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Session 18:
Computer Algebra and Computer Analysis

Symbolic Summation over Recurrences and Indefinite Nested Sums and Products

Carsten.Schneider@risc.uni-linz.ac.at
RISC-Linz
J. Kepler University Linz

A Bonus Problem in "Concrete Mathematics"

Chapter 6. Special Numbers, Bonus problem 69:

> Find a closed form for
>
> $$\sum_{k=1}^{n} \underbrace{k^2 \mathrm{H}_{n+k}}_{f(k)},$$
>
> where $\mathrm{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

Knuth's answer to the problem is

$$\frac{1}{3}n\Big(n+\frac{1}{2}\Big)\,(n+1)\,(2\mathrm{H}_{2n}-\mathrm{H}_n) - \frac{1}{36}n\,\big(10n^2+9n-1\big)$$

with the remark

> *"It would be nice to automate the derivation of formulas such as this."*

In[1]:= $<<$ **Sigma`**

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Sigma -A summation package by Carsten Schneider
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In[2]:= **Problem69 = SigmaSum[k^2 SigmaHNumber[n + k], {k, 1, n}]**

Out[2]= $\sum_{k=1}^{n} \left(k^2\ H_{k+n}\right)$

In[3]:= **SigmaReduce[Problem69]**

Out[3]= $-\frac{1}{36}\ n\ (1+n)\ (-1+10\ n+6\ (1+2\ n)\ H_n-12\ (1+2\ n)\ H_{2n})$

Telescoping

- GIVEN $f(k)$
- FIND $g(k)$:

$$\boxed{g(k+1) - g(k) = f(k)}$$

Then:

$$\boxed{g(b+1) - g(a) = \sum_{k=a}^{b} f(k)}$$

- Algebraic setting in Sigma: Karr's $\Pi\Sigma$-fields (1981)

Zeilberger's Creative Telescoping Paradigm

- GIVEN

$$\mathrm{SUM}(m) := \sum_{k=0}^{m} \underbrace{(-1)^k \binom{m}{k}^3 \Big[3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)}\Big]}_{=:\, f(m,k)}, \quad H_k^{(2)} := \sum_{i=1}^{k} \frac{1}{i^2}$$

- FIND $c_0(m)$, $c_1(m)$, $c_2(m)$, and $g(m,k)$ s.t.

$$\boxed{g(m,k+1) - g(m,k)} = \boxed{c_0(m)\, f(m,k) + c_1(m)\, f(m+1,k) + c_2(m)\, f(m+2,k)}$$

for all $0 \leq k \leq m$ and all $m \geq 0$

Sigma computes:

$$c_0(m) := 3(3m+2)(3m+4)(3m+8), \;\; c_1(m) := 0, \;\; c_2(m) := (m+2)^2(3m+8)$$

$$g(m,k) := (-1)^k \binom{m}{k}^3 \frac{p_1(k,m,H_k,H_k^{(2)},H_{m-k},H_{m-k}^{(2)})}{(m-k+1)^5(m-k+2)^5}$$

$$g(m,k+1) := (-1)^k \binom{m}{k}^3 \frac{p_2(k,m,H_k,H_k^{(2)},H_{m-k},H_{m-k}^{(2)})}{(m-k+1)^5}$$

Summing this equation over k from 0 to m gives:

$$\boxed{g(m,m+1)-g(m,0)} = \boxed{\begin{array}{l} c_0(m)\,\mathrm{SUM}(m)+ \\ c_1(m)\left[\mathrm{SUM}(m+1)-f(m+1,m+1)\right] \\ c_2(m)\left[\mathrm{SUM}(m+2)-f(m+2,m+1)-f(m+2,m+2)\right] \end{array}}$$

Quadratic Padé Approximation to the Logarithm

(A. Weideman, K. Driver, H. Prodinger, C.S.)

Theorem (Sigma 2002). For all $m \geq 0$ we have

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)} \right] = 0.$$

Proof.

In[4]:= $\mathbf{Pade2} = \sum_{\mathbf{k=0}}^{\mathbf{m}} \left(\mathbf{3}\ (-\mathbf{H_k} + \mathbf{H_{-k+m}})^{\mathbf{2}} + \mathbf{H_k^{(2)}} + \mathbf{H_{-k+m}^{(2)}}\right) \left(\left(\binom{\mathbf{m}}{\mathbf{k}} \right)^{\mathbf{3}} (-\mathbf{1})^{\mathbf{k}} \right)_{\mathbf{k}};$

In[5]:= **GenerateRecurrence[Pade2]**

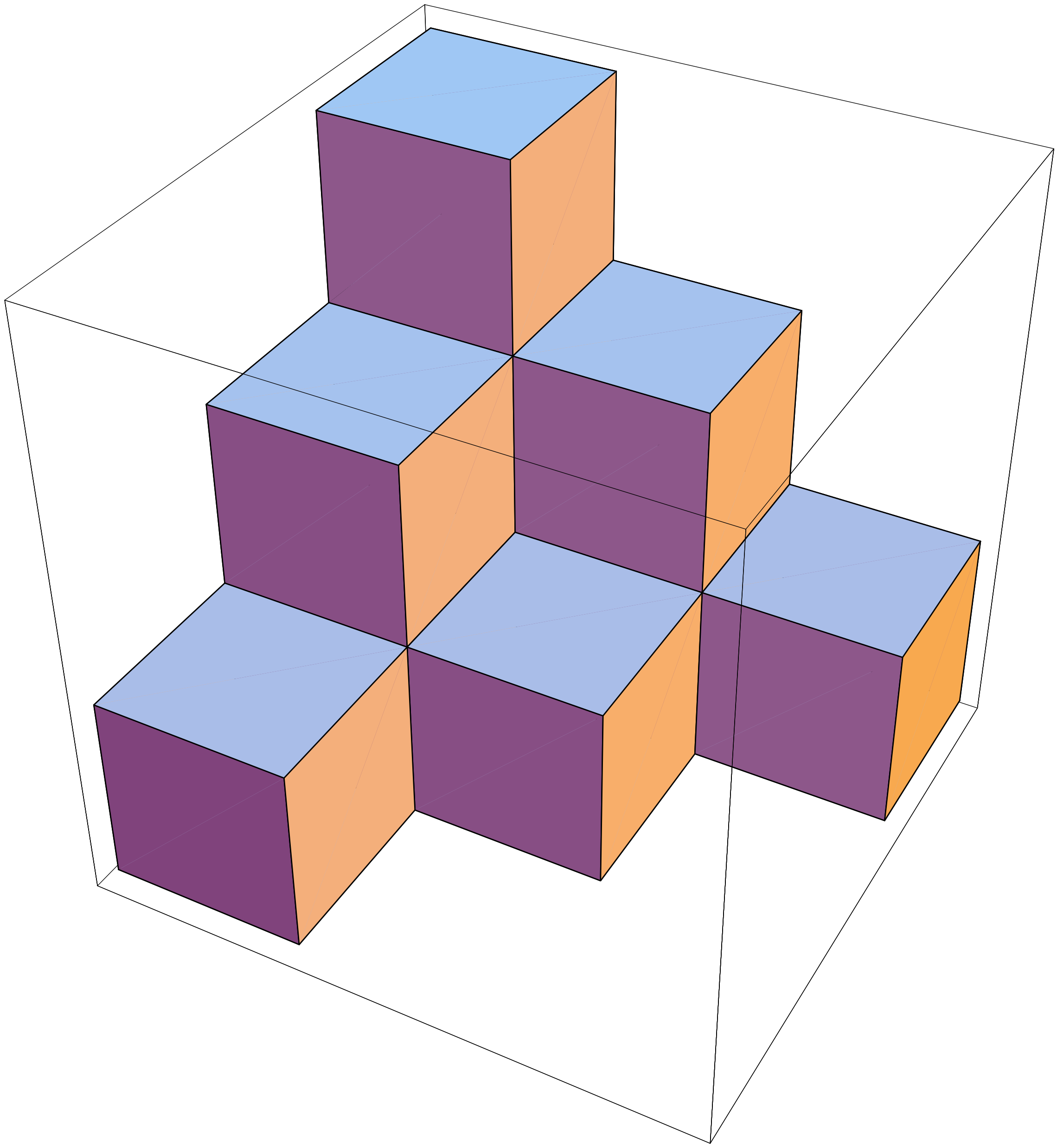
Out[5]= $\{3\ (2+3\ \mathtt{m})\ (4+3\ \mathtt{m})\ \mathtt{SUM}[\mathtt{m}] + (2+\mathtt{m})^2\ \mathtt{SUM}[2+\mathtt{m}] == 0\}$

In[6]:= **Table[Pade2, {m, 0, 10}]**

Out[6]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

■

Totally **S**ymmetric **P**lane **P**artitions (TSPP)



Theorem (Stembridge, 1995). The number of totally symmetric plane partitions of order $n \geq 1$ is

$$TSPP(n) = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2}.$$

A symbolic summation proof (G. Andrews, P. Paule, C.S.)

Example:

$$A = \sum_{k=0}^{2m} \underbrace{\sum_{s=0}^{\lfloor \frac{2m-k}{2} \rfloor} \left(\binom{m-s-1}{2m-2s-k} + \binom{m-s}{2m-2s-k} \right) \frac{(-1)^{s+k}}{2m\, 4^s} \sum_{r=0}^{s} \frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r}}_{=: \mathcal{R}(k) = \texttt{RSum}}$$

• Sigma gives:

In[7]:= $\mathbf{rec}\mathcal{R} = \mathbf{GenerateRecurrence}[\mathcal{R}\mathbf{Sum}, \mathbf{k}]/.\mathbf{SUM} \to \mathcal{R}$

10.985 Second

Out[7]= $2\ (2+k)^2\ (k-2\ m)\ (1+k+2\ m)\ \mathcal{R}[k] -$
$\left(29\ k^3 + 5\ k^4 + k\ \left(46 - 20\ m - 40\ m^2\right) - 12\ \left(-1+m+2\ m^2\right) - 2\ k^2\ \left(-29+3\ m+6\ m^2\right)\right)\ \mathcal{R}[1+k] +$
$\left(26\ k^3 + 4\ k^4 + k\ \left(55 - 14\ m - 28\ m^2\right) + k^2\ \left(59 - 6\ m - 12\ m^2\right) - 6\ \left(-3+m+2\ m^2\right)\right)\ \mathcal{R}[2+k] -$
$(1+k)^2\ (2+k-2\ m)\ (3+k+2\ m)\ \mathcal{R}[3+k] == 0$

• New Sigma gives:

In[8]:= $\mathbf{A} = \sum_{\mathbf{k=0}}^{\mathbf{2\ m}} \mathcal{R}[\mathbf{k}];$

In[9]:= $\mathbf{SigmaReduce}[\mathbf{A}, \{\{\mathbf{rec}\mathcal{R}, \mathcal{R}[\mathbf{k}]\}\}]$

Out[9]= $\dfrac{(-3\ (1+2\ m)\ \mathcal{R}[0] + (1+2\ m)\ \mathcal{R}[1] + (5+2\ m)\ \mathcal{R}[1+2\ m] - 2\ \mathcal{R}[2+2\ m])}{1+2\ m}$

Telescoping in recurrence terms

GIVEN: a linear recurrence relation

$$\mathcal{R}(k+s+1) = a_{-1}(k) + a_0(k)\mathcal{R}(k) + \cdots + a_s(k)\mathcal{R}(k+s)$$

and

$$f(k) = f_{-1}(k) + f_0(k)\mathcal{R}(k) + \cdots + f_s(k)\mathcal{R}(k+s)$$

FIND

$$g(k) = g_{-1}(k) + g_0(k)\mathcal{R}(k) + \cdots + g_s(k)\mathcal{R}(k+s)$$

s.t.

$$g(k+1) - g(k) = f(k)$$

$$\Updownarrow$$

FIND $g_{-1}(k), \ldots, g_s(k)$ s.t.

$$\begin{aligned}
&\Big[g_{-1}(k+1) \quad -g_{-1}(k) - f_{-1}(k)\Big] 1 + \\
&\qquad\qquad \Big[-\, g_0(k) \quad -f_0(k)\Big] \mathcal{R}(k) + \\
&\Big[g_0(k+1) \qquad -g_1(k) \quad -f_0(k)\Big] \mathcal{R}(k+1) + \\
&\qquad\qquad \vdots \\
&\Big[g_{s-1}(k+1) \quad -g_s(k) \quad -f_s(k)\Big] \mathcal{R}(k+s) + \\
&\qquad\qquad g_s(k+1)\mathcal{R}(k+s+1) = 0
\end{aligned}$$

$$a_{-1}(k) + a_0(k)\mathcal{R}(k) + \cdots + a_s(k)\mathcal{R}(k+s)$$

$$\Updownarrow$$

$$\boxed{\begin{aligned}&\text{FIND } g_{-1}(k),\ldots,g_s(k) \text{ s.t.}\\
&\Big[g_{-1}(k+1) \;\; -g_{-1}(k)+\boxed{a_{-1}(k)g_s(k+1)}-f_{-1}(k)\Big]1+\\
&\qquad\qquad\Big[-g_0(k) \;\; +\boxed{a_0(k)g_s(k+1)} \;\; -f_0(k)\Big]\mathcal{R}(k)+\\
&\Big[g_0(k+1) \qquad -g_1(k) \;\; +\boxed{a_1(k)g_s(k+1)} \;\; -f_0(k)\Big]\mathcal{R}(k+1)+\\
&\qquad\qquad\qquad\vdots\\
&\Big[g_{s-1}(k+1) \quad -g_s(k) \;\; +\boxed{a_s(k)g_s(k+1)} \;\; -f_s(k)\Big]\mathcal{R}(k+s)=0\end{aligned}}$$

$$\Uparrow$$

FIND $g_{-1}(k), \ldots, g_s(k)$ s.t.

$$
\begin{aligned}
g_{-1}(k+1) - g_{-1}(k) + a_{-1}(k) g_s(k+1) - f_{-1}(k) &= 0 \\
- g_0(k) + a_0(k) g_s(k+1) - f_0(k) &= 0 \\
g_{r-1}(k+1) - g_r(k) + a_r(k) g_s(k+1) - f_r(k) &= 0, \qquad 0 < r < s
\end{aligned}
$$

$$\Updownarrow$$

FIND $g_{-1}(k), \ldots, g_s(k)$ s.t.

$$\boxed{\sum_{j=0}^{s} a_{s-j}(k+j) g_s(k+j+1) - g_s(k) = \sum_{j=0}^{s} f_{s-j}(k+j)}$$

$$g_0(k) = a_0(k) g_s(k+1) - f_0(k)$$

$$g_r(k) = a_r(k) g_s(k+1) + g_{r-1}(k+1) - f_r(k), \qquad 0 < r < s$$

$$\boxed{g_{-1}(k+1) - g_{-1}(k) = f_{-1}(k) + a_{-1}(k) g_s(k+1)}$$

The main problem (TSPP)

Given

$$A_2(i,m) := \sum_{k=i}^{2m} (-1)^k \mathcal{R}(k,m)$$

$$A_0(i,m) := \sum_{k=0}^{2m} \binom{i+k-3}{i-2} \mathcal{R}(k,m)$$

with

$$\mathcal{R}(k,m) := \sum_{s=0}^{\lfloor \frac{2m-k}{2} \rfloor} \left(\binom{m-s-1}{2m-2s-k} + \binom{m-s}{2m-2s-k} \right) \frac{(-1)^{s+k}}{2m\, 4^s} \sum_{r=0}^{s} \frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r}$$

Show that

$$F(i,m) := 2\mathcal{R}(i-2,m) - 5\mathcal{R}(i-1,m) + 6(-1)^i A_2(i,m) - A_0(i,m) - \prod_{s=1}^{2m-1} \frac{2(m+s-1)}{2m+s-2} = 0$$

for all $3 \le i \le 2m+1$

Creative telescoping with recurrences

Hermite Polynomials

In[10]:= $\mathbf{recH = H[k+2] == 2x\ H[k+1] - 2(k+1)H[k];}$

In[11]:= $\mathbf{initial = \{1, 2x\}}$

In[12]:= $\mathbf{BuildEvaluation[recH, H[k], \{1, 2x\}, 0]}$

In[13]:= $\mathbf{Table[H[k], \{k, 0, 6\}]//Simplify}$

Out[13]= $\{1,$

$2\ x,$

$-2+4\ x^2,$

$4\ x\ (-3+2\ x^2),$

$4\ (3-12\ x^2+4\ x^4),$

$8\ x\ (15-20\ x^2+4\ x^4),$

$8\ (-15+90\ x^2-60\ x^4+8\ x^6)$

In[14]:= $\mathbf{mySum = \sum_{k=0}^{n} H[k]\binom{n}{k};}$

In[15]:= $\mathbf{GenerateRecurrence[mySum, n, \{\{recH, H[k]\}\}]}$

Out[15]= $\{-2\ (1+n)\ SUM[n] + (1+2\ x)\ SUM[1+n] - SUM[2+n] == 0\}$

Summation over Recurrences

In[16]:= $\mathbf{rec}\mathcal{R} = \mathbf{6\ (1 + H_k + k\ H_k)\ (3 + 2\ k + 2\ H_k + 3\ k\ H_k + k^2\ H_k)}\ \mathcal{R}[\mathbf{k}] - \mathbf{5\ (1 + k)\ H_k\ (3 + 2\ k + 2\ H_k + 3\ k\ H_k + k^2\ H_k)}\ \mathcal{R}[\mathbf{1 + k}] + \mathbf{(1 + k)\ (2 + k)\ H_k\ (1 + H_k + k\ H_k)}\ \mathcal{R}[\mathbf{2 + k}] == \mathbf{0}$

In[17]:= $\mathbf{BuildEvaluation}[\mathbf{rec}\mathcal{R}, \mathcal{R}[\mathbf{k}], \{\mathbf{1}, \mathbf{2}\}, \mathbf{1}]$

In[18]:= $\mathbf{Table}[\mathcal{R}[\mathbf{k}], \{\mathbf{k}, \mathbf{1}, \mathbf{8}\}]$

Out[18]= $\{1, 2, \frac{11}{9}, -\frac{175}{18}, -\frac{5617}{90}, -\frac{7987}{30}, -\frac{68849}{70}, -\frac{1420787}{420}\}$

In[19]:= $\mathbf{mySum} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \binom{\mathbf{n}}{\mathbf{k}}\ \mathcal{R}[\mathbf{k}];$

In[20]:= $\mathbf{rec}\mathcal{R} = \mathbf{GenerateRecurrence}[\mathbf{mySum}, \mathbf{n}, \{\{\mathbf{rec}\mathcal{R}, \mathcal{R}[\mathbf{k}]\}\}]$

Out[20]= $\{144\ (1 + n)\ (2 + n)\ (21 + 6\ n + n^2)\ \mathtt{SUM}[n] - 84\ (2 + n)\ (75 + 60\ n + 15\ n^2 + 2\ n^3)\ \mathtt{SUM}[1 + n] + (8100 + 8700\ n + 3565\ n^2 + 730\ n^3 + 73\ n^4)\ \mathtt{SUM}[2 + n] - 7\ (3 + n)\ (100 + 60\ n + 15\ n^2 + 2\ n^3)\ \mathtt{SUM}[3 + n] + (3 + n)\ (4 + n)\ (16 + 4\ n + n^2)\ \mathtt{SUM}[4 + n] == \frac{1}{3}\ (1432 + 1290\ n + 307\ n^2 + 37\ n^3)\}$

We get:

$$\sum_{k=1}^{n} \binom{n}{k} \mathcal{R}(k) = \frac{1}{18}\Big[5\,3^{n+1}\big(H_n - \sum_{i=1}^{n} \frac{1}{3^i i}\big) - 4^{n+1}\big(H_n - \sum_{i=1}^{n} \frac{1}{3^i i}\big)\Big]$$

Indefinite summation in ΠΣ-fields

Self-cooked examples

$$\sum_{k=2}^{n} \frac{1}{\mathrm{H}_k(1-k\,\mathrm{H}_k)} = \frac{1-\mathrm{H}_n}{\mathrm{H}_n},$$

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{K+k}}{K+j}}{K+i} = 3\,\mathrm{H}_K\,\mathrm{H}_{K+N}^{(2)} + \mathrm{H}_{K+N}\left(3\,\mathrm{H}_K^2 - 3\,\mathrm{H}_K^{(2)} + 3\,\mathrm{H}_{K+N}^{(2)}\right) - 2\,\mathrm{H}_K^{(3)} + 2\,\mathrm{H}_{K+N}^{(3)}\Big),$$

where K is a positive integer and $\mathrm{H}_n^{(\alpha)} := \sum_{i=1}^{n} \frac{1}{i^\alpha}$;

Vicious random walkers on a multidimensional lattice, (Essam)

$$\sum_{k_1=0}^{n} \sum_{k_2=0}^{\mathbf{k1}} \sum_{k_3=0}^{k_2} \sum_{k_4=0}^{k_3} \sum_{k_5=0}^{k_4} (k_1-k_2)\,(k_1-k_3)\,(k_2-k_3)\,(k_1-k_4)\,(k_2-k_4)\,(k_3-k_4)\,(k_1-k_5)$$
$$(k_2-k_5)\,(k_3-k_5)\,(k_4-k_5)\binom{n}{k_1}\binom{n}{k_2}\binom{n}{k_3}\binom{n}{k_4}\binom{n}{k_5}$$
$$= \frac{3\,(-3+n)\,(-2+n)^2\,(-1+n)^3\,n^5\,\binom{2n}{n}^2\,2^n}{256\,(-5+2\,n)\,(3-8\,n+4\,n^2)^2}.$$

Sigma is able to prove this series of identities up to the case 7.

Variations of Calkin's identity

$$\sum_{k=0}^{a}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2 = (n-a)\binom{n}{a}\sum_{j=0}^{a}\binom{n}{j} + (1+a-\frac{n}{2})\left(\sum_{j=0}^{a}\binom{n}{j}\right)^2 - \frac{n}{2}\sum_{j=0}^{a}\binom{n}{j}^2$$
$$\stackrel{a=n}{\rightarrow} \sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2 = (n+1)\,4^n - \frac{n}{2}\,4^n - \frac{n}{2}\binom{2\,n}{n},$$

Indefinite summation in ΠΣ-fields

Calkin's identity and variations/generalizations

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3=\frac{n}{2}\,8^n+8^n-\frac{3\,n}{4}\,2^n\binom{2\,n}{n};$$

$$\sum_{k=0}^{2\,n}\left(\sum_{j=0}^{k}(-1)^{\frac{1}{2}(j-1)\,j}\binom{2\,n}{j}\right)^2=\frac{2^{2\,n}}{4}\Big(4{+}6\,n{-}4\,n\,(-1)^n{+}3\,n\sum_{j=2}^{n}\frac{\binom{4\,j}{2\,j}}{(4\,j-3)\,2^{2\,j}}{+}3\,n\sum_{j=2}^{n}\frac{\binom{4\,j}{2\,j}}{(4\,j-1)\,2^{2\,j}}\Big),$$

$$\sum_{k=0}^{2\,n-1}(-1)^k\left(\sum_{j=0}^{k}\binom{2\,n-1}{j}\right)^3=\frac{3\,n\,\binom{2\,n}{n}\,(-1)^n\,2^{2\,n}}{8(2\,n-1)}-\frac{64^n}{16},$$

$$\sum_{k=0}^{2\,n}(-1)^k\left(\sum_{j=0}^{k}\binom{2\,n}{j}\right)^3=\frac{64^n}{2}-\frac{(-1)^n}{16\,n}\frac{64^n}{\binom{2\,n}{n}}\sum_{i=0}^{n-1}(3+11\,i)\binom{2\,i}{i}^2\binom{3\,i}{i}64^{-i}.$$

Self-cooked examples

$$\sum_{r=1}^{n}\left(\frac{\sum_{l=1}^{r}\Big(\prod_{k=1}^{l}\Big(\frac{\mathrm{H}_{-k+n}}{1+\mathrm{H}_{-k+n}}\Big)\Big)}{1+\mathrm{H}_{n-r}}\right)=-2-\frac{1}{\mathrm{H}_n}-\sum_{k=1}^{n}\frac{1}{1+\mathrm{H}_k}+\frac{1+\mathrm{H}_n}{\mathrm{H}_n}\left(\prod_{k=1}^{n}\frac{\mathrm{H}_k}{1+\mathrm{H}_k}\right)$$
$$\left(1{+}\sum_{k=1}^{n}\prod_{i=1}^{k}\frac{1+\mathrm{H}_i}{\mathrm{H}_i}{+}\sum_{k=1}^{n}\left(\prod_{i=1}^{k}\frac{1+\mathrm{H}_i}{\mathrm{H}_i}\right)\sum_{i=1}^{k}\left(\frac{1}{1+\mathrm{H}_i}\right){-}\sum_{k=1}^{n}\frac{\left(\prod_{i=1}^{k}\frac{1+\mathrm{H}_i}{\mathrm{H}_i}\right)\sum_{i=1}^{k}\frac{1}{1+\mathrm{H}_i}}{1+\mathrm{H}_k}\right).$$

"The Number of Rhombus Tilings of a Symmetric Hexagon" (Fulmek,Krattenthaler)

$$\sum_{k=1}^{n}\frac{\mathrm{H}_k\;(3+k+n)!\;(-1)^k\;(-1)^{-1+n}}{(1+k)!\;(2+k)!\;(-k+n)!}$$
$$+\frac{(n)!}{(3+n)!}\sum_{k=1}^{n}-\frac{(3+k+n)!\;(-1)^k\;(1-(2+n)\;(-1)^n)}{k\;(1+k)!^2\;(-k+n)!}=(2+n)(-1)^n-2.$$