Computer algebra,
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Proving and Finding Multisum Identities in Difference Fields

Carsten.Schneider@risc.uni-linz.ac.at
RISC-Linz
J. Kepler University Linz

A Bonus Problem in "Concrete Mathematics"

Chapter 6. Special Numbers, Bonus problem 69:

Find a closed form for $$\sum_{k=1}^{n} k^2 \mathrm{H}_{n+k},$$ where $\mathrm{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

Knuth's answer to the problem is

$$\frac{1}{3}n\big(n+\frac{1}{2}\big)\,(n+1)\,(2\mathrm{H}_{2n}-\mathrm{H}_n)-\frac{1}{36}n\,\big(10n^2+9n-1\big)$$

with the remark

> *"It would be nice to automate the derivation of formulas such as this."*

In[1]:= **<< Sigma`**

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Sigma -A summation package by Carsten Schneider
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In[2]:= **Problem69 = SigmaSum[k^2 SigmaHNumber[n + k], {k, 1, n}]**

Out[2]= $\sum_{k=1}^{n} \left(k^2\ \mathrm{H}_{k+n}\right)$

In[3]:= **SigmaReduce[Problem69]//Simplify**

Out[3]= $-\frac{1}{36}\ n\ (1+n)\ (-1+10\ n+6\ (1+2\ n)\ \mathrm{H}_n-12\ (1+2\ n)\ \mathrm{H}_{2n})$

- Based on Karr's ideas (1981) of $\Pi\Sigma$-fields

Indefinite Summation in Difference Fields

Goal: Find a closed form for

$$\sum_{k=0}^{n} k\, k!$$

A Difference Field for the Problem

Let t_1, t_2 be indeterminates where

$$\begin{aligned} t_1 &\longleftrightarrow k \\ t_2 &\longleftrightarrow k! \end{aligned}$$

Consider the **field automorphism** $\sigma : \mathbb{Q}(t_1, t_2) \to \mathbb{Q}(t_1, t_2)$ canonically defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q} & & \\ \sigma(t_1) &= t_1 + 1 & \mathrm{S}\, k &= k + 1 \\ \sigma(t_2) &= (t_1 + 1)t_2 & \mathrm{S}\, k! &= (k + 1)! \end{aligned}$$

$(\mathbb{Q}(t_1, t_2), \sigma)$ is our difference field.

The Telescoping Problem

Find $g \in \mathbb{Q}(t_1, t_2)$: $\boxed{\sigma(g) - g = t_1\, t_2}$

$\downarrow$ by *Sigma*

$g = t_2.$

The Closed Form

$$\boxed{(k + 1)! - k! = k\, k!}$$

$$\downarrow$$

$$\sum_{k=0}^{n} k\, k! = (n + 1)! - 1.$$

Examples of “ΠΣ-Terms”

- GIVEN $f(k)$
- FIND $g(k)$:

$$\boxed{g(k+1)-g(k)=f(k)}$$

$f(k), g(k)$: **rational terms** *in arbitrarily* **indefinite nested** *sums and products*:

$$k! = \prod_{i=1}^{k} i \qquad \boxed{k\,k!} \qquad \frac{k}{k!^3+k!+k^4+1}$$

$$\mathrm{H}_k := \sum_{i=1}^{k} \frac{1}{i} \qquad \boxed{k^2\,\mathrm{H}_{n-k}} \qquad s_1(k) := \frac{\mathrm{H}_k^3+\mathrm{H}_k\,k!}{k\,\mathrm{H}_k^2+k!} \qquad \prod_{i=1}^{k} s_1(i)$$

$$\binom{n}{k} = \prod_{i=1}^{k} \frac{n+1-i}{i} \qquad \mathrm{H}_k^{(r)} := \sum_{i=1}^{k} \frac{1}{i^r}, \quad r \in \mathbb{N}$$

$$\textbf{?} \qquad \sum_{i=0}^{k} \binom{k}{i} \qquad \sum_{i=1}^{k} \frac{\sum_{j=1}^{i} \frac{\sum_{l=1}^{j} \frac{1}{l}}{j}}{(i+k)^4}$$

Quadratic Padé Approximation to the Logarithm

FIND $r_m(x) = \sum_{k=0}^{m} a_k x^k, s_m(x) = \sum_{k=0}^{m} b_k x^k, t_m(x) = \sum_{k=0}^{m} c_k x^k$:

$$\boxed{r_m(x)\,(\log x)^2 + s_m(x)\,\log(x) + t_m(x) = O((x-1)^{3m+2})}$$

$$\downarrow$$

Theorem (*Sigma* 2002). For all $m \geq 0$ we have

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)}\right] = 0.$$

Attempt.

In[4]:= $\mathbf{Pade2} = \sum_{\mathbf{k=0}}^{\mathbf{m}} \left(\mathbf{3}\,(-\mathbf{H_k} + \mathbf{H_{-k+m}})^{\mathbf{2}} + \mathbf{H_k^{(2)}} + \mathbf{H_{-k+m}^{(2)}}\right) \left(\left(\binom{\mathbf{m}}{\mathbf{k}}\right)^{\mathbf{3}} (-\mathbf{1})^{\mathbf{k}}\right)_{\mathbf{k}}$;

In[5]:= **SigmaReduce[Pade2]**

Out[5]= $\sum_{\iota_1=0}^{m} \left(3\,H_{m-\iota_1}^2 - 6\,H_{m-\iota_1}\,H_{\iota_1} + 3\,H_{\iota_1}^2 + H_{m-\iota_1}^{(2)} + H_{\iota_1}^{(2)}\right) \left(\left(\binom{m}{\iota_1}\right)^{3} (-1)^{\iota_1}\right)_{\iota_1}$

Quadratic Padé Approximation to the Logarithm

Theorem (*Sigma* 2002). For all $m \geq 0$ we have

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^3\left[3(H_{m-k}-H_k)^2+H^{(2)}_{m-k}+H^{(2)}_k\right]=0.$$

Proof.

In[6]:= $\mathbf{Pade2} = \sum_{\mathbf{k=0}}^{\mathbf{m}} \left(\mathbf{3}\ (-\mathbf{H_k}+\mathbf{H_{-k+m}})^{\mathbf{2}}+\mathbf{H_k^{(2)}}+\mathbf{H^{(2)}_{-k+m}}\right)\left(\left(\binom{\mathbf{m}}{\mathbf{k}}\right)^{\mathbf{3}}(-\mathbf{1})^{\mathbf{k}}\right)_{\mathbf{k}};$

In[7]:= **GenerateRecurrence[Pade2]**

Out[7]= $\{3\ (2+3\ m)\ (4+3\ m)\ \mathtt{SUM}[m]+(2+m)^2\ \mathtt{SUM}[2+m]==0\}$

In[8]:= **Table[Pade2, {m, 0, 10}]**

Out[8]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

■

Z's Creative Telescoping Trick

- GIVEN

$$\mathrm{SUM}(m) := \sum_{k=0}^{m} \underbrace{(-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)}\right]}_{=:\, f(m,k)}$$

- FIND $c_0(m)$, $c_1(m)$, $c_2(m)$, and $g(m,k)$ s.t.

$$\boxed{g(m,k+1) - g(m,k)} = \boxed{c_0(m)\, f(m,k) + c_1(m)\, f(m+1,k) + c_2(m)\, f(m+2,k)}$$

for all $0 \leq k \leq m$ and all $m \geq 0$

Sigma computes:

$$c_0(m) := 3(3m+2)(3m+4)(3m+8), \quad c_1(m) := 0, \quad c_2(m) := (m+2)^2(3m+8)$$

$$g(m,k) := (-1)^k \binom{m}{k}^3 \frac{p_1(k, m, H_k, H_k^{(2)}, H_{m-k}, H_{m-k}^{(2)})}{(m-k+1)^5(m-k+2)^5}$$

$$g(m,k+1) := (-1)^k \binom{m}{k}^3 \frac{p_2(k, m, H_k, H_k^{(2)}, H_{m-k}, H_{m-k}^{(2)})}{(m-k+1)^5}$$

Z's Creative Telescoping Trick

- GIVEN

$$\mathrm{SUM}(m) := \sum_{k=0}^{m} \underbrace{(-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)}\right]}_{=:\, f(m,k)}$$

- FIND $c_0(m)$, $c_1(m)$, $c_2(m)$, and $g(m,k)$ s.t.

$$\boxed{g(m,k+1) - g(m,k)} = \boxed{c_0(m)\, f(m,k) + c_1(m)\, f(m+1,k) + c_2(m)\, f(m+2,k)}$$

  for all $0 \le k \le m$ and all $m \ge 0$

Summing this equation over k from 0 to m gives:

$$\boxed{g(m,m+1) - g(m,0)} = \boxed{\begin{array}{l} c_0(m)\,\mathrm{SUM}(m) + \\ c_1(m)\,\big[\mathrm{SUM}(m+1) - f(m+1,m+1)\big] \\ c_2(m)\,\big[\mathrm{SUM}(m+2) - f(m+2,m+1) - f(m+2,m+2)\big] \end{array}}$$

- Linear Padé

$$r_m(x)\ \log(x) + s_m(x) = O((x-1)^{2m+1})$$

$$\sum_{k=0}^{m} \binom{m}{k}^2 \left[1 + 2k(\mathrm{H}_{m-k} - \mathrm{H}_k)\right] = 0$$

- Quadratic Padé

$$r_m(x)\,(\log x)^2 + s_m(x)\ \log(x) + t_m(x) = O((x-1)^{3m+2})$$

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H^{(2)}_{m-k} + H^{(2)}_k\right] = 0,$$

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}^3 \left[k(3(H_{m-k} - H_k)^2 + H^{(2)}_{m-k} + H^{(2)}_k) + 2(H_{m-k} - H_k)\right] = 0$$

∞

- Cubic Padé

$$r_m(x)\,(\log x)^3 + s_m(x)\,(\log x)^2 + t_m(x)\,\log(x) + u_m(x) = O((x-1)^{4m+3})$$

$$\sum_{k=0}^{m} \binom{m}{k}^4 \Big[3(H_{m-k}-H_k)^2 + H^{(2)}_{m-k} + H^{(2)}_k$$
$$+\,4k(H_{m-k}-H_k)^3 + 6(H_{m-k}-H_k)(H^{(2)}_{m-k}+H^{(2)}_k) + H^{(3)}_{m-k} - H^{(3)}_k\Big] = 0$$

- Padé of order 4

$$r_m(x)\,(\log x)^4 + s_m(x)\,(\log x)^3 + t_m(x)\,(\log x)^2 + u_m(x)\,\log(x) + v_m = O((x-1)^{5m+4})$$

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}^5\Big[125\big(H_k - H^{(1)}_{m-k}\big)^4 + 150\big(H_k - H^{(1)}_{m-k}\big)^2\big(H^{(2)}_k + H^{(2)}_{m-k}\big) + 15\big(H^{(2)}_k + H^{(2)}_{m-k}\big)^2$$
$$+\,40\big(H_k - H_{m-k}\big)\big(H^{(3)}_k - H^{(3)}_{m-k}\big) + 6H^{(4)}_k + 6H^{(4)}_{m-k}\Big] = 0$$

$$
\begin{aligned}
&\sum_{k=0}^{m}(-1)^k\binom{m}{k}^5\Big[-60\big(H_k-H_{m-k}\big)\big(H_k^{(2)}+H_{m-k}^{(2)}\big)+4\big(25\big(-H_k+H_{m-k}\big)^3-2H_k^{(3)}+2H_{m-k}^{(3)}\big)\\
&\qquad+5k\big(-H_k+H_{m-k}\big)\big(25\big(-H_k+H_{m-k}\big)^3-8H_k^{(3)}+8H_{m-k}^{(3)}\big)\\
&\qquad+3k\big(5\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big(10\big(H_k-H_{m-k}\big)^2+H_k^{(2)}+H_{m-k}^{(2)}\big)+2\big(H_k^{(4)}+H_{m-k}^{(4)}\big)\big)\Big]=0
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=0}^{m}(-1)^k\binom{m}{k}^5\Big[125\big(-1+k\big)kH_k^4+100\big(-1+2k\big)H_{m-k}^3+125\big(-1+k\big)kH_{m-k}^4+100H_k^3\big(1-2k\\
&\qquad-5\big(-1+k\big)kH_{m-k}\big)+30H_{m-k}^2\big(2+5\big(-1+k\big)k\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big)+30H_k^2\big(2+5\big(\big(-2+4k\big)H_{m-k}\\
&\qquad+5\big(-1+k\big)kH_{m-k}^2+\big(-1+k\big)k\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big)\big)+20H_k\big(\big(15-30k\big)H_{m-k}^2-25\big(-1+k\big)kH_{m-k}^3\\
&\qquad+\big(3-6k\big)H_k^{(2)}+3H_{m-k}^{(2)}-3H_{m-k}\big(2+5\big(-1+k\big)k\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big)+2k\big(-3H_{m-k}^{(2)}+\big(-1+k\big)\big(H_k^{(3)}\\
&\qquad-H_{m-k}^{(3)}\big)\big)\big)+20H_{m-k}\big(\big(-3+6k\big)H_k^{(2)}+\big(-3+6k\big)H_{m-k}^{(2)}-2\big(-1+k\big)k\big(H_k^{(3)}-H_{m-k}^{(3)}\big)\big)\\
&\qquad+4\big(3H_k^{(2)}+3H_{m-k}^{(2)}+2H_k^{(3)}-2H_{m-k}^{(3)}\big)+k\big(15\big(-1+k\big){H_k^{(2)}}^2+30\big(-1+k\big)H_k^{(2)}H_{m-k}^{(2)}\\
&\qquad+15\big(-1+k\big){H_{m-k}^{(2)}}^2+2\big(-8H_k^{(3)}+8H_{m-k}^{(3)}+3\big(-1+k\big)\big(H_k^{(4)}+H_{m-k}^{(4)}\big)\big)\big)\Big]=0
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=0}^{m}(-1)^k\binom{m}{k}^5\Big[125\big(-2+k\big)\big(-1+k\big)kH_k^4+100\big(2+3\big(-2+k\big)k\big)H_{m-k}^3+125\big(-2+k\big)\big(-1+k\big)kH_{m-k}^4+\\
&\qquad100H_k^3\big(-2-3\big(-2+k\big)k-5\big(-2+k\big)\big(-1+k\big)kH_{m-k}\big)+30\big(-1+k\big)H_{m-k}^2\big(6+5\big(-2+k\big)k\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big)\\
&\qquad+30H_k^2\big(10\big(2+3\big(-2+k\big)k\big)H_{m-k}+25\big(-2+k\big)\big(-1+k\big)kH_{m-k}^2+\big(-1+k\big)\big(6+5\big(-2+k\big)k\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big)\big)\\
&\qquad+4H_{m-k}\big(6+15\big(2+3\big(-2+k\big)k\big)H_k^{(2)}+5\big(3\big(2+3\big(-2+k\big)k\big)H_{m-k}^{(2)}-2\big(-2+k\big)\big(-1+k\big)k\big(H_k^{(3)}-H_{m-k}^{(3)}\big)\big)\big)\\
&\qquad-4\big(9H_{m-k}^{(2)}+4H_k^{(3)}-4H_{m-k}^{(3)}\big)+4H_k\big(-6\big(1+5H_k^{(2)}+5H_{m-k}^{(2)}\big)-5\big(3+5\big(-1+k\big)H_{m-k}\big)\big(6\big(-1+k\big)H_{m-k}\\
&\qquad+5\big(-2+k\big)kH_{m-k}^2+3\big(-2+k\big)k\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big)+10\big(-2+k\big)\big(-1+k\big)kH_k^{(3)}-10\big(-2+k\big)\big(-1+k\big)kH_{m-k}^{(3)}\big)\\
&\qquad+3\big(5\big(-2+k\big)\big(-1+k\big)k{H_k^{(2)}}^2+2\big(-1+k\big)H_k^{(2)}\big(6+5\big(-2+k\big)kH_{m-k}^{(2)}\big)+k\big(12H_{m-k}^{(2)}+5\big(-2+k\big)\big(-1+k\big){H_{m-k}^{(2)}}^2\\
&\qquad+2\big(-2+k\big)\big(-4H_k^{(3)}+4H_{m-k}^{(3)}+\big(-1+k\big)\big(H_k^{(4)}+H_{m-k}^{(4)}\big)\big)\big)\big)\Big]=0
\end{aligned}
$$

Difference Equations and Symbolic Summation

Let $(\mathbb{F}, \sigma)$ be a difference field and

$$\mathbb{K} = \{k \in \mathbb{F} \,|\, \sigma(k) = k\}$$

be the constant field. Assume $\mathbb{Q} \subseteq \mathbb{K}$.

Telescoping

- GIVEN $f \in \mathbb{F}$
- FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$$\downarrow \qquad \uparrow$$

Parameterized Telescoping

- GIVEN $f_0, \dots, f_d \in \mathbb{F}$, $a_0, a_1 \in \mathbb{F}$
- FIND ALL $c_0, \dots, c_d \in \mathbb{K}$, $h \in \mathbb{F}$:

$$\boxed{a_1\, \sigma(h) - a_0\, h = c_0\, f_0 + \cdots + c_d\, f_d}$$

Remark: Z's "Creative Telescoping"

- GIVEN $f_i = \text{summand}(n + i, k) \in \mathbb{F}$
- FIND ALL $c_0, \dots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0\, f_0 + \cdots + c_d\, f_d}$$

Linear Difference Equations

- GIVEN $f, a_0, \ldots, a_m \in \mathbb{F}$
- FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

Parameterized Linear Difference Equations

- GIVEN $a_0, \ldots, a_m \in \mathbb{F}, f_0, \ldots, f_d \in \mathbb{F}$.
- FIND ALL $g \in \mathbb{F}$, $c_0, \ldots, c_d \in \mathbb{K}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = c_0\,f_0 + \cdots + c_d\,f_d}$$

Solving Difference Equations

In[9]:= $\mathbf{rec = 144\ (1+n)\ (2+n)\ \left(21+6\ n+n^2\right)\ SUM[n]-}$
$\mathbf{84\ (2+n)\ \left(75+60\ n+15\ n^2+2\ n^3\right)\ SUM[1+n]+}$
$\mathbf{\left(8100+8700\ n+3565\ n^2+730\ n^3+73\ n^4\right)\ SUM[2+n]-}$
$\mathbf{7\ (3+n)\ \left(100+60\ n+15\ n^2+2\ n^3\right)\ SUM[3+n]+}$
$\mathbf{(3+n)\ (4+n)\ \left(16+4\ n+n^2\right)\ SUM[4+n] ==}$
$\mathbf{\frac{1}{3}\left(1432+1290\ n+307\ n^2+37\ n^3\right);}$

In[10]:= **SolveRecurrence[rec, SUM[n]]**

Out[10]= $\{\{0, 0\}\}$

The underlying difference field is too small!

Find product extensions

In[11]:= **FindProductExtensions[rec, SUM[n], Solutions → All]**

`I use M. Petkovsek's package Hyper!`

Out[11]= $\left\{\prod_{i=1}^{n} 3, \prod_{i=1}^{n} 4\right\}$

In[12]:= **SolveRecurrence[rec, SUM[n], Tower → {3^n, 4^n}]**

Out[12]= $\{\{0, 3^{n}\}, \{0, 4^{n}\}\}$

Find sum extensions

In[13]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n], Tower \to \{3^{n}, 4^{n}\}, NestedSumExt \to \infty]}$

Out[13]= $\left\{\{0, 3^{n}\}, \{0, 4^{n}\}, \left\{0, \frac{4\ 3^{n} + n\ 3^{n} \sum_{\iota_1=2}^{n} \frac{3+\iota_1}{(-1+\iota_1)\ \iota_1}}{n}\right\},\right.$

$\left.\left\{0, -\frac{4^{n}\left(4 + n \sum_{\iota_1=2}^{n} \frac{3+\iota_1}{(-1+\iota_1)\ \iota_1}\right)}{n}\right\}, \left\{1, \frac{1}{18}\ 4^{n} \sum_{\iota_1=0}^{n} \frac{3^{\iota_1} \sum_{\iota_2=2}^{\iota_1} \frac{11+37\ \iota_2}{\left(-\iota_2+\iota_2^2\right)\ 3^{\iota_2}}}{4^{\iota_1}}\right\}\right\}$

Simplification by indefinite summation

In[14]:= $\mathbf{recSol = SigmaReduce[recSol, n, SimplifyByExt \to Depth]}$

Out[14]= $\left\{\{0, 3^{n}\}, \{0, 4^{n}\}, \left\{0, -\frac{4^{n}\left(1 + 3\ n + n \sum_{\iota_1=2}^{n} \frac{1}{-1+\iota_1}\right)}{n}\right\},\right.$

$\left\{0, \frac{(1+3\ n)\ 3^{n} + n\ 3^{n} \sum_{\iota_1=2}^{n} \frac{1}{-1+\iota_1}}{n}\right\}, \left\{1, \frac{1}{18\ n}\left(-11 - 11\ n\ 3^{n} + \right.\right.$

$\left.\left.\left.11\ n\ 4^{n} - 45\ n\ 3^{n} \sum_{\iota_1=2}^{n} \frac{1}{(-1+\iota_1)\ 3^{\iota_1}} + 16\ n\ 4^{n} \sum_{\iota_1=2}^{n} \frac{1}{(-1+\iota_1)\ 4^{\iota_1}}\right)\right\}\right\}$

"Cosmetics"

In[15]:= $\mathbf{tower = \left\{H_n, \sum_{k=1}^{n} \frac{1}{k\ 3^{k}}, \sum_{k=1}^{n} \frac{1}{k\ 4^{k}}\right\};}$

In[16]:= $\mathbf{recSol = SigmaReduce[recSol, n, Tower \to tower]}$

Out[16]= $\left\{\{0, 3^{n}\}, \{0, 4^{n}\}, \{0, -(3+H_n)\ 4^{n}\}, \{0, (3+H_n)\ 3^{n}\},\right.$

$\left.\left\{1, \frac{1}{18}\left(-11\ 3^{n} + 11\ 4^{n} - 15\ 3^{n} \sum_{\iota_1=1}^{n} \frac{1}{\iota_1\ 3^{\iota_1}} + 4\ 4^{n} \sum_{\iota_1=1}^{n} \frac{1}{\iota_1\ 4^{\iota_1}}\right)\right\}\right\}$

Summation over Recurrences (Hermite Polynomials)

In[17]:= $\mathbf{rec = H[l+2] == 2x\ H[l+1] - 2(l+1)H[l];}$

In[18]:= $\mathbf{initial = \{1, 2x\}}$

In[19]:= $\mathbf{BuildEvaluation[recH, H[l], \{1, 2x\}, 0]}$

In[20]:= $\mathbf{Table[H[i], \{i, 0, 6\}]//Simplify}$

Out[20]= $\{1,$
$2\ x,$
$-2 + 4\ x^2,$
$4\ x\ (-3 + 2\ x^2),$
$4\ (3 - 12\ x^2 + 4\ x^4),$
$8\ x\ (15 - 20\ x^2 + 4\ x^4),$
$8\ (-15 + 90\ x^2 - 60\ x^4 + 8\ x^6)$

Definite summation (creative telescoping)

In[21]:= $\mathbf{mySum = \sum_{i=0}^{n} H[i] \binom{n}{i};}$

In[22]:= $\mathbf{GenerateRecurrence[mySum, n, \{\{recH, H[i]\}\}]}$

Out[22]= $\{-2\ (1+n)\ \mathtt{SUM}[n] + (1 + 2\ x)\ \mathtt{SUM}[1+n] - \mathtt{SUM}[2+n] == 0\}$

Summation over Recurrences

In[23]:= $\mathbf{recS = 6\ (1 + H_k + k\ H_k)\ (3 + 2\ k + 2\ H_k + 3\ k\ H_k + k^2\ H_k)\ S[k] - 5\ (1 + k)\ H_k\ (3 + 2\ k + 2\ H_k + 3\ k\ H_k + k^2\ H_k)\ S[1 + k] + (1 + k)\ (2 + k)\ H_k\ (1 + H_k + k\ H_k)\ S[2 + k] == 0}$

In[24]:= $\mathbf{BuildEvaluation[recS, S[k], \{1, 2\}, 1]}$

In[25]:= $\mathbf{Table[S[k], \{k, 1, 8\}]}$

Out[25]= $\{1, 2, \frac{11}{9}, -\frac{175}{18}, -\frac{5617}{90}, -\frac{7987}{30}, -\frac{68849}{70}, -\frac{1420787}{420}\}$

In[26]:= $\mathbf{mySum = \sum_{k=1}^{n} \binom{n}{k}\ S[k];}$

In[27]:= $\mathbf{rec = GenerateRecurrence[mySum, n, \{\{recS, S[k]\}\}]}$

Out[27]= $\{144\ (1 + n)\ (2 + n)\ (21 + 6\ n + n^2)\ \mathrm{SUM}[n] - 84\ (2 + n)\ (75 + 60\ n + 15\ n^2 + 2\ n^3)\ \mathrm{SUM}[1 + n] + (8100 + 8700\ n + 3565\ n^2 + 730\ n^3 + 73\ n^4)\ \mathrm{SUM}[2 + n] - 7\ (3 + n)\ (100 + 60\ n + 15\ n^2 + 2\ n^3)\ \mathrm{SUM}[3 + n] + (3 + n)\ (4 + n)\ (16 + 4\ n + n^2)\ \mathrm{SUM}[4 + n] == \frac{1}{3}\ (1432 + 1290\ n + 307\ n^2 + 37\ n^3)\}$

Hence

$$\sum_{k=1}^{n} \binom{n}{k} S(k) = \frac{1}{18}\Big[5\,3^{n+1}\Big(H_n - \sum_{i=1}^{n} \frac{1}{3^i i}\Big) - 4^{n+1}\Big(H_n - \sum_{i=1}^{n} \frac{1}{3^i i}\Big)\Big]$$

Plane partitions

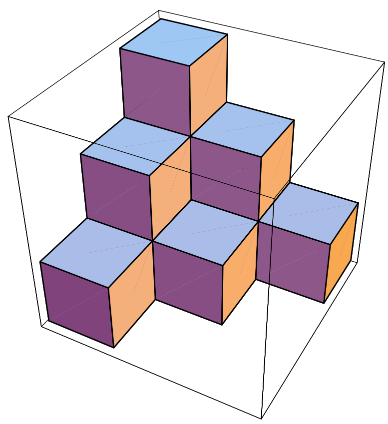
A plane partition of order n is

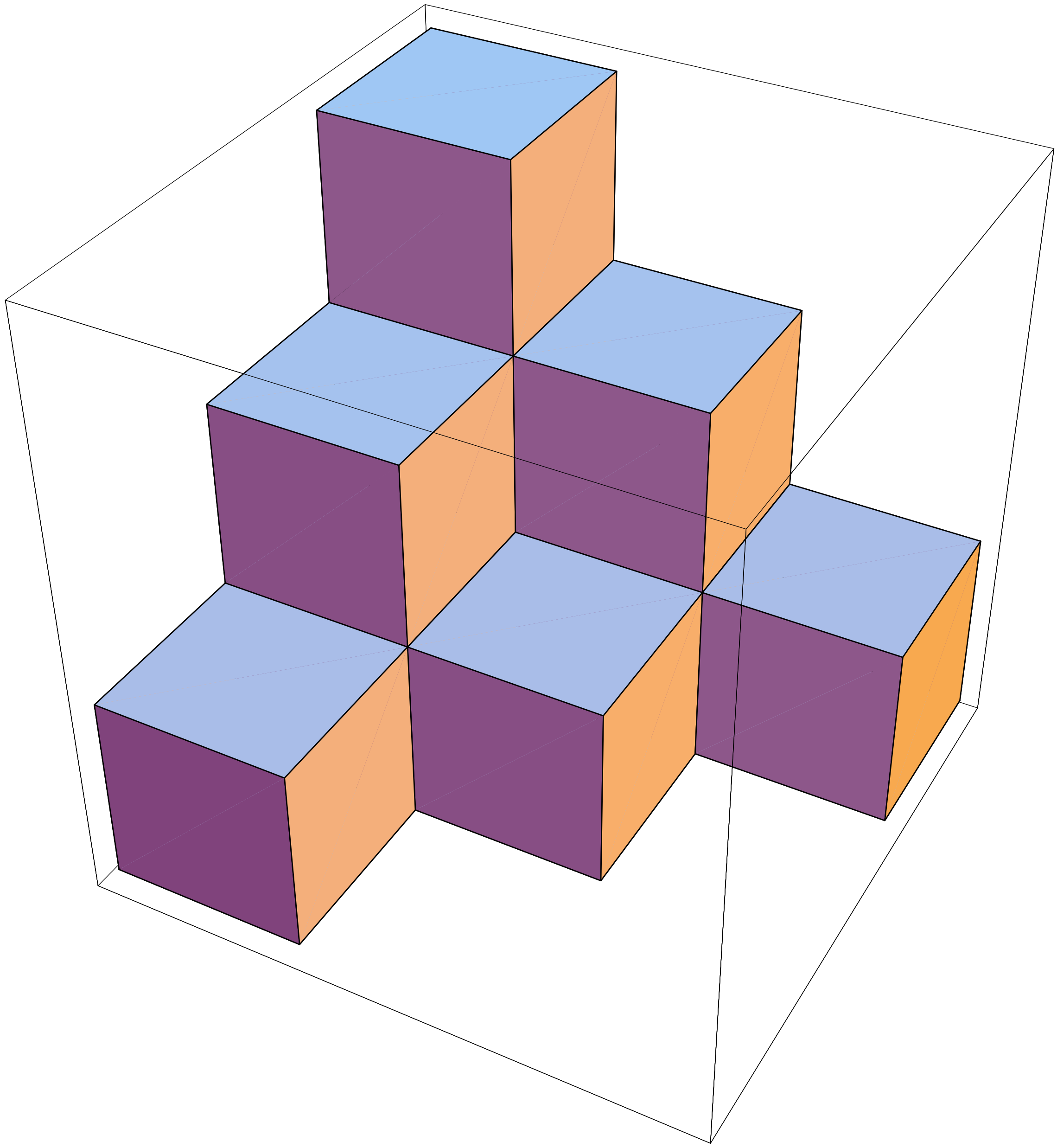
$$
\begin{array}{ccccc}
n & & & & \\
a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\
a_{21} & a_{22} & & \dots & a_{2n} \\
a_{31} & & & & a_{3n} \\
\vdots & & & & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \\
 & & & & & 0
\end{array}
$$

Example:

$$
\begin{array}{ccc}
3 & 2 & 1 \\
2 & 1 & 0 \\
1 & 0 & 0
\end{array}
$$

$$\updownarrow$$





Theorem (Stembridge, 1995). The number of totally symmetric plane partitions of order $n \geq 0$ is

$$TSPP(n) = \prod_{j=0}^{n-1} \frac{(2j)!(n+2j+1)!}{(3j+1)!(n+j)!}.$$

A symbolic summation proof

Given

$$A_2(i,m) := \sum_{k=i}^{2m} (-1)^k h(k,m)$$

$$A_0(i,m) := \sum_{k=0}^{2m} \binom{i+k-3}{i-2} h(k,m)$$

with

$$h(k,m) := \sum_{s=0}^{\lfloor \frac{2m-k}{2} \rfloor} \left(\binom{m-s-1}{2m-2s-k} + \binom{m-s}{2m-2s-k} \right) \frac{(-1)^{s+k}}{2m\,4^s} \sum_{r=0}^{s} \frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r}$$

Show that

$$F(i,m) := 2h(i-2,m) - 5h(i-1,m) + 6(-1)^i A_2(i,m) - A_0(i,m) - \prod_{s=1}^{2m-1} \frac{2(m+s-1)}{2m+s-2} = 0$$

for all $3 \leq i \leq 2m+1$

Finding a recurrence for $\boldsymbol{h}(k,m)$

In[28]:= $$\mathbf{hSum} = \sum_{s=0}^{\text{Floor}\left[-\frac{k}{2}+m\right]} \frac{1}{2\ (m-s)}\left(k\left(\frac{\binom{m-s}{-k+2\ m-2\ s}^{.}\ (-1)^{s.}}{4^{s.}}\right)_{s.}\ (-1)^{k.}\ \sum_{r=0}^{s}\frac{(m-r)\left(\prod_{ll=1}^{r}(-2+ll-3\ m)\right)\left(\prod_{ll=1}^{r}\frac{1}{-\frac{1}{2}+ll-2\ m}\right)\ \prod_{ll=1}^{r}(-1+ll+m)\frac{1}{r!.}}{m}\right);$$

In[29]:= **recH = GenerateRecurrence[hSum, k, FiniteSupport → True]/. SUM → h**

`10.985 Second`

Out[29]= $$2\ (2+k)^2\ (k-2\ m)\ (1+k+2\ m)\ h[k]-\left(29\ k^3+5\ k^4+k\left(46-20\ m-40\ m^2\right)-12\left(-1+m+2\ m^2\right)-2\ k^2\left(-29+3\ m+6\ m^2\right)\right)\ h[1+k]+\left(26\ k^3+4\ k^4+k\left(55-14\ m-28\ m^2\right)+k^2\left(59-6\ m-12\ m^2\right)-6\left(-3+m+2\ m^2\right)\right)\ h[2+k]-(1+k)^2\ (2+k-2\ m)\ (3+k+2\ m)\ h[3+k]==0;$$

Finding a recurrence for $\boldsymbol{A}_0(i,m)$

In[30]:= $$\mathbf{A0} = \sum_{k=0}^{2\ m} h[k]\binom{-3+i+k}{-2+i}^{.};$$

In[31]:= **GenerateRecurrence[A0, i, {{recH, h[k]}}, FiniteSupport → True]**

`15.332 Second`

Out[31]= $$-\left(2+i+i^2\right)\ (-2+i-2\ m)\ (-1+i+2\ m)\ \text{SUM}[i]+(3+i)\left(-2+2\ i-i^2+i^3+2\ m+4\ m^2\right)\ \text{SUM}[1+i]+(-3+i)\left(2+2\ i+i^2+i^3-2\ m-4\ m^2\right)\ \text{SUM}[2+i]-\left(2-i+i^2\right)\ (1+i-2\ m)\ (2+i+2\ m)\ \text{SUM}[3+i]==0$$

Proof.

$$h(k,m) := \sum_{s=0}^{\lfloor \frac{2m-k}{2} \rfloor - 1} \overbrace{\frac{k}{m-s}\binom{m-s}{2m-2s-k}\frac{(-1)^{s+k}}{2m\,4^s}\sum_{r=0}^{s}\frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r}}^{=:\, q(s,k,m)}$$

Creative telescoping equation:

$$\boxed{g(s+1,k,m) - g(s,k,m) = c_0(k,m)q(s,k,m) + \cdots + c_3 q(s,k+3,m)}$$

with

$$\begin{aligned}
c_0(k,m) &= m2(2+k)^2(k-2m)(1+k+2m),\\
c_1(k,m) &= -m(29k^3+5k^4+k(46-20m-40m^2)-12(-1+m+2m^2)-2k^2(-29+3m+6m^2)),\\
c_2(k,m) &= m(26k^3+4k^4+k(55-14m-28m^2)+k^2(59-6m-12m^2)-6(-3+m+2m^2)),\\
c_3(k,m) &= -m(1+k)^2(2+k-2m)(3+k+2m)h(3+k) = 0,
\end{aligned}$$

$$g(s,k,m) := \binom{m-s}{2m-2s-k}\frac{-2(-1)^{s+k}}{4^s(1+k-m+s)(1+k-m+s)}, \qquad p_i(s,k,m) \in \mathbb{Z}[s,k,m]$$

$$\left[p_1(s,k,m)\frac{(m)_s(-3m-1)_s}{s!(\frac{1}{2}-2m)_s} + p_2(s,k,m)\sum_{r=0}^{s}\frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r}\right].$$

TRICK:

$$\binom{m-s}{2m-2s-k} = \frac{(m-s-k-1)(m-s-k-2)}{(2m-2s-k)(2m-2s-k-1)}\binom{m-s}{2m-2s-k-2}$$

■

Proof.

$$A_0(i,m) = \sum_{k=0}^{2m} \overbrace{\binom{i+k-3}{i-2} h(k,m)}^{=:\, q(i,k,m)}$$

Creative telescoping equation:

$$\boxed{g(k+1,i,m) - g(k,i,m) = c_0(i,m)\, q(k,i,m) + \cdots + c_3(i,m)\, q(k,i+3,m)} \qquad (1)$$

with

$$\begin{aligned}
c_0(i,m) &= (1-i)i(1+i)(2+i+i^2)(-2+i-2m)(-1+i+2m),\\
c_1(i,m) &= (-1+i)i(1+i)(3+i)(-2+2i-i^2+i^3+2m+4m^2),\\
c_2(i,m) &= (-3+i)(-1+i)i(1+i)(2+2i+i^2+i^3-2m-4m^2),\\
c_3(i,m) &= (1-i)i(1+i)(2-i+i^2)(1+i-2m)(2+i+2m),
\end{aligned}$$

$$g(k,i,m) := -\big[p_0(i,m)h(i,m) + p_1(i,m)h(i+1,m) + p_2(i,m)h(i+2,m)\big]\frac{k-1}{(k+1)^2}\binom{i+k-3}{i-2}, \quad p_j(i,m) \in \mathbb{Z}[i,m].$$

VERIFICATION:

$$g(k,i,m) \xleftarrow{\text{depends on}} \tbinom{i+k-3}{i-2}, h(k,m), h(k+1,m), h(k+2,m)$$

$$g(k+1,i,m) \xleftarrow{\text{depends on}} \tbinom{i+k-2}{i-2}, h(k+1,m), h(k+2,m), h(k+3,m)$$

Check equality of (1) with relations

$$\boxed{\begin{gathered}\tbinom{i+k-2}{i-2} = \tfrac{k+i-2}{k}\tbinom{i+k-3}{i-2},\\
h(k+3,m) = \alpha_0(k,m)\, h(k,m) + \alpha_1(k,m)\, h(k+1,m) + \alpha_2(k,m)\, h(k+2,m), \quad \alpha_j(i,m) \in \mathbb{Q}(i,m). \blacksquare\end{gathered}}$$

For all $3 \le i \le 2m+1$ we compute (with a proof)

- a recurrence for $h(i-2,m)$

$$
\begin{aligned}
&2i^2(-1+i+2m)(i-2(1+m))h(-2+i,m)\\
&-(-11i^3+5i^4+4m(1+2m)-2i^2(-2+3m+6m^2)+i(2+4m+8m^2))h(-1+i,m)\\
&+(-6i^3+4i^4-2m(1+2m)-i^2(1+6m+12m^2)+i(3+10m+20m^2))h(i,m)\\
&-(-1+i)^2(i-2m)(1+i+2m)h(1+i,m)=0
\end{aligned}
$$

- a recurrence for $h(i-1,m)$

$$
\begin{aligned}
&2(1+i)^2(-1+i-2m)(i+2m)h(-1+i,m)\\
&-(9i^3+5i^4+2m(1+2m)+i^2(1-6m-12m^2)-i(3+8m+16m^2))h(i,m)\\
&+(10i^3+4i^4+2m(1+2m)+i^2(5-6m-12m^2)-i(1+2m+4m^2))h(1+i,m)\\
&-i^2(1+i-2m)(2+i+2m)h(2+i,m)=0
\end{aligned}
$$

- a recurrence for $A_0(i,m)$

$$
\begin{aligned}
&(-2-i-i^2)(2-3i+i^2-2m-4m^2)A_0(i,m)\\
&+(3+i)(-2+2i-i^2+i^3+2m+4m^2)A_0(i+1,m)\\
&+(-3+i)(2+2i+i^2+i^3-2m-4m^2)A_0(i+2,m)\\
&-(2-i+i^2)(2+3i+i^2-2m-4m^2)A_0(i+3,m)=0
\end{aligned}
$$

- a recurrence for $A_2(i,m)$

$$
\begin{aligned}
&2(6+5i+i^2)(i+i^2-2m(1+2m))A_2(i,m)\\
&+(3+i)(3i^2+i^3+8m(1+2m)+2i(1+m+2m^2))A_2(1+i,m)\\
&-2(1+i)(24+14i^2+2i^3-5m-10m^2-i(-32+m+2m^2))A_2(2+i,m)\\
&-2(2+i)(6+6i^2+i^3+i(11-2m-4m^2))A_2(3+i,m)\\
&+2(3+4i+i^2)(8+6i+i^2-m-2m^2)A_2(4+i)\\
&+(2+3i+i^2)(7i+i^2-2(-6+m+2m^2))A_2(5+i,m)=0
\end{aligned}
$$

$\downarrow$

`gfun` for Maple (INRIA, B. Salvy)

`GeneratingFunctions` for Mathematica (RISC, C. Mallinger)

$\downarrow$

A **homogeneous** recurrence for

$$
F(i,m)=2h(i-2,m)-5h(i-1,m)+6(-1)^iA_2(i,m)-A_0(i,m)-\prod_{s=1}^{2m-1}\frac{2(m+s-1)}{2m+s-2}
$$

for all $3 \le i \le 2m+1$

Proof strategy

Show that

$$F(i,m) := 2h(i-2,m) - 5h(i-1,m) + 6(-1)^i A_2(i,m) - A_0(i,m) - \prod_{s=1}^{2m-1} \frac{2(m+s-1)}{2m+s-2} = 0$$

for all $3 \leq i \leq 2m+1$

Compute linear recurrences in i for

$$h(i-2,m) \qquad h(i-1,m) \qquad 1$$
$$A_2(i,m) \qquad A_0(i,m)$$

that hold for $3 \leq i \leq 2m+1$.

Initial values for $3 \leq i \leq 12$

Verify that

$$F(i,m) = 0$$

for all $m \geq \lceil \frac{i-1}{2} \rceil$.

Compute a **homogeneous** recurrence in i of order 10 for $F(i,m)$ for $3 \leq i \leq 2m+1$.

- We have the **recurrence relation**

$$c_0(i,m)\,F(i,m) + \cdots + c_9(i+9,m)\,F(i+9,m) \\ = c_{10}(i,m)\,F(i+10,m)$$

for $3 \le i \le 2m+1$.

- We have

$$F(i,m) = 0, \quad m \ge \left\lceil \frac{i-1}{2} \right\rceil$$

for **initial values** $3 \le i \le 12$.

Then

$$F(i,m) = 0, \quad \forall 3 \le i \le 2m+1 \ \textbf{?}$$

$$\Uparrow$$

$$c_{10}(i,m) \neq 0, \quad 3 \le i \le 2m-9$$

$$\Uparrow$$

- $c_{10}(3,m) \neq 0$ for all integers m,
- there are no real numbers i, m with

$$4 \le i \le 2m-9 \quad \wedge \quad c_{10}(i,m) = 0.$$

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                Quantifier Elimination
                          in
            Elementary Algebra and Geometry
                          by
      Partial Cylindrical Algebraic Decomposition

               Version B 1.21, 13 Aug 2003

                          by
                       Hoon Hong
                  (hhong@math.ncsu.edu)

With contributions by: Christopher W. Brown, George E. Collins,
Mark J. Encarnacion, Jeremy R. Johnson Werner Krandick, Richard
Liska, Scott McCallum, Nicolas Robidoux, and Stanly Steinberg
=======================================================
Enter an informal description  between '[' and ']': [TSSP Id13]
Enter a variable list: (i,m)
Enter the number of free variables: 0
Enter a prenex formula:
(E i) (E m) [4<=i /\ i<=2 m /\ 14770123200
i + 46888055280 i^2 + 62160803568 i^3 + 46322739108 i^4 +
22503191904 i^5 + 8432977716 i^6 + 3061588104 i^7 + 1145499084 i^8
+ 367147728 i^9 + 85849956 i^10 + 13547736 i^11 + 1356912 i^12 +
77760 i^13 + 1944 i^14 - 65498025600 m - 267811777440 i m -
521411333760 i^2 m - 633525786432 i^3 m - 534780465156 i^4 m -
332152252932 i^5 m - 156775135692 i^6 m - 57169208532 i^7 m -
16212163008 i^8 m - 3576942156 i^9 m - 611436504 i^10 m - 80207340
i^11 m - 7941156 i^12 m - 579264 i^13 m - 30300 i^14 m - 1104 i^15
m - 24 i^16 m + 80406950400 m^2 + 194229937584 i m^2 +
161013433932 i^2 m^2 - 7535713356 i^3 m^2 - 134657608947 i^4 m^2 -
142969324102 i^5 m^2 - 87924082051 i^6 m^2 - 37079032168 i^7 m^2 -
11261778679 i^8 m^2 - 2494071430 i^9 m^2 - 398363869 i^10 m^2 -
43876856 i^11 m^2 - 2898494 i^12 m^2 - 46232 i^13 m^2 + 9388 i^14
m^2 + 736 i^15 m^2 + 16 i^16 m^2 + 765767237472 m^3 +
2687946582024 i m^3 + 4492612561644 i^2 m^3 + 4753055125296 i^3
m^3 + 3562541526128 i^4 m^3 + 2003926064888 i^5 m^3 + 874078680870
i^6 m^3 + 301408021628 i^7 m^3 + 83070544202 i^8 m^3 + 18387877496
i^9 m^3 + 3267631470 i^10 m^3 + 463411132 i^11 m^3 + 51757014 i^12
m^3 + 4441312 i^13 m^3 + 279760 i^14 m^3 + 11776 i^15 m^3 + 256
i^16 m^3 + 381046008672 m^4 + 1562910502680 i m^4 + 2913758492784
i^2 m^4 + 3351788246604 i^3 m^4 + 2688518848839 i^4 m^4 +
1600771966610 i^5 m^4 + 732941866733 i^6 m^4 + 263444140352 i^7
m^4 + 75196579709 i^8 m^4 + 17133210434 i^9 m^4 + 3115737635 i^10
m^4 + 449715496 i^11 m^4 + 50865868 i^12 m^4 + 4401952 i^13 m^4 +
278800 i^14 m^4 + 11776 i^15 m^4 + 256 i^16 m^4 - 843027873696 m^5
- 2520580180104 i m^5 - 3595390118652 i^2 m^5 - 3230426402112 i^3
```

```
m^5 - 2034357606108 i^4 m^5 - 945668220120 i^5 m^5 - 333171884628
i^6 m^5 - 90085262976 i^7 m^5 - 18761682060 i^8 m^5 - 2998863600
i^9 m^5 - 363808272 i^10 m^5 - 32840496 i^11 m^5 - 2137944 i^12
m^5 - 94464 i^13 m^5 - 2304 i^14 m^5 - 299803386528 m^6 -
1090307192040 i m^6 - 1757026624092 i^2 m^6 - 1714714819488 i^3
m^6 - 1146361033736 i^4 m^6 - 557715814152 i^5 m^6 - 203638093204
i^6 m^6 - 56634294912 i^7 m^6 - 12055193520 i^8 m^6 - 1958117904
i^9 m^6 - 240082408 i^10 m^6 - 21796896 i^11 m^6 - 1422608 i^12
m^6 - 62976 i^13 m^6 - 1536 i^14 m^6 + 427713831360 m^7 +
978435680544 i m^7 + 1071755079216 i^2 m^7 + 739831166592 i^3 m^7
+ 355345859424 i^4 m^7 + 123577341984 i^5 m^7 + 31493767056 i^6
m^7 + 5850050688 i^7 m^7 + 775151424 i^8 m^7 + 70499136 i^9 m^7 +
4211040 i^10 m^7 + 165888 i^11 m^7 + 4608 i^12 m^7 + 159585281856
m^8 + 406267594848 i m^8 + 472738768464 i^2 m^8 + 338496696768 i^3
m^8 + 166581998736 i^4 m^8 + 59037031872 i^5 m^8 + 15296514528 i^6
m^8 + 2882093184 i^7 m^8 + 385763472 i^8 m^8 + 35249568 i^9 m^8 +
2105520 i^10 m^8 + 82944 i^11 m^8 + 2304 i^12 m^8 - 71653572864
m^9 - 110979416448 i m^9 - 84485395776 i^2 m^9 - 41844810240 i^3
m^9 - 14729217600 i^4 m^9 - 3657732480 i^5 m^9 - 599874240 i^6 m^9
- 57242880 i^7 m^9 - 2416320 i^8 m^9 - 27192886272 m^10 -
45093407616 i m^10 - 34385602368 i^2 m^10 - 16643847168 i^3 m^10 -
5774306304 i^4 m^10 - 1440853632 i^5 m^10 - 238714176 i^6 m^10 -
22897152 i^7 m^10 - 966528 i^8 m^10 + 1733428224 m^11 - 621513216
i m^11 - 621561600 i^2 m^11 + 102629376 i^3 m^11 + 128051712 i^4
m^11 + 24261120 i^5 m^11 + 1347840 i^6 m^11 + 798935040 m^12 +
32684544 i m^12 - 167770368 i^2 m^12 + 34209792 i^3 m^12 +
42683904 i^4 m^12 + 8087040 i^5 m^12 + 449280 i^6 m^12 + 208760832
m^13 + 221405184 i m^13 + 36384768 i^2 m^13 + 66281472 m^14 +
63258624 i m^14 + 10395648 i^2 m^14 + 5308416 m^15 + 1327104
m^16=0].

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Before Normalization > finish.

An equivalent quantifier-free formula:

FALSE


=====================  The End  =======================
```