - (s + w) = \mathbf{c}\mathbf{f}$ for some $w \in \mathbb{E}$. Observe that there also does not exist an $h \in \mathbb{G}$ with $\sigma(h) - h = \beta \in \mathbb{F}$. Otherwise we would have $\sigma(h + w) - (h + w) = \mathbf{c}\mathbf{f}$ with $h + w \in \mathbb{G}$, a contradiction. Consequently, by Theorem 1 also $(\mathbb{G}(s), \sigma)$ is a Σ^* -extension of (\mathbb{G}, σ) with $\sigma(s) = s + \beta$ and therefore a Σ^* -extension of (\mathbb{E}, σ) over \mathbb{F} . Since $\Pi_n(V((1, -1), \mathbf{f}, \mathbb{G}))$ is a proper subspace of $\Pi_n(V((1, -1), \mathbf{f}, \mathbb{G}(s)))$ and those

spaces have dimension at most n , this argument can be repeated at most n times before an \mathbb{F} -complete Σ^* -extension is reached. \square

In the following we will represent the $\Pi\Sigma$ -field (\mathbb{E}, σ) in such a way that one can find a single-nested δ -complete extension of (\mathbb{E}, σ) for \mathbf{f} by finding an \mathbb{F} -complete extension over a certain subfield $\mathbb{F} \subseteq \mathbb{E}$.

Let $\mathbb{G} := \mathbb{F}(s_1) \dots (s_u)(x)(t_1) \dots (t_v)$ be a field of rational functions. Then the field $\mathbb{H} := \mathbb{F}(x)(s_1) \dots (s_u)(t_1) \dots (t_v)$ is isomorphic with \mathbb{G} by the field isomorphism $\tau : \mathbb{G} \rightarrow \mathbb{H}$ with $\tau(f) = f$ for all $f \in \mathbb{F}$, $\tau(s_i) = s_i$, $\tau(x) = x$ and $\tau(t_i) = t_i$. More sloppily, we write $\tau(f) = f$ for $f \in \mathbb{G}$, or $\mathbb{G} = \mathbb{H}$. Now suppose that in addition we consider a $\Pi\Sigma$ -extension (\mathbb{G}, σ) of (\mathbb{F}, σ) . Then we can define the automorphism $\sigma' : \mathbb{H} \rightarrow \mathbb{H}$ with $\sigma'(f) = \tau(\sigma(\tau^{-1}(f)))$ for all $f \in \mathbb{H}$. In a more sloppy way, we write $\sigma = \sigma'$. Then obviously, (\mathbb{H}, σ) is a difference field extension of (\mathbb{F}, σ) with $\text{const}_\sigma \mathbb{G} = \text{const}_\sigma \mathbb{H} = \text{const}_\sigma \mathbb{F}$. But in general, (\mathbb{H}, σ) is not anymore a $\Pi\Sigma$ -extension of (\mathbb{F}, σ) . But if we have $\sigma(x) = \alpha x + \beta$ with $\alpha, \beta \in \mathbb{F}$ then this reordering of the variables gives us again a $\Pi\Sigma$ -extension which is isomorphic to the original one with the trivial difference field isomorphism $\tau : \mathbb{G} \rightarrow \mathbb{H}$ with $\tau(f) = f$ and $\sigma(\tau(f)) = \tau(\sigma(f))$. The proof of this statement can be carried out rigorously with techniques used in [9, Section 2.4]. Observe that one can reorder a $\Pi\Sigma$ -field (\mathbb{E}, σ) over \mathbb{K} with depth d and $1 \leq \delta \leq d + 1$ to a $\Pi\Sigma$ -field $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ with $\text{depth}(\mathbb{F}) = \delta - 1$ and $\text{depth}(t_i) \geq \delta$ for all $1 \leq i \leq e$. This construction is possible, since any $\Pi\Sigma$ -extension in \mathbb{F} has smaller depth than the t_i and is therefore free of the t_i in the definition of σ . In addition, we obtain the *difference field isomorphism* $\tau : \mathbb{E} \rightarrow \mathbb{F}(t_1) \dots (t_e)$ where $\tau(f) = f$ for all $f \in \mathbb{E}$. With this reordered $\Pi\Sigma$ -field one obtains

Lemma 2. *Let $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ be a $\Pi\Sigma$ -field with $\delta := \text{depth}(\mathbb{F}) + 1$ and $\text{depth}(t_i) \geq \delta$ for $1 \leq i \leq e$, and let (\mathbb{H}, σ) be a single-nested $\Pi\Sigma$ -extension of $(\mathbb{F}(t_1) \dots (t_e), \sigma)$. Then this extension has maximal depth δ iff it is over \mathbb{F} .*

Proof. Write $\mathbb{H} := \mathbb{F}(t_1) \dots (t_e)(s_1) \dots (s_u)$. First assume that the extension is over \mathbb{F} , i.e., $\sigma(s_i) = \alpha_i s_i + \beta_i$ with $\alpha_i, \beta_i \in \mathbb{F}$. Then, because of $\text{depth}(\mathbb{F}) = \delta - 1$ it follows that $\text{depth}(\beta_i) \leq \delta - 1$ and $\text{depth}(\alpha_i) \leq \delta - 1$, thus $\text{depth}(s_i) = \max(\text{depth}(\alpha_i), \text{depth}(\beta_i)) + 1 \leq \delta$, and therefore the extension has maximal depth δ . Conversely, suppose that this extension has maximal depth δ , i.e. $\text{depth}(s_i) \leq \delta$. Then $\text{depth}(\alpha_i) \leq \delta - 1$ and $\text{depth}(\beta_i) \leq \delta - 1$, and consequently $\alpha_i, \beta_i \in \mathbb{F}$. \square

Theorem 3. *Let (\mathbb{E}, σ) with $\mathbb{E} := \mathbb{F}(t_1) \dots (t_e)$ be a $\Pi\Sigma$ -field where $\delta := \text{depth}(\mathbb{F}) + 1$ and $\text{depth}(t_i) \geq \delta$ for $1 \leq i \leq e$, and $\mathbf{f} \in \mathbb{E}^n$. Then a $\Pi\Sigma$ -extension (\mathbb{G}, σ) of (\mathbb{E}, σ) over \mathbb{F} which is \mathbb{F} -complete for \mathbf{f} has maximal depth δ and is single-nested δ -complete for \mathbf{f} .*

Proof. Assume such an extension (\mathbb{G}, σ) of (\mathbb{E}, σ) is not single-nested δ -complete for \mathbf{f} . Then take a single-nested $\Pi\Sigma$ -extension (\mathbb{H}, σ) of (\mathbb{E}, σ) with maximal depth δ and $\mathbf{c} \in \Pi_n(\mathbb{V}((1, -1), \mathbf{f}, \mathbb{H})) \setminus \Pi_n(\mathbb{V}((1, -1), \mathbf{f}, \mathbb{G}))$. Since $\delta = \text{depth}(\mathbb{F}) + 1$ and $\text{depth}(t_i) \geq \delta$, (\mathbb{H}, σ) is an extension of (\mathbb{E}, σ) over \mathbb{F} by Lemma 2, and thus the extension (\mathbb{G}, σ) of (\mathbb{E}, σ) is not \mathbb{F} -complete for \mathbf{f} , a contradiction. Moreover, the extension (\mathbb{G}, σ) of (\mathbb{E}, σ) is single-nested with maximal depth δ by Lemma 2. \square

In Section 5 we will develop an algorithm that computes an \mathbb{F} -complete Σ^* -extension of $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ over \mathbb{F} for \mathbf{f} . Then by Theorem 3 this extension will be also single-nested δ -complete for \mathbf{f} with maximal depth δ .

4. A REDUCTION STRATEGY

We develop a streamlined version of Karr's summation algorithm [6] based on results of [2] and [9, 12, 10, 11] that solves Problem *PFLDE*. In particular, this approach will assist in finding \mathbb{F} -complete extensions over \mathbb{F} .

More precisely, let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$ -extension of (\mathbb{F}, σ) with $\sigma(t) = \alpha t + \beta$, $\mathbb{K} = \text{const}_\sigma \mathbb{F}$, $\mathbf{0} \neq \mathbf{a} = (a_1, a_2) \in \mathbb{F}(t)^2$ and $\mathbf{f} \in \mathbb{F}(t)^n$. We will introduce a simplified version of Karr's reduction strategy [6] that helps in finding a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$ over \mathbb{K} . If (\mathbb{F}, σ) is a $\Pi\Sigma$ -field, this reduction turns into a complete algorithm. Moreover, this reduction technique will deliver all the information to compute an \mathbb{F} -complete extension.

A special case. If $a_1 a_2 = 0$, we have $\mathbf{g} = \mathbf{c} \sigma^{-1}(\frac{\mathbf{f}}{a_1})$ with $a_1 \neq 0$ or $\mathbf{g} = \mathbf{c} \frac{\mathbf{f}}{a_2}$ with $a_2 \neq 0$. Then it follows with $\mathbf{g} = (g_1, \dots, g_n)$ and the i -th unit vector $(0, \dots, 1, \dots, 0) \in \mathbb{K}^n$ that $\{(0, \dots, 1, \dots, 0, g_i)\}_{1 \leq i \leq n} \subseteq \mathbb{K}^n \times \mathbb{F}(t)$ is a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$. Hence from now on we suppose $\mathbf{a} \in (\mathbb{F}(t)^*)^2$.

Clearing denominators and cancelling common factors. Compute $\mathbf{a}' = (a'_1, a'_2) \in (\mathbb{F}[t]^*)^2$, $\mathbf{f}' = (f'_1, \dots, f'_n) \in \mathbb{F}[t]^n$ such that $\text{gcd}_{\mathbb{F}[t]}(f'_1, \dots, f'_n, a'_1, a'_2) = 1$ and $\mathbf{a}' = \mathbf{a} q$, $\mathbf{f}' = \mathbf{f} q$ for some $q \in \mathbb{F}(t)^*$. Then we have $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t)) = V(\mathbf{a}', \mathbf{f}', \mathbb{F}(t))$. Therefore we may suppose that $\mathbf{a} \in (\mathbb{F}[t]^*)^2$ and $\mathbf{f} \in \mathbb{F}[t]^n$ where the entries have no common factors.

In Karr's original approach [6] the solutions $g = p+q \in \mathbb{F}(t)$ in $(c_1, \dots, c_n, g) \in V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$ are computed by deriving first the polynomial part $p \in \mathbb{F}[t]$ and afterwards the fractional part $q \in \mathbb{F}(t)$, i.e., the degree of the numerator is smaller than the degree of the denominator. We simplify this approach substantially by first computing a common denominator of all the possible solutions in $\mathbb{F}(t)$ and afterwards computing the "numerator" of the solutions over this common denominator.

Denominator bounding. In the first important reduction step one looks for a *denominator bound* d of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$, i.e. a polynomial $d \in \mathbb{F}[t]^*$ that fulfills

$$\forall (c_1, \dots, c_n, g) \in V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t)) : dg \in \mathbb{F}[t].$$

Since $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$ is a finite dimensional vector space over \mathbb{K} , a denominator bound must exist. Now suppose that we have given such a d and define $\mathbf{a}' := (\frac{a_1}{\sigma(d)}, \frac{a_2}{d})$. Then it follows that $\{(c_{i1}, \dots, c_{in}, g_i)\}_{1 \leq i \leq r}$ is a basis of $V(\mathbf{a}', \mathbf{f}, \mathbb{F}[t])$ if and only if $\{(c_{i1}, \dots, c_{in}, \frac{g_i}{d})\}_{1 \leq i \leq r}$ is a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$. For a proof we refer to [9, 12]. Hence, given a denominator bound d of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$, we can reduce the problem to search for a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$ to look for a basis of $V(\mathbf{a}', \mathbf{f}, \mathbb{F}[t])$. By clearing denominators and cancelling common factors in \mathbf{a} and \mathbf{f} , as above, we may also suppose that $\mathbf{a} \in (\mathbb{F}[t]^*)^2$ and $\mathbf{f} \in \mathbb{F}[t]^n$.

Polynomial degree bounding. The next step consists of bounding the polynomial degrees in $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$. For convenience we introduce $\mathbb{F}[t]_b := \{f \in \mathbb{F}[t] \mid \deg(f) \leq b\}$ for $b \in \mathbb{N}_0$ and $\mathbb{F}[t]_{-1} := \{0\}$. Moreover, we define $\|b\| := \deg b$ for $b \in \mathbb{F}[t]^*$, $\|0\| := -1$, and $\|\mathbf{b}\| := \max_{1 \leq i \leq l} \|b_i\|$ for $\mathbf{b} = (b_1, \dots, b_l) \in \mathbb{F}[t]^l$. Then we look for a *polynomial degree bound* b of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$, i.e., a $b \in \mathbb{N}_0 \cup \{-1\}$ such that

$$V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_b) = V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]), \quad b \geq \max(-1, \|\mathbf{f}\| - \|\mathbf{a}\|). \quad (3)$$

Again, since $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$ is finite dimensional over \mathbb{K} , a degree bound must exist. Note that by the second condition in (3) it follows that $\mathbf{f} \in \mathbb{F}[t]_{\|\mathbf{a}\|+b}$ which is needed to proceed with the degree elimination technique below.

Due to [6, 7, 2] the problem to determine a denominator bound or degree bound is completely constructive if (\mathbb{F}, σ) is a $\Pi\Sigma$ -field over a σ -computable \mathbb{K} . The proofs and sub-algorithms of these results can be found in [2, 10, 11].

Theorem 4. *If $(\mathbb{F}(t), \sigma)$ is a $\Pi\Sigma$ -field over a σ -computable \mathbb{K} , $\mathbf{a} \in (\mathbb{F}[t]^*)^2$ and $\mathbf{f} \in \mathbb{F}[t]^n$ then there are algorithms that compute a denominator bound of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$ or a degree bound of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$.*

Polynomial degree reduction. Finally we have to deal with the problem to compute a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_\delta)$ for some $\delta \in \mathbb{N}_0 \cup \{-1\}$ where $\mathbf{f} \in \mathbb{F}[t]_{\delta+l}^n$ with $l := \|\mathbf{a}\|$; this is guaranteed if δ is a polynomial degree bound of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t])$. Here we follow exactly the idea in [6]. Namely, we first find the candidates of the leading coefficients $g_\delta \in \mathbb{F}$ for the solutions $(c_1, \dots, c_n, g) \in V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_\delta)$ with $g = \sum_{i=0}^{\delta} g_i t^i$, plugging back its solution space and go on recursively to derive the candidates of the missing coefficients $g_i \in \mathbb{F}$.

This reduction idea is graphically illustrated in Figure 1 which has to be read as follows. The problem of finding a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_\delta)$ is reduced to **(i)** searching for a basis of $V(\tilde{\mathbf{a}}_\delta, \tilde{\mathbf{f}}_\delta, \mathbb{F})$ for some specifically determined $\mathbf{0} \neq \tilde{\mathbf{a}}_\delta \in \mathbb{F}^2$ and $\tilde{\mathbf{f}}_\delta \in \mathbb{F}^n$ and **(ii)** finding a basis of $V(\mathbf{a}, \mathbf{f}_{\delta-1}, \mathbb{F}[t]_{\delta-1})$ for some particular chosen $\mathbf{f}_{\delta-1} \in \mathbb{F}[t]_{\delta-1}^\lambda$. Then **(iii)**, the original problem $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_\delta)$ can be reconstructed by the two bases of the corresponding subproblems. Intuitively, the solution in $\mathbb{F}[t]_\delta$ is reconstructed by sub-solutions in \mathbb{F} (the leading coefficients) and $\mathbb{F}[t]_{\delta-1}$ (the polynomial with the remaining coefficients) which is reflected by the vector space isomorphism $\mathbb{F}[t]_\delta \simeq \mathbb{F}[t]_{\delta-1} \oplus t^\delta \mathbb{F}$. In the sequel we explain this reduction in more details. Define

$$\begin{aligned} \tilde{\mathbf{a}}_\delta &= (\tilde{a}_1, \tilde{a}_2) := (\text{coeff}(a_1, l) \alpha^\delta, \text{coeff}(a_2, l)) \\ \tilde{\mathbf{f}}_\delta &:= (\text{coeff}(f_1, \delta + l), \dots, \text{coeff}(f_n, \delta + l)). \end{aligned} \tag{4}$$

where $\mathbf{0} \neq \tilde{\mathbf{a}}_\delta \in \mathbb{F}^2$ and $\tilde{\mathbf{f}}_\delta \in \mathbb{F}^n$; $\text{coeff}(p, l)$ gives the l -th coefficient of $p \in \mathbb{F}[t]$. Then there is the following crucial observation for a solution $\mathbf{c} \in \mathbb{K}^n$ and $g = \sum_{i=0}^{\delta} g_i t^i \in \mathbb{F}[t]_\delta$ of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_\delta)$; see [9, 12]: Since t is transcendental over \mathbb{F} , it follows by coefficient comparison that $\tilde{a}_1 \sigma(g_\delta) + \tilde{a}_2 g_\delta = \mathbf{c} \tilde{\mathbf{f}}_\delta$ which means that $(c_1, \dots, c_n, g_\delta) \in V(\tilde{\mathbf{a}}_\delta, \tilde{\mathbf{f}}_\delta, \mathbb{F})$. Therefore, the right linear combinations of a basis of $V(\tilde{\mathbf{a}}_\delta, \tilde{\mathbf{f}}_\delta, \mathbb{F})$ enable one to construct partially the solutions $(c_1, \dots, c_n, g) \in V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_\delta)$, namely $(c_1, \dots, c_n) \in \mathbb{K}^n$ with the δ -th coefficient g_δ in $g \in \mathbb{F}[t]_\delta$. So, the basic idea is to find first a basis B_1 of $V(\tilde{\mathbf{a}}_\delta, \tilde{\mathbf{f}}_\delta, \mathbb{F})$.

• **CASE A:** $B_1 = \{\}$. Then there are no $g \in \mathbb{F}[t]_\delta$ and $\mathbf{0} \neq \mathbf{c} \in \mathbb{K}^n$ with $a_1 \sigma(g) + a_2 g = \mathbf{c} \mathbf{f}$, and thus $\mathbf{c} = \mathbf{0}$ and $g \in \mathbb{F}[t]_{\delta-1}$ with $a_1 \sigma(g) + a_2 g = 0$ give the only solutions; see [12]. Hence, take a basis B_2 of $V(\mathbf{a}, \mathbf{f}_{\delta-1}, \mathbb{F}[t]_{\delta-1})$ with

$$\mathbf{f}_{\delta-1} := (0)$$

and try to extract such a $g \in \mathbb{F}[t]_{\delta-1}^*$ from B_2 . If possible, a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_\delta)$ is $(0, \dots, 0, g)$. Otherwise, $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_\delta) = \{\mathbf{0}_{n+1}\}$.

• **CASE B:** $B_1 \neq \{\}$, say $B_1 = \{(c_{i1}, \dots, c_{in}, w_i)\}_{1 \leq i \leq \lambda}$. Then define $\mathbf{C} := (c_{ij}) \in \mathbb{K}^{\lambda \times n}$, $\mathbf{g} := (w_1 t^\delta, \dots, w_\lambda t^\delta) \in t^\delta \mathbb{F}^\lambda$ and consider

$$\mathbf{f}_{\delta-1} := \mathbf{C} \mathbf{f} - (a_1 \sigma(\mathbf{g}) + a_2 \mathbf{g}). \tag{5}$$

By construction it follows that $\mathbf{f}_{\delta-1} \in \mathbb{F}[t]_{\delta+l-1}^\lambda$. Now we proceed as follows. We try to determine exactly those $h \in \mathbb{F}[t]_{\delta-1}$ and $\mathbf{d} \in \mathbb{K}^\lambda$ that fulfill $a_1 \sigma(h + \mathbf{d} \mathbf{g}) + a_2 (h + \mathbf{d} \mathbf{g}) = \mathbf{d} \mathbf{C} \mathbf{f}$ which is equivalent to $a_1 \sigma(h) + a_2 h = \mathbf{d} \mathbf{f}_{\delta-1}$. For this task, we take a basis B_2 of

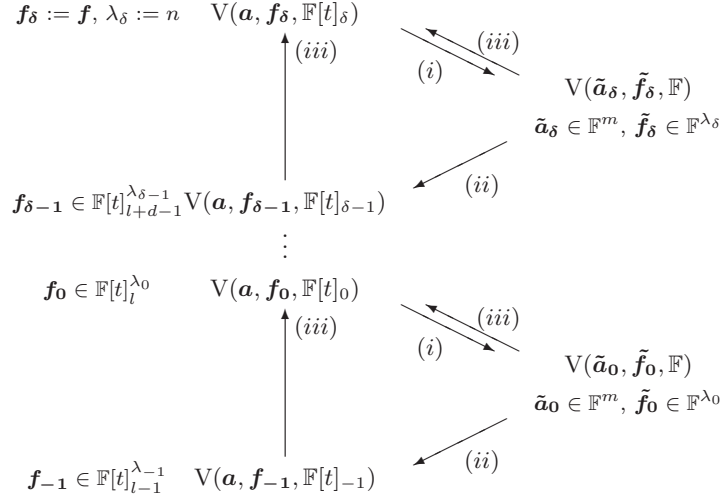


FIGURE 1. Incremental reduction

$V(\mathbf{a}, \mathbf{f}_{\delta-1}, \mathbb{F}[t]_{\delta-1})$.

★ CASE B.a: $B_2 = \{\}$. Then $V(\mathbf{a}, \mathbf{f}_{\delta}, \mathbb{F}[t]_{\delta}) = \{\mathbf{0}_{n+1}\}$.

★ CASE B.b: $B_2 \neq \{\}$, say $B_2 = \{(d_{i1}, \dots, d_{i\lambda}, h_i)\}_{1 \leq i \leq \mu}$. Then define $\mathbf{D} := (d_{ij}) \in \mathbb{K}^{\mu \times \lambda}$, $\mathbf{h} := (h_1, \dots, h_{\mu}) \in \mathbb{F}[t]_{\delta-1}^{\mu}$ which gives $a_1 \sigma(\mathbf{h} + \mathbf{D}\mathbf{g}) + a_2(\mathbf{h} + \mathbf{D}\mathbf{g}) = \mathbf{D}\mathbf{C}\mathbf{f}$. Now define $\kappa_{ij} \in \mathbb{K}$ and $p_i \in \mathbb{F}[t]_{\delta}^{\mu}$ with

$$\begin{pmatrix} \kappa_{11} & \dots & \kappa_{1n} \\ \vdots & & \vdots \\ \kappa_{\mu 1} & \dots & \kappa_{\mu n} \end{pmatrix} := \mathbf{D}\mathbf{C}, \quad (p_1, \dots, p_{\mu}) := \mathbf{h} + \mathbf{D}\mathbf{g}. \quad (6)$$

Then by the above considerations it follows that $B_3 := \{(\kappa_{i1}, \dots, \kappa_{in}, p_i)\}_{1 \leq i \leq \mu}$ spans a subspace of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_{\delta})$ over \mathbb{K} . By linear algebra arguments one can even show that B_3 is a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_{\delta})$ over \mathbb{K} . This polynomial degree reduction is the inner core of Karr's summation algorithm given in [6]. A complete proof can be found in [12].

Summarizing, let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma$ -extension of (\mathbb{F}, σ) , $\mathbf{a} \in (\mathbb{F}[t]^*)^2$ with $l := \|\mathbf{a}\|$ and $\mathbf{f} \in \mathbb{F}[t]_{\delta+l}^n$ for some $\delta \in \mathbb{N}_0 \cup \{-1\}$. Then we can apply this reduction technique step by step and obtain an *incremental reduction* of $(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_{\delta})$ given in Figure 1. We call $\{(\mathbf{a}, \mathbf{f}_{\delta}, \mathbb{F}[t]_{\delta}), \dots, (\mathbf{a}, \mathbf{f}_{-1}, \mathbb{F}[t]_{-1})\}$ the *incremental tuples* and $\{(\tilde{\mathbf{a}}_{\delta}, \tilde{\mathbf{f}}_{\delta}, \mathbb{F}), \dots, (\tilde{\mathbf{a}}_0, \tilde{\mathbf{f}}_0, \mathbb{F})\}$ the *coefficient tuples* of such an incremental reduction.

Base case I. In the incremental reduction we finally reach the problem to find a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_{-1})$ with $\mathbb{F}[t]_{-1} = \{0\}$. Then we have $V(\mathbf{a}, \mathbf{f}, \{0\}) = \text{Nullspace}_{\mathbb{K}}(\mathbf{f}) \times \{0\}$ where $\text{Nullspace}_{\mathbb{K}}(\mathbf{f}) = \{\mathbf{k} \in \mathbb{K}^n \mid \mathbf{f}\mathbf{k} = 0\}$. Note that a basis of $V(\mathbf{a}, \mathbf{f}, \{0\})$ can be computed by linear algebra if (\mathbb{F}, σ) is a $\Pi\Sigma$ -field over a σ -computable \mathbb{K} ; for more details see [12].

Example 2. Take the $\Pi\Sigma$ -field $(\mathbb{Q}(t_1)(t_2), \sigma)$ over \mathbb{Q} from Example 1 and write $\mathbb{F} := \mathbb{Q}(t_1)$. With our reduction strategy we will find a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t_2))$ for $\mathbf{a} = (1, -1) \in \mathbb{F}(t_2)^2$ and $\mathbf{f} = (\sigma(t_2/t_1)) = (\frac{1+(t_1+1)t_2}{(t_1+1)^2}) \in \mathbb{F}(t_2)^1$. Clearing denominators gives the vectors $\mathbf{a} = ((t_1+1)^2, -(t_1+1)^2) \in \mathbb{F}[t_2]^2$, $\mathbf{f} = (1 + (t_1+1)t_2) \in \mathbb{F}[t_2]^1$. A denominator bound of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t_2))$ is 1, and a degree bound of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t_2])$ is 2. Now we start the incremental reduction of $(\mathbf{a}, \mathbf{f}, \mathbb{F}[t_2]_2)$. For the incremental tuple $(\mathbf{a}, \mathbf{f}_2, \mathbb{F}[t_2]_2)$ with $\mathbf{f}_2 := \mathbf{f} \in \mathbb{F}[t_2]_2^1$

we obtain the coefficient tuple $(\mathbf{a}, (0), \mathbb{F})$. The basis $\{(1, 0), (0, 1)\}$ of $V(\mathbf{a}, (0), \mathbb{F})$ gives $\mathbf{C} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{K}^{2 \times 1}$, $\mathbf{g} = (0, t_2^2) \in \mathbb{F}[t_2]_2^2$. This defines the incremental tuple $(\mathbf{a}, \mathbf{f}_1, \mathbb{F}[t_2]_1)$ with $\mathbf{f}_1 = (1 + (t_1 + 1)t_2, -1 - 2(t_1 + 1)t_2) \in \mathbb{F}[t_2]_1^2$ and the coefficient tuple $(\mathbf{a}, (t_1 + 1, -2(t_1 + 1)), \mathbb{F})$. Then taking the basis $\{(2, 1, 0), (0, 0, 1)\}$ of $V(\mathbf{a}, (1, -2), \mathbb{F})$, one obtains $\mathbf{f}_0 = (1, -t_1 - 1) \in \mathbb{F}[t_2]_0^2$, the incremental tuple $(\mathbf{a}, \mathbf{f}_0, \mathbb{F}[t_2]_0)$ and the coefficient tuple $(\mathbf{a}, \mathbf{f}_0, \mathbb{F})$. A basis of the solution space $V(\mathbf{a}, \mathbf{f}_0, \mathbb{F})$ is $\{(0, 0, 1)\}$ which defines $\mathbf{f}_{-1} = (0)$. Finally, a basis of $V(\mathbf{a}, \mathbf{f}_{-1}, \{0\})$ is $\{(1, 0)\}$. This gives the basis $\{(0, 0, 1)\}$ of $V(\mathbf{a}, \mathbf{f}_i, \mathbb{F}[t_2]_i)$ for $i \in \{0, 1\}$ and therefore the basis $\{(0, 1)\}$ of $V(\mathbf{a}, \mathbf{f}_2, \mathbb{F}[t_2]_2)$ and $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t_2))$.

A reduction to \mathbb{F} . Suppose that we have given not only a single but a nested $\Pi\Sigma$ -extension $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ of (\mathbb{F}, σ) where we write $\mathbb{F}_i := \mathbb{F}(t_1) \dots (t_i)$ for $0 \leq i \leq e$, i.e., $\mathbb{F}_0 = \mathbb{F}$. Let $\mathbf{0} \neq \mathbf{a} = (a_1, a_2) \in \mathbb{F}_e$ and $\mathbf{f} \in \mathbb{F}_e^n$. Then we understand by a *reduction of $(\mathbf{a}, \mathbf{f}, \mathbb{F}_e)$ to \mathbb{F}* a recursive application of the above reductions. More precisely, if $e = 0$, we do nothing. Otherwise, suppose that $e > 0$. If $a_1 a_2 = 0$, we just apply the special case from above. Otherwise, within our reduction there is a denominator bound $d \in \mathbb{F}_{e-1}[t_e]^*$ which reduces the problem to find a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}_e)$ to find one for $V(\mathbf{a}', \mathbf{f}', \mathbb{F}_{e-1}[t_e])$ for some $\mathbf{a}' \in (\mathbb{F}_{e-1}[t_e]^*)^2$ and $\mathbf{f}' \in \mathbb{F}_{e-1}[t_e]^n$; those are given by setting $\mathbf{a}' := (a_1/\sigma(d), a_2/d)$, $\mathbf{f}' := \mathbf{f}$ and clearing denominators and cancelling common factors. Next, with a degree bound b of $V(\mathbf{a}', \mathbf{f}', \mathbb{F}_{e-1}[t_e])$ the incremental reduction of $(\mathbf{a}', \mathbf{f}', \mathbb{F}_{e-1}[t_e]_b)$ is applied. Within this reduction the coefficient tuples $(\mathbf{a}_i, \mathbf{f}_i, \mathbb{F}_{e-1})$ for $0 \leq i \leq b$ give the subreductions of $(\mathbf{a}_i, \mathbf{f}_i, \mathbb{F}_{e-1})$ to \mathbb{F} for $0 \leq i \leq b$ that define recursively the whole reduction of $(\mathbf{a}, \mathbf{f}, \mathbb{F}_e)$ to \mathbb{F} .

We call T the *tuple set* of a reduction of $(\mathbf{a}, \mathbf{f}, \mathbb{F}_e)$ to \mathbb{F} if besides $(\mathbf{a}, \mathbf{f}, \mathbb{F}_e) \in T$ the set T contains exactly all those coefficient tuples that occur in the recursively applied incremental reductions. Moreover, for $\mathbf{a}_e := \mathbf{a}$ and $\mathbf{f}_e := \mathbf{f}$ we call $\{(\mathbf{a}_i, \mathbf{f}_i, \mathbb{F}_i)\}_{r \leq i \leq e} \subseteq T$ *path-tuples* of $(\mathbf{a}_r, \mathbf{f}_r, \mathbb{F}_r) \in T$ if in the subreduction of $(\mathbf{a}_{i+1}, \mathbf{f}_{i+1}, \mathbb{F}_{i+1})$ to \mathbb{F} the coefficient tuple $(\mathbf{a}_i, \mathbf{f}_i, \mathbb{F}_i)$ occurs for each $r \leq i < e$ in the incremental reduction. Finally, we introduce the \mathbb{F}_r -*critical tuple set* S in a reduction of $(\mathbf{a}, \mathbf{f}, \mathbb{F}_e)$ to \mathbb{F} as that subset of the tuple set T of the reduction to \mathbb{F} that contains all $(\mathbf{a}', \mathbf{f}', \mathbb{F}_r) \in T$ with the following property: for its path-tuples $\{(\mathbf{a}_i, \mathbf{f}_i, \mathbb{F}_i)\}_{r \leq i \leq e}$ we have that \mathbf{a}_i is homogeneous for all $r \leq i \leq e$. Summarizing, we obtain the following method that generates a reduction to \mathbb{F} .

Algorithm 1. `SolveSolutionSpace` $((\mathbb{F}(t_1) \dots (t_e), \sigma), \mathbf{a}, \mathbf{f})$

Input: A $\Pi\Sigma$ -extension $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ of (\mathbb{F}, σ) with $\mathbb{K} := \text{const}_{\sigma}\mathbb{F}$; $\mathbf{0} \neq \mathbf{a} = (a_1, a_2) \in \mathbb{F}(t_1) \dots (t_e)^2$ and $\mathbf{f} \in \mathbb{F}(t_1) \dots (t_e)^n$.

Output: A basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t_1) \dots (t_e))$ over \mathbb{K} .

- (1) IF $e = 0$ compute a basis B of $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$ and RETURN B . FI
Let $\mathbb{H} := \mathbb{F}(t_1) \dots (t_{e-1})$, i.e. $(\mathbb{H}(t_e), \sigma)$ is a $\Pi\Sigma$ -ext. of (\mathbb{H}, σ) .
- (2) IF $a_1 a_2 = 0$ THEN set $\mathbf{g} := \frac{\mathbf{f}}{a_2}$ if $a_2 \neq 0$, otherwise set $\mathbf{g} := \frac{\sigma(\mathbf{f})}{a_2}$; with $\mathbf{g} = (g_1, \dots, g_n)$ RETURN $\{(0 \dots, 1, \dots, 0, g_i)\}_{1 \leq i \leq n}$. FI
- (3) Clear denominators and common factors s.t. $\mathbf{a} \in (\mathbb{H}[t_e]^*)^2$, $\mathbf{f} \in \mathbb{H}[t_e]^n$.
- (4) Compute a denominator bound $d \in \mathbb{H}[t_e]^*$ of $V(\mathbf{a}, \mathbf{f}, \mathbb{H}(t_e))$.
- (5) Set $\mathbf{a}' := (a_1/\sigma(d), a_2/d) \in \mathbb{H}(t_e)^2$, $\mathbf{f}' := \mathbf{f}$ and clear denominators and common factors s.t. $\mathbf{a}' \in (\mathbb{H}[t_e]^*)^2$ and $\mathbf{f}' \in \mathbb{H}[t_e]^n$.
- (6) Compute a degree bound b of $V(\mathbf{a}', \mathbf{f}', \mathbb{H}[t_e])$.
- (7) Compute a basis $B := \text{IncrementalReduction}((\mathbb{H}(t_e), \sigma), b, \mathbf{a}', \mathbf{f}')$ by using Algorithm 2; say $B = \{(\kappa_{i1}, \dots, \kappa_{in}, p_i)\}_{1 \leq i \leq \mu}$.

(8) IF $B = \{\}$ THEN RETURN $\{\}$ ELSE RETURN $\{(\kappa_{i1}, \dots, \kappa_{in}, \frac{p_i}{d})\}_{1 \leq i \leq \mu}$. FI

Algorithm 2. $\text{IncrementalReduction}((\mathbb{F}(t_1) \dots (t_e), \sigma), \delta, \mathbf{a}, \mathbf{f})$

Input: A $\Pi\Sigma$ -extension $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ of (\mathbb{F}, σ) with $\mathbb{K} := \text{const}_\sigma \mathbb{F}$; $\delta \in \mathbb{N}_0 \cup \{-1\}$; $\mathbf{a} = (a_1, a_2) \in (\mathbb{F}(t_1) \dots (t_{e-1})[t_e]^*)^2$ with $l := \|\mathbf{a}\|$ and $\mathbf{f} \in \mathbb{F}(t_1) \dots (t_{e-1})[t_e]_{l+\delta}^n$.

Output: A basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}[t]_\delta)$ over \mathbb{K} .

(1) IF $d = -1$, RETURN a basis of $\text{Nullspace}_{\mathbb{K}}(\mathbf{f}) \times \{0\}$ over \mathbb{K} . FI

Let $\mathbb{H} := \mathbb{F}(t_1) \dots (t_{e-1})$, i.e. $(\mathbb{H}(t_e), \sigma)$ is a $\Pi\Sigma$ -ext. of (\mathbb{H}, σ) .

(2) Define $\mathbf{0} \neq \tilde{\mathbf{a}}_\delta \in \mathbb{H}^2$ and $\tilde{\mathbf{f}}_\delta \in \mathbb{H}^n$ as in (4).

(3) Compute $B_1 := \text{SolveSolutionSpace}((\mathbb{H}, \sigma), \tilde{\mathbf{a}}_\delta, \tilde{\mathbf{f}}_\delta)$ with Alg. 1.

(4) IF $B_1 = \{\}$ THEN

(5) Compute $B_2 := \text{IncrementalReduction}((\mathbb{H}(t_e), \sigma), \delta - 1, \mathbf{a}, (0))$.

(6) IF an $h \in \mathbb{H}[t_e]_{\delta-1}$ with $a_1 \sigma(h) + a_2 h = 0$ is found THEN

RETURN $\{(0, \dots, 0, h)\} \subset \mathbb{K}^n \times \mathbb{H}[t_e]_{\delta-1}$ ELSE RETURN $\{\}$ FI

FI

(7) With $B_1 = \{(c_{i1}, \dots, c_{in}, w_i)\}_{1 \leq i \leq \lambda}$ define $\mathbf{C} := (c_{ij}) \in \mathbb{K}^{\lambda \times n}$, $\mathbf{g} := (w_1 t_e^\delta, \dots, w_\lambda t_e^\delta) \in t_e^\delta \mathbb{H}^\lambda$, and $\mathbf{f}_{\delta-1} \in \mathbb{H}[t_e]_{\delta-1}^\lambda$ as in (5).

(8) Compute $B_2 := \text{IncrementalReduction}((\mathbb{H}(t_e), \sigma), \delta - 1, \mathbf{a}, \mathbf{f}_{\delta-1})$.

(9) IF $B_2 = \{\}$ THEN RETURN $\{\}$ FI

(10) Let $B_2 = \{(d_{i1}, \dots, d_{i\lambda}, h_i)\}_{1 \leq i \leq \mu}$ and define $\mathbf{D} := (d_{ij}) \in \mathbb{K}^{\mu \times \lambda}$, $\mathbf{h} := (h_1, \dots, h_\mu) \in \mathbb{H}[t_e]_{\delta-1}^\mu$. Define $\kappa_{ij} \in \mathbb{K}$ for $1 \leq i \leq \mu$, $1 \leq j \leq n$ and $p_i \in \mathbb{H}[t_e]_\delta$ for $1 \leq i \leq \mu$ as in (6).

(11) RETURN $\{(\kappa_{i1}, \dots, \kappa_{in}, p_i)\}_{1 \leq i \leq \mu}$

If the denominator bound problem and polynomial degree bound problem can be solved in the $\Pi\Sigma$ -extensions (\mathbb{F}_i, σ) of $(\mathbb{F}_{i-1}, \sigma)$ for $1 \leq i \leq e$ and one can compute a basis of any solution space in (\mathbb{F}, σ) , Algorithms 1 and 2 give an algorithm to compute a basis of a solution space $V(\mathbf{a}, \mathbf{f}, \mathbb{F}_e)$. In particular these algorithms give a reduction of $(\mathbf{a}, \mathbf{f}, \mathbb{F}_e)$ to \mathbb{F} . Moreover, by taking all $(\mathbf{a}, \mathbf{f}, \mathbb{F}_i)$ when calling Algorithm 1, one gets the reduction tuple set of this reduction. Furthermore, if one stops collecting tuples in the subreductions of $(\mathbf{a}, \mathbf{f}, \mathbb{F}_i)$ to \mathbb{F} when \mathbf{a} is inhomogeneous, one can extract the \mathbb{F}_r -critical tuples in this reduction.

Now assume that (\mathbb{F}, σ) is a $\Pi\Sigma$ -field over a σ -computable \mathbb{K} , i.e., $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ is a $\Pi\Sigma$ -field over \mathbb{K} . Then by Theorem 4 there are algorithms to solve the denominator and polynomial degree bound problem. Moreover, for the special case $\mathbb{F} = \mathbb{K}$ there is the following

Base case II. If $\text{const}_\sigma \mathbb{K} = \mathbb{K}$, $\mathbf{0} \neq \mathbf{a} = (a_1, a_2) \in \mathbb{K}^2$ and $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{K}^n$ then $V(\mathbf{a}, \mathbf{f}, \mathbb{K}) = \text{Nullspace}_{\mathbb{K}}(\mathbf{f}')$ for $\mathbf{f}' = (f_1, \dots, f_n, -(a_1 + a_2))$. A basis can be computed by linear algebra; see [10].

Hence, with Algorithms 1 and 2 one can compute a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t_1) \dots (t_e))$ in a $\Pi\Sigma$ -field $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ over a σ -computable \mathbb{K} and can extract the \mathbb{F} -critical tuples of the corresponding reduction of $(\mathbf{a}, \mathbf{f}, \mathbb{F}(t_1) \dots (t_e))$ to \mathbb{F} .

Finally, we introduce reductions to \mathbb{F} that are extension-stable. Let $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ be a $\Pi\Sigma$ -extension of (\mathbb{F}, σ) , $\mathbf{a} \in (\mathbb{H}[t_e]^*)^2$ and $\mathbf{f} \in \mathbb{H}[t_e]^n$ for $\mathbb{H} := \mathbb{F}(t_1) \dots (t_{e-1})$. We call a denominator bound $d \in \mathbb{H}[t_e]^*$ of $V(\mathbf{a}, \mathbf{f}, \mathbb{H}(t_e))$ or a degree bound b of $V(\mathbf{a}, \mathbf{f}, \mathbb{H}[t_e])$ extension-stable over \mathbb{F} if \mathbf{a} is inhomogeneous over $\mathbb{H}(t_e)$ or the following holds. Take any Σ^* -extension $(\mathbb{F}(t_1) \dots (t_e)(s), \sigma)$ of $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ over \mathbb{F} , and embed \mathbf{a}, \mathbf{f} in the reordered

$\Pi\Sigma$ -ext. $(\mathbb{F}(s)(t_1)\dots(t_e), \sigma)$ of (\mathbb{F}, σ) . Then also d embedded in $\mathbb{F}(s)(t_1)\dots(t_e)$ must be a denominator bound of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(s)(t_1)\dots(t_e))$. Similarly, b must be a degree bound of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(s)(t_1)\dots(t_{e-1})[t_e])$.

We call a reduction of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(t_1)\dots(t_e))$ to \mathbb{F} *extension-stable* if all denominator and degree bounds within the reduction to \mathbb{F} are extension-stable over \mathbb{F} .

It has been shown in [10, Theorem 8.2] and [11, Theorem 7.3] that the algorithms proposed in [6] already compute extension-stable denominator and degree bounds in a $\Pi\Sigma$ -field. Summarizing, we obtain

Theorem 5. *Let (\mathbb{E}, σ) be a $\Pi\Sigma$ -field over a σ -computable \mathbb{K} , $\mathbf{0} \neq \mathbf{a} \in \mathbb{E}^2$ and $\mathbf{f} \in \mathbb{E}^n$. Then with Algorithms 1 and 2 one can compute a basis of $V(\mathbf{a}, \mathbf{f}, \mathbb{E})$ with an extension-stable reduction of $(\mathbf{a}, \mathbf{f}, \mathbb{E})$ to \mathbb{F} . Moreover, during this computation, one can extract the \mathbb{F} -critical tuples.*

Example 3. *In Example 2 the denominator and degree bounds are extension-stable. Consequently, this reduction of $((1, -1), (\sigma(t_2/t_1)), \mathbb{F}(t_2))$ to \mathbb{F} is extension-stable. The \mathbb{F} -critical tuples are $((t_1 + 1)^2, -(t_1 + 1)^2), \mathbf{f}, \mathbb{F}$ for $\mathbf{f} \in \{(0), (t_1 + 1, -2(t_1 + 1)), (1, -(t_1 + 1))\}$.*

5. REFINED SUMMATION ALGORITHMS

In the sequel let (\mathbb{E}, σ) with $\mathbb{E} := \mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma$ -field over a σ -computable \mathbb{K} and $\mathbf{f} \in \mathbb{E}^n$. Then in Theorem 6 we will develop a constructive criterium which tells us if a Σ^* -extension of (\mathbb{E}, σ) over \mathbb{F} is \mathbb{F} -complete for \mathbf{f} and how such an extension can be constructed. For this task we first compute a basis of $V := V((1, -1), \mathbf{f}, \mathbb{E})$ with Algorithms 1 and 2 together with an extension-stable reduction of $((1, -1), \mathbf{f}, \mathbb{E})$ to \mathbb{F} ; see Theorem 5. If the dimension of V is $n + 1$, the trivial extension (\mathbb{E}, σ) of (\mathbb{E}, σ) is clearly \mathbb{F} -complete for \mathbf{f} . Otherwise, we extract the \mathbb{F} -critical tuple set in our extension-stable reduction; see Theorem 5. Then the crucial observation is stated in Proposition 1 that depends on Lemma 3. This lemma is a special case of Karr's Fundamental Theorem [6, 7]; for a proof see [9, Proposition 4.1.2].

Lemma 3. *If (\mathbb{E}, σ) is a Σ^* -extension of (\mathbb{F}, σ) , $\mathbf{0} \neq \mathbf{a} \in \mathbb{F}^2$ inhomogeneous over \mathbb{F} and $\mathbf{f} \in \mathbb{F}^n$ then $V(\mathbf{a}, \mathbf{f}, \mathbb{E}) = V(\mathbf{a}, \mathbf{f}, \mathbb{F})$.*

Proposition 1. *Let $(\mathbb{E}(s), \sigma)$ with $\mathbb{E} := \mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma$ -extension of (\mathbb{F}, σ) with $\sigma(s) - s \in \mathbb{F}$ and consider the reordered $\Pi\Sigma$ -extension $(\mathbb{F}(s)(t_1)\dots(t_e), \sigma)$ of (\mathbb{F}, σ) . Let $\mathbf{a} \in \mathbb{E}^2$ be homogeneous over \mathbb{E} , $\mathbf{f} \in \mathbb{E}^n$, and let S be an \mathbb{F} -critical tuple set of an extension-stable reduction of $(\mathbf{a}, \mathbf{f}, \mathbb{E})$ to \mathbb{F} . If for all $(\mathbf{a}', \mathbf{f}', \mathbb{F}) \in S$ we have $V(\mathbf{a}', \mathbf{f}', \mathbb{F}) = V(\mathbf{a}', \mathbf{f}', \mathbb{F}(s))$ then $V(\mathbf{a}, \mathbf{f}, \mathbb{E}) = V(\mathbf{a}, \mathbf{f}, \mathbb{E}(s)) = V(\mathbf{a}, \mathbf{f}, \mathbb{F}(s)(t_1)\dots(t_e))$.*

Proof. The proof will be done by induction on the number e of extensions $\mathbb{F}(t_1)\dots(t_e)$. First consider the case $e = 0$. Since \mathbf{a} is homogeneous, $(\mathbf{a}, \mathbf{f}, \mathbb{F}) \in S$ and therefore $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(s)) = V(\mathbf{a}, \mathbf{f}, \mathbb{F})$. Now assume that the proposition holds for $e \geq 0$. Let $(\mathbb{F}(t_1)\dots(t_e)(t_{e+1})(s), \sigma)$ be a $\Pi\Sigma$ -extension of (\mathbb{F}, σ) with $\sigma(s) - s \in \mathbb{F}$ and consider the reordered $\Pi\Sigma$ -extension $(\mathbb{F}(s)(t_1)\dots(t_e)(t_{e+1}), \sigma)$ of (\mathbb{F}, σ) . We write $\mathbb{E} := \mathbb{F}(t_1)\dots(t_e)$ and $\mathbb{H} := \mathbb{F}(s)(t_1)\dots(t_e)$ as shortcut. Let $\mathbf{a} \in \mathbb{E}(t_{e+1})^2$ be homogeneous over $\mathbb{E}(t_{e+1})$, $\mathbf{f} \in \mathbb{E}(t_{e+1})^n$, and take any extension-stable reduction of $(\mathbf{a}, \mathbf{f}, \mathbb{E}(t_{e+1}))$ to \mathbb{F} with the \mathbb{F} -critical tuple set S . Now suppose that $V(\mathbf{a}', \mathbf{f}', \mathbb{F}) = V(\mathbf{a}', \mathbf{f}', \mathbb{F}(s))$ for all $(\mathbf{a}', \mathbf{f}', \mathbb{F}) \in S$. Then we will show that

$$V(\mathbf{a}, \mathbf{f}, \mathbb{E}(t_{e+1})) = V(\mathbf{a}, \mathbf{f}, \mathbb{H}(t_{e+1})). \quad (7)$$

In the extension-stable reduction let $d \in \mathbb{E}[t_{e+1}]^*$ be the denominator bound of the solution space $V(\mathbf{a}, \mathbf{f}, \mathbb{E}(t_{e+1}))$. Since \mathbf{a} is homogeneous over $\mathbb{E}(t_{e+1})$, $d \in \mathbb{H}[t_{e+1}]$ is also a denominator

bound of $V(\mathbf{a}, \mathbf{f}, \mathbb{H}(t_{e+1}))$. After clearing denominators and cancelling common factors, we get $\mathbf{a}' := (a_1/\sigma(d), a_2/d)q \in \mathbb{E}[t_{e+1}]^2$ and $\mathbf{f}' := \mathbf{f}q \in \mathbb{E}[t_{e+1}]^n$ for some $q \in \mathbb{E}(t_{e+1})^*$ in our reduction. Note that \mathbf{a}' is still homogeneous over $\mathbb{E}(t_{e+1})$. This follows from the fact that if for $h \in \mathbb{E}(t_{e+1})$ we have $a_1 \sigma(h) + a_2 h = 0$ then $a'_1 \sigma(hd) + a'_2 hd = 0$. Now it suffices to show that $V(\mathbf{a}', \mathbf{f}', \mathbb{H}[t_{e+1}]) = V(\mathbf{a}', \mathbf{f}', \mathbb{E}[t_{e+1}])$, in order to show (7). In the given reduction let b be the degree bound of $V(\mathbf{a}', \mathbf{f}', \mathbb{E}[t_{e+1}])$. Since \mathbf{a}' is homogeneous over $\mathbb{E}(t_{e+1})$, b is a degree bound of $V(\mathbf{a}', \mathbf{f}', \mathbb{H}[t_{e+1}])$ too. Hence, if $V(\mathbf{a}', \mathbf{f}', \mathbb{E}[t_{e+1}]_b) = V(\mathbf{a}', \mathbf{f}', \mathbb{H}[t_{e+1}]_b)$, also (7) is proven. Let $((\mathbf{a}, \mathbf{f}_b, \mathbb{E}[t_{e+1}]_b), \dots, (\mathbf{a}, \mathbf{f}_{-1}, \mathbb{E}[t_{e+1}]_{-1}))$ be the incremental tuples and $((\tilde{\mathbf{a}}_b, \tilde{\mathbf{f}}_b, \mathbb{E}), \dots, (\tilde{\mathbf{a}}_0, \tilde{\mathbf{f}}_0, \mathbb{E}))$ be the coefficient-tuples in the incr. reduction of $(\mathbf{a}, \mathbf{f}, \mathbb{E}[t_{e+1}]_b)$. We show that $V(\tilde{\mathbf{a}}_i, \tilde{\mathbf{f}}_i, \mathbb{E}) = V(\tilde{\mathbf{a}}_i, \tilde{\mathbf{f}}_i, \mathbb{H})$ for all $0 \leq i \leq b$. By reordering of the difference field $(\mathbb{F}(t_1) \dots (t_{e+1})(s), \sigma)$ we get the $\Pi\Sigma$ -extension $(\mathbb{F}(t_1) \dots (t_e)(s)(t_{e+1}), \sigma)$ of (\mathbb{F}, σ) . First suppose that $\tilde{\mathbf{a}}_i$ is inhomogeneous over \mathbb{E} . Hence, $V(\tilde{\mathbf{a}}_i, \tilde{\mathbf{f}}_i, \mathbb{E}) = V(\tilde{\mathbf{a}}_i, \tilde{\mathbf{f}}_i, \mathbb{E}(s))$ by Lemma 3, and therefore $V(\tilde{\mathbf{a}}_i, \tilde{\mathbf{f}}_i, \mathbb{E}) = V(\tilde{\mathbf{a}}_i, \tilde{\mathbf{f}}_i, \mathbb{H})$ by $(\mathbb{F}(t_1) \dots (t_e)(s), \sigma) \simeq (\mathbb{F}(s)(t_1) \dots (t_e), \sigma)$. Otherwise, assume that $\tilde{\mathbf{a}}_i$ is homogeneous over \mathbb{E} . Then the extension-stable reduction of $(\mathbf{a}, \mathbf{f}, \mathbb{E}(t_{e+1}))$ to \mathbb{F} contains an extension-stable reduction of $(\tilde{\mathbf{a}}_i, \tilde{\mathbf{f}}_i, \mathbb{E})$ to \mathbb{F} and the \mathbb{F} -critical tuple set of the reduction of $(\tilde{\mathbf{a}}_i, \tilde{\mathbf{f}}_i, \mathbb{E})$ is a subset of S . Hence with the induction assumption it follows that $V(\tilde{\mathbf{a}}_i, \tilde{\mathbf{f}}_i, \mathbb{E}) = V(\tilde{\mathbf{a}}_i, \tilde{\mathbf{f}}_i, \mathbb{H})$. Since $\mathbb{E}[t_{e+1}]_{-1} = \mathbb{H}[t_{e+1}]_{-1} = \{0\}$, we have $V(\mathbf{a}, \mathbf{f}_{-1}, \mathbb{E}[t_{e+1}]_{-1}) = V(\mathbf{a}, \mathbf{f}_{-1}, \mathbb{H}[t_{e+1}]_{-1})$. Then by the construction of the incremental reduction we can conclude that $V(\mathbf{a}, \mathbf{f}_i, \mathbb{E}[t_{e+1}]_i) = V(\mathbf{a}, \mathbf{f}_i, \mathbb{H}[t_{e+1}]_i)$ for all $-1 \leq i \leq b$ and therefore we have proven (7). With reordering $(\mathbb{F}(s)(t_1) \dots (t_{e+1}), \sigma) \simeq (\mathbb{F}(t_1) \dots (t_{e+1})(s), \sigma)$, it follows $V(\mathbf{a}, \mathbf{f}, \mathbb{E}(t_{e+1})) = V(\mathbf{a}, \mathbf{f}, \mathbb{E}(t_{e+1})(s))$. \square

Consequently we have $V((1, -1), \mathbf{f}, \mathbb{E}) \subsetneq V((1, -1), \mathbf{f}, \mathbb{E}(s))$ for a Σ^* -extension $(\mathbb{E}(s), \sigma)$ of (\mathbb{E}, σ) over \mathbb{F} if $V(\mathbf{a}', \mathbf{f}', \mathbb{F}) \subsetneq V(\mathbf{a}', \mathbf{f}', \mathbb{F}(s))$ in one of its \mathbb{F} -critical tuples $(\mathbf{a}', \mathbf{f}', \mathbb{F})$ in an extension-stable reduction to \mathbb{F} .

Example 4. Consider the $\Pi\Sigma$ -fields from Example 1, 2 and 3. By Example 1 it follows that $V((1, -1), (\frac{\sigma(t_2)}{t_1}), \mathbb{F}(t_2))$ is a proper subset of $V((1, -1), (\frac{\sigma(t_2)}{t_1}), \mathbb{F}(t_2)(s))$. Hence looking at the \mathbb{F} -critical tuples of our extension stable reduction in Example 3, we know by Proposition 1 that there is an $\mathbf{f} \in \{(0, (t_1 + 1, -2(t_1 + 1)), (1, -(t_1 + 1)))\}$ such that $V(\mathbf{a}, \mathbf{f}, \mathbb{F})$ with $\mathbf{a} = ((t_1 + 1)^2, -(t_1 + 1)^2)$ is a proper subset of $V(\mathbf{a}, \mathbf{f}, \mathbb{F}(s))$. Indeed, we can choose $\mathbf{f} = (1, -(t_1 + 1))$ since there does not exist a $g \in \mathbb{F}$ with $\sigma(g) - g = \frac{1}{(t_1 + 1)^2}$, but there is the solution $g = s \in \mathbb{F}(s)$.

Next we provide a sufficient condition in Proposition 2 which tells us if a Σ^* -extension cannot contribute further to a given solution space.

Proposition 2. Let (\mathbb{F}, σ) be a difference field with $\mathbf{a} = (a_1, a_2) \in \mathbb{F}^2$ homogeneous over \mathbb{F} and $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. If for all $1 \leq i \leq n$ there is a $g \in \mathbb{F}^*$ with $a_1 \sigma(g) + a_2 g = f_i$ then for any difference field (ring) extension (\mathbb{E}, σ) of (\mathbb{F}, σ) with $\text{const}_\sigma \mathbb{E} = \text{const}_\sigma \mathbb{F}$ we have $V(\mathbf{a}, \mathbf{f}, \mathbb{F}) = V(\mathbf{a}, \mathbf{f}, \mathbb{E})$.

Proof. Let $g_i \in \mathbb{F}$ with $a_1 \sigma(g_0) + a_2 g_0 = 0$ and $a_1 \sigma(g_i) + a_2 g_i = f_i$ for $1 \leq i \leq n$. Then observe that $(0, \dots, 0, g_0), (1, 0, \dots, 0, g_1), \dots, (0, \dots, 0, 1, g_n)$ forms a basis of $\mathbb{V} := V(\mathbf{a}, \mathbf{f}, \mathbb{F})$ over $\mathbb{K} := \text{const}_\sigma \mathbb{F}$. Since \mathbb{V} is a subspace of $\mathbb{W} := V(\mathbf{a}, \mathbf{f}, \mathbb{E})$ over \mathbb{K} and the dimension of \mathbb{W} is at most $n + 1$, it follows that $\mathbb{V} = \mathbb{W}$. \square

This result allows us to specify a criterium in Theorem 6 if a Σ^* -extension of (\mathbb{E}, σ) over \mathbb{F} is \mathbb{F} -complete for \mathbf{f} .

Theorem 6. *Let (\mathbb{E}, σ) with $\mathbb{E} := \mathbb{F}(t_1) \dots (t_e)$ be a $\Pi\Sigma$ -extension of (\mathbb{F}, σ) and $\mathbf{f} \in \mathbb{E}^n$. Let $\{(\mathbf{a}_i, \mathbf{f}_i, \mathbb{F})\}_{1 \leq i \leq k}$ with $\mathbf{a}_i = (a_{i1}, a_{i2})$ and $\mathbf{f}_i = (f_{i1}, \dots, f_{ir_i}) \in \mathbb{F}^{r_i}$ be the \mathbb{F} -critical tuple set of an extension-stable reduction of $V((1, -1), \mathbf{f}, \mathbb{E})$ to \mathbb{F} . If (\mathbb{G}, σ) is a Σ^* -extension of (\mathbb{E}, σ) over \mathbb{F} where for any $1 \leq i \leq k$ and $1 \leq j \leq r_i$ there is a $g \in \mathbb{G}^*$ with $a_{i1} \sigma(g) - a_{i2} g = f_{ij}$ then the extension is \mathbb{F} -complete for \mathbf{f} .*

Proof. Suppose such an extension (\mathbb{G}, σ) of (\mathbb{E}, σ) over \mathbb{F} is not \mathbb{F} -complete for \mathbf{f} . Then we can take a $\mathbf{c} \in \mathbb{K}^n$ such that $\sigma(g) - g = \mathbf{c} \mathbf{f}$ has a solution in some $\Pi\Sigma$ -extension of (\mathbb{E}, σ) , but no solution in (\mathbb{G}, σ) and therefore no solution in (\mathbb{E}, σ) . Hence, by Lemma 1 there is a Σ^* -extension $(\mathbb{E}(s), \sigma)$ of (\mathbb{E}, σ) over \mathbb{F} and a $g \in \mathbb{E}(s)$ with $\sigma(g) - g = \mathbf{c} \mathbf{f}$. Consequently, by Proposition 1 there exists an i with $1 \leq i \leq k$ such that $V(\mathbf{a}_i, \mathbf{f}_i, \mathbb{F}) \subsetneq V(\mathbf{a}_i, \mathbf{f}_i, \mathbb{F}(s))$ holds for the Σ^* -extension $(\mathbb{F}(s), \sigma)$ of (\mathbb{F}, σ) . But by Proposition 2 we have $V(\mathbf{a}_i, \mathbf{f}_i, \mathbb{F}) = V(\mathbf{a}_i, \mathbf{f}_i, \mathbb{F}(s))$, a contradiction. \square

Example 5. *Consider Examples 2 and 3. Since for any $f \in \{0, t_1 + 1, -2(t_1 + 1), 1, -(t_1 + 1)\}$ there is a $g \in \mathbb{F}(t_2)(s)$ with $\sigma(g) - g = f$, it follows that the Σ^* -extension $(\mathbb{F}(t_2)(s), \sigma)$ of $(\mathbb{F}(t_2), \sigma)$ is \mathbb{F} -complete for $(\sigma(t_2)/t_1)$.*

Finally, in Proposition 3 we show that such an extension can be constructed that fulfills our sufficient criterium.

Proposition 3. *Let (\mathbb{E}, σ) be a $\Pi\Sigma$ -extension of (\mathbb{F}, σ) , $(a_{i1}, a_{i2}) \in \mathbb{F}^2$ be homogeneous over \mathbb{F} and $f_i \in \mathbb{F}$ for $1 \leq i \leq n$. Then there is a Σ^* -extension (\mathbb{G}, σ) of (\mathbb{E}, σ) over \mathbb{F} such that there is a $g \in \mathbb{G}^*$ with $a_{i1} \sigma(g) + a_{i2} g = f_i$ for all $1 \leq i \leq n$. If (\mathbb{F}, σ) is a $\Pi\Sigma$ -field over a σ -computable \mathbb{K} , such a $\Pi\Sigma$ -field (\mathbb{G}, σ) can be computed.*

Proof. Suppose that we have shown the existence for such a Σ^* -extension (\mathbb{G}, σ) of (\mathbb{E}, σ) over \mathbb{F} for $1 \leq i \leq n$. Now let $(a_1, a_2) \in \mathbb{F}^2$ be homogeneous over \mathbb{F} and $f \in \mathbb{F}$. If there is a $g \in \mathbb{G}$ with $a_1 \sigma(g) + a_2 g = f$, we have shown the induction step. Otherwise, construct the extension $(\mathbb{G}(s), \sigma)$ of (\mathbb{G}, σ) with s transcendental over \mathbb{F} and $\sigma(s) = s - \frac{f}{h a_2} \in \mathbb{F}$ where $h \in \mathbb{F}^*$ with $a_1 \sigma(h) + a_2 h = 0$. Now suppose there is a $g' \in \mathbb{G}^*$ with $\sigma(g') - g' = -\frac{f}{h a_2}$. Then for $w := h g' \in \mathbb{G}^*$ we have $f = -a_2 h(\sigma(g') - g') = a_1 \sigma(h) \sigma(g') + a_2 h g' = a_1 \sigma(w) + a_2 w$, a contradiction. Hence by Theorem 1 $(\mathbb{G}(s), \sigma)$ is a Σ^* -extension of (\mathbb{G}, σ) over \mathbb{F} . Furthermore, for $v := h s \in \mathbb{G}(s)$ we have that $a_1 \sigma(v) + a_2 v = f$, which follows by similar arguments as above for w . This closes the induction step.

Now suppose that (\mathbb{F}, σ) is a $\Pi\Sigma$ -field over a σ -computable \mathbb{K} . Then by Theorem 5 one can decide if there exists a $g \in \mathbb{G}^*$ with $a_1 \sigma(g) + a_2 g = f$ and can compute an $h \in \mathbb{F}^*$ with $a_1 \sigma(h) + a_2 h = 0$. This shows, that the proof above becomes completely constructive. \square

Summarizing, we first compute a basis of $V((1, -1), \mathbf{f}, \mathbb{E})$ with an extension-stable reduction and extract the \mathbb{F} -critical tuples; this is possible by Theorem 5. Next we construct with Proposition 3 a Σ^* -extension of (\mathbb{E}, σ) over \mathbb{F} that fulfills the criterium in Theorem 6.

Example 6. *Looking at Example 3 we obtain immediately the Σ^* -extension $(\mathbb{F}(t_2)(s), \sigma)$ of $(\mathbb{F}(t_2), \sigma)$ with $\sigma(s) = s + \frac{1}{(t_1+1)^2}$ which is \mathbb{F} -complete for $(\sigma(t_2)/t_1) \in \mathbb{F}(t_2)^1$ by following this strategy. Finally we restart our computation in this extension and obtain for $V((1, -1), (\sigma(t_2)/t_1), \mathbb{F}(t_2)(s))$ the basis $\{(0, 1), (2, t_2 + s)\}$ which gives the result $g = \frac{t_2 + s}{2}$ in Example 1.*

Now we proceed, and try to find a $g' \in \mathbb{F}(t_2)(s)$ such that $\sigma(g') - g' = \sigma(g/t_1)$, but we fail.

Therefore, we extract the \mathbb{F} -critical tuples $((t_1 + 1)^3, -(t_1 + 1)^3), \mathbf{f}, \mathbb{F}$ with

$$\mathbf{f} \in \left\{ \left(-(t_1 + 1)^2, \frac{t_1 + 1}{2}, -2(t_1 + 1) \right), \left(2(t_1 + 1)^2, 0, 0 \right), \right. \\ \left. (0, 0), (-3(t_1 + 1)^2, (t_1 + 1)^2, 0), ((t_1 + 1), 2, (t_1 + 1)^2) \right\} \quad (8)$$

from our extension stable reduction to \mathbb{F} . Following Theorem 6 we construct a Σ^* -extension (\mathbb{G}, σ) of $(\mathbb{F}(t_2)(s), \sigma)$ over \mathbb{F} such that there are $h \in \mathbb{G}$ with $\sigma(h) - h = \frac{f}{(t_1+1)^2}$ for all $f \in \mathbb{F}$ from (8). Following the algorithm given in the proof of Proposition 3 we obtain the Σ^* -extension $(\mathbb{F}(t_2)(s)(s'), \sigma)$ of $(\mathbb{F}(t_2)(s), \sigma)$ with $\sigma(s') = s' + \frac{2}{(t_1+1)^3}$; afterwards we cancel the constant factor 2. By Theorem 6 this extension is \mathbb{F} -complete for $(\sigma(g/t_1)) \in \mathbb{F}(t_2)(s)^1$. To this end we compute for the solution space $V((1, -1), (\sigma(g/t_1)), \mathbb{F}(t_2)(s)(s'))$ the basis $\{(0, 1), (6, (t_2^3 + 3t_2s + 2s'))\}$ which gives the final result in Example 1.

Let $I \subseteq \{0, \dots, e\}$. Restricting Algorithm 3 to $I = \{0\}$ gives just the above strategy. In addition, $\mathbb{F}_i := \mathbb{F}(t_1) \dots (t_i)$ -complete extensions can be searched for all $i \in I$. This can be motivated as follows. \mathbb{F}_i -complete extensions (\mathbb{E}_i, σ) of (\mathbb{E}, σ) with bigger i can give more solutions $\mathbb{W}_i := \Pi_n(V((1, -1), \mathbf{f}, \mathbb{E}_i))$; but they might be also more complicated, since they depend on more t_j (which are usually more nested). Hence, one should look for extensions with smallest possible i that give still interesting solutions in \mathbb{W}_i . Algorithm 3 enables one to search in one stroke for all those \mathbb{F}_i -complete extensions with $i \in I$.

Algorithm 3. SingleNestedCompleteExtensions $((\mathbb{E}_0, \sigma), \mathbf{f})$

Input: A $\Pi\Sigma$ -field (\mathbb{E}_0, σ) with $\mathbb{E}_0 := \mathbb{F}(t_1) \dots (t_e)$ over a σ -computable \mathbb{K} ,

$I = \{j_1 < \dots < j_\lambda\} \subseteq \{0, \dots, e\}$ and $\mathbf{f} \in \mathbb{E}_0^n$.

Output: Σ^* -extensions (\mathbb{E}_i, σ) of $(\mathbb{E}_{i-1}, \sigma)$ over $\mathbb{F}(t_1) \dots (t_{j_i})$ which are single-nested $\mathbb{F}(t_1) \dots (t_{j_i})$ -complete for \mathbf{f} for $1 \leq i \leq \lambda$; a basis of $V((1, -1), \mathbf{f}, \mathbb{E}_\lambda)$.

- (1) Compute a basis B of $V((1, -1), \mathbf{f}, \mathbb{E}_0)$ with an extension-stable reduction to \mathbb{F} . Let $d := \dim V((1, -1), \mathbf{f}, \mathbb{E}_0)$.
- (2) IF $d = n + 1$ RETURN $((\mathbb{E}_0, \sigma), B)$ FI
- (3) FOR $i = 1$ TO λ DO
- (4) Extract the $\mathbb{F}(t_1) \dots (t_{j_\lambda})$ -critical tuple set, say $\{\mathbf{a}_i, \mathbf{f}_i, \mathbb{F}\}_{1 \leq i \leq k}$ where $\mathbf{a}_i = (a_{i1}, a_{i2})$ and $\mathbf{f}_i = (f_{i1}, \dots, f_{ir_i}) \in \mathbb{F}^{r_i}$ with $r_i > 0$. Construct a single-nested Σ^* -extension (\mathbb{E}_i, σ) of $(\mathbb{E}_{i-1}, \sigma)$ over $\mathbb{F}(t_1) \dots (t_{j_\lambda})$ such that for any $1 \leq i \leq k$ and $1 \leq j \leq r_i$ there exists a $g \in \mathbb{E}_i^*$ with $a_{i1} \sigma(g) - a_{i2} g = f_{ij}$. OD
- (5) IF $(\mathbb{E}_\lambda, \sigma) = (\mathbb{E}_0, \sigma)$ RETURN $((\mathbb{E}_0, \sigma), B)$ FI
- (6) Compute a basis B' of $V((1, -1), \mathbf{f}, \mathbb{E}_\lambda)$ with dimension d' .
- (7) IF $d = d'$ then RETURN $((\mathbb{E}_0, \sigma), B)$ else RETURN $((\mathbb{E}_\lambda, \sigma), B')$ FI

Theorem 7. Let (\mathbb{E}_0, σ) with $\mathbb{E}_0 := \mathbb{F}(t_1) \dots (t_e)$ be a $\Pi\Sigma$ -field over a σ -computable \mathbb{K} , $I = \{j_1 < \dots < j_\lambda\} \subseteq \{0, \dots, e\}$ and $\mathbf{f} \in \mathbb{E}_0^n$. Then with Algorithm 3 Σ^* -extensions (\mathbb{E}_i, σ) of $(\mathbb{E}_{i-1}, \sigma)$ over $\mathbb{F}(t_1) \dots (t_{j_i})$ can be computed which are $\mathbb{F}(t_1) \dots (t_{j_i})$ -complete for \mathbf{f} for $1 \leq i \leq \lambda$.

The Σ^* -extension $(\mathbb{E}_\lambda, \sigma)$ of (\mathbb{E}, σ) over \mathbb{F} produced by Algorithm 3 can be reduced to a more compact extension that delivers the same solutions $\Pi_n(V((1, -1), \mathbf{f}, \mathbb{E}_\lambda))$. Namely, if $\mathbb{E}_\lambda := \mathbb{E}(s_1) \dots (s_e)$, remove those s_i that do not occur in $\mathbb{W}_\lambda = V((1, -1), \mathbf{f}, \mathbb{E}_\lambda)$. Moreover, join all those s_i 's to one single Σ^* -extension which occur in a basis element of \mathbb{W}_i ; see

Lemma 1. Furthermore, cancel constants from \mathbb{K} that may occur in the summand $\sigma(s_i) - s_i$; see Example 6.

Observe that recursively applied indefinite summation can be treated more efficiently, if one reduces these extensions after each application of Algorithm 3.

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