SYNASC 2004

A Collection of Denominator Bounds to Solve Parameterized Linear Difference Equations in ΠΣ-Extensions

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Indefinite Summation Examples

$$\sum_{k=0}^{n} H_k = (H_{n+1} - 1)(n+1)$$

where $H_k = \sum_{i=1}^{k} \frac{1}{i}$

$$\sum_{k=3}^{n} \frac{H_k\Big(3k(k+1)H_k + 3k - 1\Big)\prod_{i=3}^{k}\frac{iH_i-3}{iH_i}}{k^2(kH_k-3)} = \frac{81}{8} - \Big(H_n^3 + 3H_n^2\Big)\prod_{i=3}^{n}\frac{iH_i-3}{iH_i}$$

$$\prod_{k=2}^{n} \frac{(k+1)(k+2)^2(k^2-k+k!)(2k-3k^2+k^3+k!)}{(k-1)^5k^6(k!+1)(k+k!)} = \frac{n^5(n+1)^3(n+2)^2}{9n!^4(n!+1)(n+n!)^2(n^2-n+n!)}$$

Indefinite Summation in Difference Fields

Goal: Find a closed form for

$$\sum_{k=0}^{n} H_k$$

where $H_k = \sum_{i=1}^{k} \frac{1}{i}$.

A difference field for the problem

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(t)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q},$$

$$\sigma(k) = k + 1, \qquad \mathcal{S}\, k = k + 1,$$

$$\sigma(t) = t + \frac{1}{k+1}, \qquad \mathcal{S}\, H_k = H_k + \frac{1}{k+1}.$$

$(\mathbb{Q}(k)(t), \sigma)$ is our difference field.

The telescoping problem

$$\text{Find } g \in \mathbb{Q}(k)(t) : \boxed{\sigma(g) - g = t}$$

$$\downarrow$$

$$g = (t-1)\, k.$$

The closed form

$$\boxed{\mathcal{S}\,(H_k - 1)(k) - (H_k - 1)k = H_k}$$

$$\downarrow$$

$$\sum_{k=0}^{n} H_k = (H_{n+1} - 1)(n+1).$$

FIND $g \in \mathbb{Q}(k)(t)$:

$$\sigma(g) - g = t.$$

Denominator bounding: COMPUTE a polynomial $d \in \mathbb{Q}(k)[t]^*$ s.t. $d = 1$

$$\forall g \in \mathbb{Q}(k)(t): \quad \sigma(g) - g = t \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[t].$$

FIND $g' \in \mathbb{Q}(k)[t]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = t.$$

Degree bounding: COMPUTE $b \geq 0$ s.t. $b = 2$

$$\forall g \in \mathbb{Q}(k)[t] \quad \sigma(g) - g = t \quad \Rightarrow \quad \deg(g) \leq b.$$

COMPUTE coefficients of $g = g_2\,t^2 + g_1\,t + g_0$.

$g = k\,t - k$

$$\left[\sigma(g_2)\left(t + \frac{1}{k+1}\right)^2 + \sigma(g_1 t + g_0)\right] - \left[g_2\,t^2 + g_1 t + g_0\right] = t$$

coeff. comp. →

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,t + g_0) - (g_1\,t + g_0) = t - c\left[\frac{2t(k+1)+1}{(k+1)^2}\right]$$

coeff comp. →

$$\boxed{\sigma(g_1) - g_1 = 1 - c\frac{2}{k+1}}$$

$c = 0,\;\; g_1 = k + d,$
$d \in \mathbb{Q}$

$$g_0 = -k, d = 0 \rightarrow \boxed{\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}}$$

Summation and Linear Difference Equations

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- $\mathbb{F} = \mathbb{K}(t_1)\dots(t_e)$ rational function field,
- for all $1 \le i \le e$:

$$\sigma(t_i) = \alpha_i\, t_i + \beta_i, \quad \alpha_i, \beta_i \in \mathbb{K}(t_1)\dots(t_{i-1}),$$

- $\text{const}_\sigma \mathbb{K}(t_1)\dots(t_e) = \mathbb{K}$ plus other constraints.

Telescoping

- GIVEN $f \in \mathbb{F}$
- FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$$\downarrow \qquad \uparrow$$

Parameterized Telescoping

- GIVEN $f_0, \dots, f_d \in \mathbb{F}$, $a_0, a_1 \in \mathbb{F}$
- FIND ALL $c_0, \dots, c_d \in \mathbb{K}$, $h \in \mathbb{F}$:

$$\boxed{a_1\, \sigma(h) - a_0\, h = c_0\, f_0 + \dots + c_d\, f_d}$$

Example 2

Consider the ΠΣ-field $(\mathbb{Q}(k)(t)(p),\sigma)$ with

$$\sigma(k)=k+1,\quad \sigma(t)=t+\frac{1}{k+1},\qquad \sigma(p)=\frac{t(k+1)-2}{t(k+1)+1}p.$$

•Find $g\in\mathbb{Q}(k)(t)(p)$:

$$\sigma(g)-g=\frac{t(3k(k+1)t+3k-1)}{k^2(k\,t-3)}\,p.$$

Den. bound:	$d=1$	
Deg. bound:	$b=1$	(numerator)

$$g=\frac{g'\,p}{1}$$

$$\downarrow$$

•Find $g' \in \mathbb{Q}(k)(t)$:

$$\frac{t(k+1)-2}{t(k+1)+1}\sigma(g') - g' = \frac{t(3k(k+1)t+3k-1)}{k^2(k\,t-3)}.$$

Den. bound:	$d = t\,k - 3$
Deg. bound:	$b = 4$ (numerator)

$$g' = \frac{g''}{t\,k-3},$$

$$g'' = \sum_{i=0}^{4} g_i\, t^i$$

$$\downarrow$$

- We compute:

$$g'' = -\frac{t(tk-1)^2(k(t+3)-1)}{k^2}.$$

Hence,

$$\boxed{\sum_{k=3}^{n} \frac{H_k\big(3k(k+1)H_k+3k-1\big)\prod_{i=3}^{k}\frac{iH_i-3}{iH_i}}{k^2(kH_k-3)} = \frac{81}{8} - \big(H_n^3+3H_n^2\big)\prod_{i=3}^{n}\frac{iH_i-3}{iH_i}.}$$

Example 3

Consider the ΠΣ-field $(\mathbb{Q}(k)(t), \sigma)$ with

$$\sigma(k) = k + 1, \quad \sigma(t) = (k+1)t.$$

•Find $g \in \mathbb{Q}(k)(t)$:

$$\frac{\sigma(g)}{g} = -\frac{(k+2)(k+3)^2(k+t)(k^2-k+t)}{k^5(k+1)^5(t+1)(1+(k+1)t)} =: f \qquad \Leftrightarrow \qquad \sigma(g) - f\,g = 0$$

$$\boxed{\begin{array}{l}\text{Den. bound } d = t^4\,(t+1)\,(t+k)^2\,(t-k+k^2)\\ \text{Deg. bound } b = 0 \quad \text{(numerator)}\end{array}} \quad g = \frac{g'}{\boxed{t^4}\boxed{(t+1)\,(t+k)^2\,(t-k+k^2)}},$$

$g' \in \mathbb{Q}(k)$

$$\downarrow$$

•We compute

$$g' = k^5\,(1+k)^3\,(2+k)^2.$$

Hence,

$$\boxed{\prod_{k=2}^{n} \frac{(k+1)(k+2)^2(k^2-k+k!)(2k-3k^2+k^3+k!)}{(k-1)^5k^6(k!+1)(k+k!)} = \frac{n^5(n+1)^3(n+2)^2}{9n!^4(n!+1)(n+n!)^2(n^2-n+n!)}}$$

Denominator Bounding

$$\boldsymbol{\sigma}\Big(\frac{\boldsymbol{g'}}{\boldsymbol{d}}\Big) - (k+2)(k+3)^2 \frac{(k+t)(k^2-k+t)}{k^5(k+1)^5(t+1)(1+(k+1)t)} \frac{\boldsymbol{g'}}{\boldsymbol{d}} = 0$$

[Bronstein00] [SYNASC]

$\boxed{d = d_0\, d_1}$

[Karr81] [SYNASC]

$\boxed{d_0 = (t+1)(t+k)^2(t-k+k^2)}$

$\boxed{d_1 = t^4}$

"Good cancellation"

$$\frac{\sigma(\frac{1}{d_0})}{\frac{1}{d_0}} = \frac{(k+t)(k^2-k+t)}{k^5(k+1)^5(t+1)(1+(k+1)t)}$$

"Bad cancellation"

$$\frac{\sigma(\frac{1}{d_1})}{\frac{1}{d_1}} = \frac{t^4}{\sigma(t^4)} = \frac{1}{(k+1)^4}$$

Remark: Z's "Creative Telescoping"

- GIVEN $f_i = \mathrm{summand}(n+i,k) \in \mathbb{F}$
- FIND ALL $c_0, \ldots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0\, f_0 + \cdots + c_d\, f_d}$$

Zeilberger's Creative Telescoping Paradigm

• GIVEN

$$\mathrm{SUM}(m) := \sum_{k=0}^{m} \underbrace{(-1)^k \binom{m}{k}^3 \Big[3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)}\Big]}_{=:\, f(m,k)}, \qquad H_k^{(2)} := \sum_{i=1}^{k} \frac{1}{i^2}$$

• FIND $c_0(m)$, $c_1(m)$, $c_2(m)$, and $g(m,k)$ s.t.

$$\boxed{g(m,k+1) - g(m,k)} = \boxed{c_0(m)\, f(m,k) + c_1(m)\, f(m+1,k) + c_2(m)\, f(m+2,k)}$$

for all $0 \leq k \leq m$ and all $m \geq 0$

Denominator bounding + reduction gives:

$$c_0(m) := 3(3m+2)(3m+4)(3m+8), \;\; c_1(m) := 0, \;\; c_2(m) := (m+2)^2(3m+8)$$

$$g(m,k) := (-1)^k \binom{m}{k}^3 \frac{p_1(k, m, H_k, H_k^{(2)}, H_{m-k}, H_{m-k}^{(2)})}{(m-k+1)^5(m-k+2)^5}$$

$$g(m,k+1) := (-1)^k \binom{m}{k}^3 \frac{p_2(k, m, H_k, H_k^{(2)}, H_{m-k}, H_{m-k}^{(2)})}{(m-k+1)^5}$$

Summing this equation over k from 0 to m gives:

$$\boxed{g(m, m+1) - g(m, 0)} = \boxed{\begin{array}{l} c_0(m)\,\mathrm{SUM}(m) + \\ c_1(m)\,\big[\mathrm{SUM}(m+1) - f(m+1, m+1)\big] \\ c_2(m)\,\big[\mathrm{SUM}(m+2) - f(m+2, m+1) - f(m+2, m+2)\big] \end{array}}$$

Quadratic Padé Approximation to the Logarithm

(A. Weideman, K. Driver, H. Prodinger, C.S.)

Theorem (Sigma **2002**)**.** For all $m \geq 0$ we have

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^3\left[3(H_{m-k}-H_k)^2+H_{m-k}^{(2)}+H_k^{(2)}\right]=0.$$

Proof.

In[1]:= $\mathbf{Pade2} = \sum_{k=0}^{m}\left(3\ (-H_k + H_{-k+m})^2 + H_k^{(2)} + H_{-k+m}^{(2)}\right)\left(\left(\binom{m}{k}\right)^3 (-1)^k\right)_k;$

In[2]:= **GenerateRecurrence[Pade2]**

Out[2]= $\{3\ (2+3\ m)\ (4+3\ m)\ \mathrm{SUM}[m] + (2+m)^2\ \mathrm{SUM}[2+m] == 0\}$

In[3]:= **Table[Pade2, {m, 0, 10}]**

Out[3]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

■

Linear Difference Equations

- GIVEN $f, a_0, \ldots, a_m \in \mathbb{F}$
- FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

Parameterized Linear Difference Equations

- GIVEN $a_0, \ldots, a_m \in \mathbb{F}$,$f_0, \ldots, f_d \in \mathbb{F}$.
- FIND ALL $g \in \mathbb{F}$, $c_0, \ldots, c_d \in \mathbb{K}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = c_0\,f_0 + \cdots + c_d\,f_d}$$