ISSAC 2004

Symbolic Summation with Single-Nested Sum Extensions

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Telescoping:

- **Given** $f(k)$;
- **Find** $g(k)$ s.t.

$$g(k+1) - g(k) = f(k+1) \tag{1}$$

THEN

$$g(n) - g(0) = \sum_{k=1}^{n} f(k)$$

Example:

- **Given** $f(k) = H_k = \sum_{i=1}^{k} \frac{1}{i}$
- **Find** $g(k)$ s.t.

$$g(k+1) - g(k) = H_{k+1}$$

SIGMA: $g(k) = (k+1)H_k - k$

THEN

$$(n+1)H_n - n = \sum_{k=1}^{n} H_k$$

The Problem in Difference Fields

1. Consider rat. function field

$$\mathbb{F} := \mathbb{Q}(k)(t),$$

field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ canonically defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q} & & \\ \sigma(k) &= k+1 & \qquad \mathrm{S}\, k &= k+1 \\ \sigma(t) &= t + \frac{1}{k+1} & \qquad \mathrm{S}\, H_k &= H_k + \frac{1}{k+1} \end{aligned}$$

$(\mathbb{Q}(k)(t), \sigma)$ is our difference field.

2. Find $g \in \mathbb{F}$ s.t.

$$\sigma(g) - g = t + \frac{1}{k+1}$$

SIGMA:

$$g = (k+1)\,t - k$$

Find $g \in \mathbb{Q}(k)(t)$:

$$\sigma(g) - g = t + \frac{1}{k+1} \tag{2}$$

Denominator Bounding: Find $d \in \mathbb{Q}(k)[t]$:

$$g \in \mathbb{Q}(k)(t) \text{ with (2)} \Leftrightarrow \quad g' = g\,d \in \mathbb{F}[t] \text{ and}$$

$$\frac{1}{\sigma(d)}\sigma(g') - \frac{1}{d}g' = t + \frac{1}{k+1} \tag{3}$$

Degree Bounding: Find $b \geq 0$:

$$g' \in \mathbb{Q}(k)[t] \text{ with (5)} \quad \Rightarrow \quad \deg(g') \leq b$$

$$\left[\sigma(g_2)\left(t + \frac{1}{k+1}\right)^2 + \sigma(g_1 t + g_0)\right] - \left[g_2 t^2 + g_1 t + g_0\right] = t + \frac{1}{k+1}$$

coeff. comp. $\rightarrow$

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$ $\rightarrow$

$$\sigma(g_1\,t + g_0) - (g_1\,t + g_0) = \left[t + \frac{1}{k+1}\right] - c\left[\frac{2t(k+1)+1}{(k+1)^2}\right]$$

coeff comp. $\rightarrow$

$$\sigma(g_1) - g_1 = t - c\frac{2}{k+1}$$

$c = 0,\ g_1 = k + d,$ $d \in \mathbb{Q}$ $\rightarrow$

$$\sigma(g_0) - g_0 = -\frac{k}{k+1} - d\frac{1}{k+1}$$

$\uparrow\downarrow$

$g_0 = -k, d = 1$

HENCE $g = (k+1)\,t - k$

Let $(\mathbb{F}(t),\sigma)$ be a difference field where

$$\sigma(t)=\alpha\,t+\beta,\quad \alpha,\beta\in\mathbb{F},$$

and

$$\mathbb{K}=\mathrm{const}_{\sigma}\mathbb{F}:=\{f\in\mathbb{F}(t)\,|\,\sigma(f)=f\}$$

plus some technical conditions.

Parameterized Linear Difference Equations (*PLDE*)

- GIVEN $\mathbf{a}=(a_1,a_2)\in\mathbb{F}(t)^2$, $\mathbf{f}=(f_1,\dots,f_n)\in\mathbb{F}(t)^n$
- FIND

$$\mathrm{V}(\mathbf{a},\mathbf{f},\mathbb{F}(t))=\{(c_1,\dots,c_n,g)\in\mathbb{K}^n\times\mathbb{F}(t):\ \boxed{a_1\,\sigma(g)+a_2\,g=c_1\,f_1+\dots+c_n\,f_n}\}\quad (4)$$

Reduction process

1. Denominator elimination

   FIND $d\in\mathbb{F}[t]^*$: (4) holds

$$\Leftrightarrow\quad g'=g\,d\in\mathbb{F}[t]\text{ and}$$

$$\frac{a_1}{\sigma(d)}\sigma(g')+\frac{a_2}{d}g'=c_1\,f_1+\dots+c_n\,f_n \qquad (5)$$

2. Degree bounding

   FIND $b\geq 0$: (5) holds with $g'\in\mathbb{F}[t]$ then

$$\deg(g')\leq b$$

3. FIND $c_i\in\mathbb{K}$ and $g'=g_b\,t^b+\dots+g_0$:

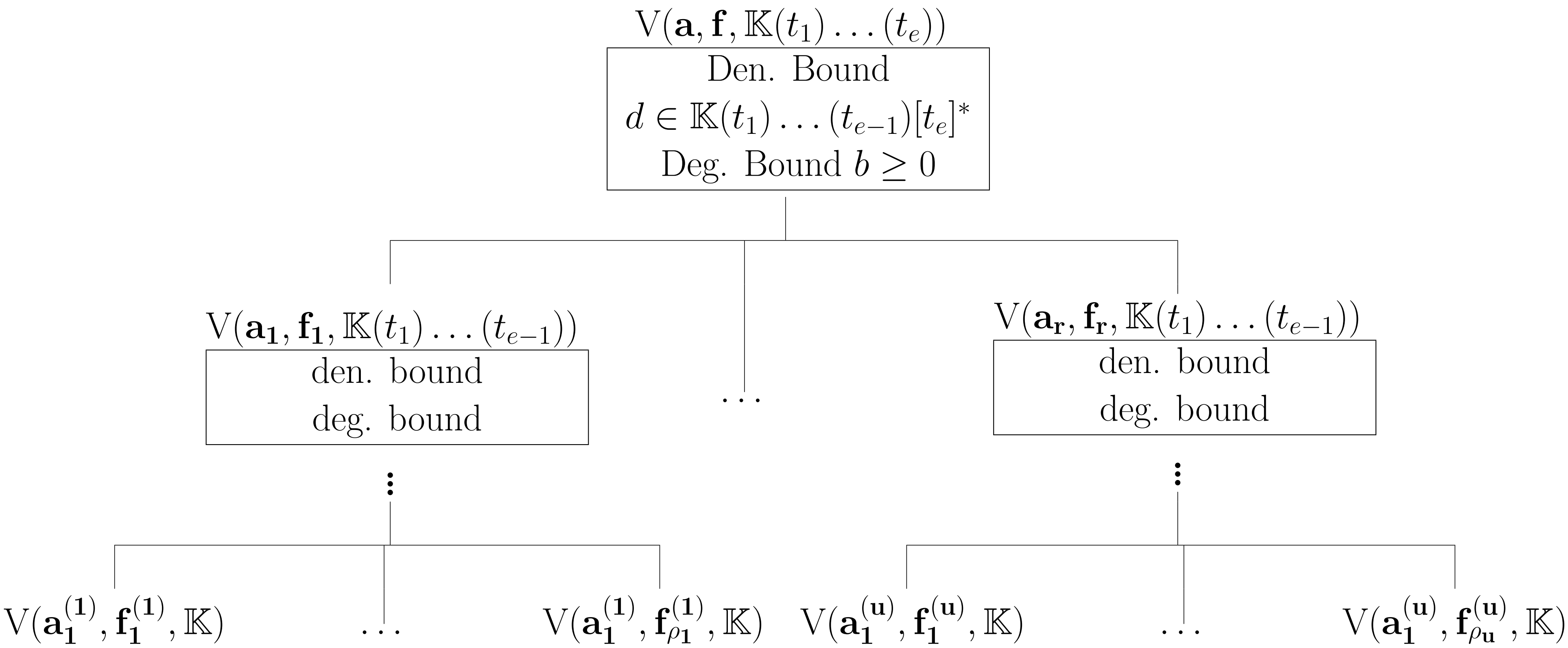
   Solving *PLDE* problems in $(\mathbb{F},\sigma)$

A $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:

- $\mathbb{F} = \mathbb{K}(t_1)\dots(t_e)$ rational function field,
- for all $1 \leq i \leq e$:

$$\sigma(t_i) = \alpha_i\, t_i + \beta_i, \quad \alpha_i, \beta_i \in \mathbb{K}(t_1)\dots(t_{i-1}),$$

- plus some technical conditions, like $\mathrm{const}_\sigma \mathbb{K}(t_1)\dots(t_e) = \mathbb{K}$.

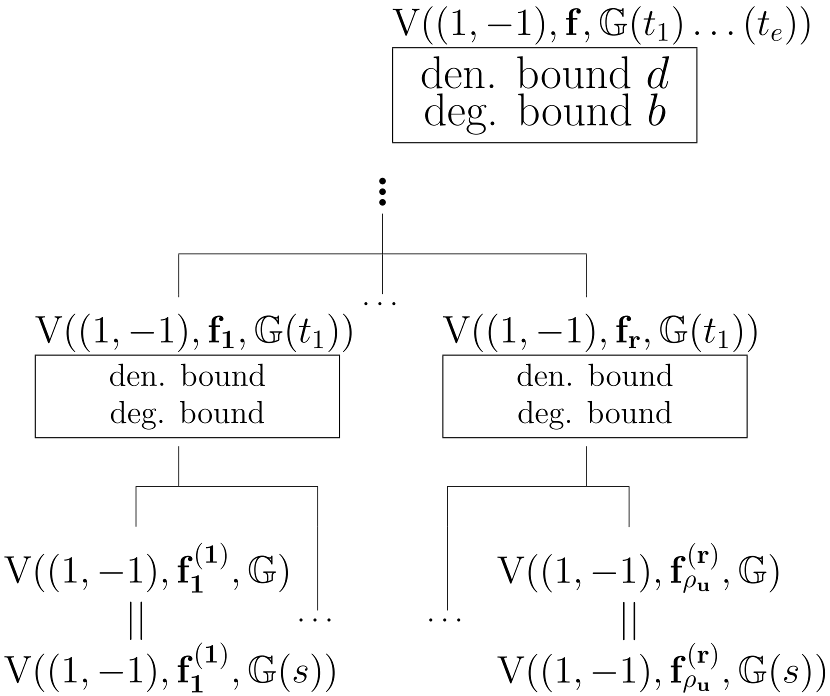


Extension stable reductions

A $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ where

$$\mathbb{F} = \underbrace{\mathbb{K}(\tau_1) \dots (\tau_\epsilon)}_{=: \mathbb{G}}(t_1) \dots (t_e)$$

GIVEN $\mathbf{a} \in \mathbb{F}^2$, $\mathbf{f} \in \mathbb{F}^n$.
FIND a basis of



FIND $c_i \in \mathbb{K}$, g

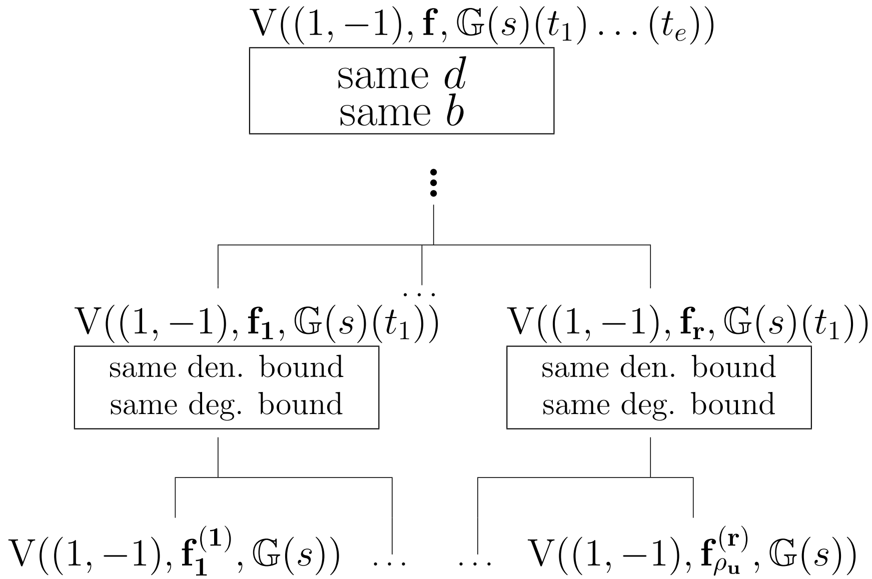
$$\sigma(g) - g = \sum_{i=1}^{n} c_i f_i$$

An extension $(\mathbb{F}(s), \sigma)$ with

$$\sigma(s) = \alpha\, s + \beta, \quad \alpha, \beta \in \mathbb{G}$$

Reordering gives

$$(\mathbb{G}(t_1) \dots (t_e)(s), \sigma) = (\mathbb{G}(s)(t_1) \dots (t_e), \sigma)$$



$$\mathrm{V}(\mathbf{a}, \mathbf{f}, \mathbb{F}(t))$$

Den. Reduction $d \in \mathbb{F}[t]^*$

Deg. Bounding $b \geq 0$

$$\mathrm{V}(\tilde{\mathbf{a}}, \tilde{\mathbf{f}}, \mathbb{F}[t]_b)$$

$\mathrm{V}(\mathbf{a_1}, \mathbf{f_1}, \mathbb{F})$ $\quad \dots \quad$ $\mathrm{V}(\mathbf{a_r}, \mathbf{f_r}, \mathbb{F})$

GIVEN $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{G}^n$

Construct $\Pi\Sigma$-field $(\mathbb{G}(s_1) \dots (s_\lambda), \sigma)$ with

$$\sigma(s_i) - s_i \in \mathbb{G}$$

s.t.

$$\dim \mathrm{V}((1, -1), \mathbf{f}, \mathbb{G}(s_1) \dots (s_\lambda)) = n + 1.$$

Namely: adjoin extension s_i with

$$\sigma(s_i) = s_i + f_j$$

if necessary.