Symbolic Summation in ΠΣ-Fields

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A Bonus Problem in "Concrete Mathematics"

Chapter 6. Special Numbers, Bonus problem 69:

Find a closed form for
$$\sum_{k=1}^{n} k^2 \mathrm{H}_{n+k},$$
where $\mathrm{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

Knuth's answer to the problem is
$$\frac{1}{3}n\big(n+\frac{1}{2}\big)\,(n+1)\,(2\mathrm{H}_{2n}-\mathrm{H}_n)-\frac{1}{36}n\left(10n^2+9n-1\right)$$
with the remark

> *"It would be nice to automate the derivation of formulas such as this."*

In[1]:= **<< Sigma`**

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Sigma -A summation package by Carsten Schneider
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In[2]:= **Problem69 = SigmaSum[k^2 SigmaHNumber[n + k], {k, 1, n}]**

Out[2]= $\sum_{k=1}^{n}\left(k^2\,\mathrm{H}_{k+n}\right)$

In[3]:= **SigmaReduce[Problem69]//Simplify**

Out[3]= $-\frac{1}{36}\,n\,(1+n)\,(-1+10\,n+6\,(1+2\,n)\,\mathrm{H}_n-12\,(1+2\,n)\,\mathrm{H}_{2n})$

- Based on Karr's ideas (1981) of $\Pi\Sigma$-fields

Indefinite Summation in Difference Field

Goal: Find a closed form for

$$\sum_{k=0}^{n} k\,k!$$

A Difference Field for the Problem

Let t_1, t_2 be indeterminates where

$$\begin{aligned} t_1 &\longleftrightarrow k \\ t_2 &\longleftrightarrow k! \end{aligned}$$

Consider the **field automorphism** $\sigma : \mathbb{Q}(t_1, t_2) \to \mathbb{Q}(t_1, t_2)$ canonically defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q} & & \\ \sigma(t_1) &= t_1 + 1 & \mathrm{S}\,k &= k+1 \\ \sigma(t_2) &= (t_1+1)t_2 & \mathrm{S}\,k! &= (k+1)! \end{aligned}$$

$(\mathbb{Q}(t_1, t_2), \sigma)$ is our difference field.

The Telescoping Problem

Find $g \in \mathbb{Q}(t_1, t_2)$: $\boxed{\sigma(g) - g = t_1\,t_2}$

$\downarrow$ by *Sigma*

$g = t_2.$

The Closed Form

$$\boxed{(k+1)! - k! = k\,k!}$$

$$\downarrow$$

$$\sum_{k=0}^{n} k\,k! = (n+1)! - 1.$$

Calkin's Identity and Variations

Find a closed form for

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^{3}$$

Case 1:

In[4]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{a}}\left(\sum_{\mathbf{j=0}}^{\mathbf{k}}\left(\binom{\mathbf{n}}{\mathbf{j}}^{.}\right)\right);$

In[5]:= **SigmaReduce[mySum]**

Out[5]= $\frac{1}{2}\left((-a+n)\binom{n}{a}^{.} + (2+2\,a-n)\sum_{\iota_1=0}^{a}\left(\binom{n}{\iota_1}^{.}\right)\right)$

Case 2:

In[6]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{a}}\left(\left(\sum_{\mathbf{j=0}}^{\mathbf{k}}\left(\binom{\mathbf{n}}{\mathbf{j}}^{.}\right)\right)^{\mathbf{2}}\right);$

In[7]:= **SigmaReduce[mySum]**

Out[7]= $\sum_{\iota_1=0}^{a}\left(\left(\sum_{\iota_2=0}^{\iota_1}\left(\binom{n}{\iota_2}^{.}\right)\right)^{2}\right)$

In[8]:= **SigmaReduce[mySum, SimplifyByExt → Depth]**

Out[8]= $(-a+n)\binom{n}{a}^{.}\sum_{\iota_1=0}^{a}\left(\binom{n}{\iota_1}^{.}\right) + \left(1+a-\frac{n}{2}\right)\left(\sum_{\iota_1=0}^{a}\left(\binom{n}{\iota_1}^{.}\right)\right)^{2} +$

$\sum_{\iota_1=0}^{a}\left(-\frac{1}{2}\,n\left(\binom{n}{\iota_1}^{.}\right)^{2}\right)$

Case 3: (Definite Summation)

In[9]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{n}} \left(\left(\sum_{\mathbf{j=0}}^{\mathbf{k}} \left(\binom{\mathbf{n}}{\mathbf{j}}^{\cdot} \right) \right)^{\mathbf{3}} \right)$

• Finding a recurrence

In[10]:= $\mathbf{rec = GenerateRecurrence[mySum][[1]]}$

Out[10]= $-4\ (1+2\ n)\ \mathtt{SUM}[n] - (12+7\ n)\ \mathtt{SUM}[1+n]$
$+(1+n)\ \mathtt{SUM}[2+n] == 2\ \left(-10\ \ +9\ n\ \right) \left(\sum_{\iota_1=0}^{n} \left(\binom{n}{\iota_1}^{\cdot} \right) \right)^3$

In[11]:= $\mathbf{rec = rec/.}\{ \sum_{\iota_1=\mathbf{0}}^{\mathbf{n}} \left(\binom{\mathbf{n}}{\iota_1}^{\cdot} \right) \to (\mathbf{2})^{\mathbf{n}\cdot} \}$

Out[11]= $-4\ (1+2\ n)\ \mathtt{SUM}[n] - (12+7\ n)\mathtt{SUM}[1+n]$
$+(1+n)\ \mathtt{SUM}[2+n] == 2\ \left(-10\ ((2)^{n\cdot})^3 + 9\ n\ ((2)^{n\cdot})^3\right)$

• Solving the recurrence

In[12]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n],}$
$\mathbf{Tower} \to \{ \binom{\mathbf{2n}}{\mathbf{n}}^{\cdot} \}]$

Out[12]= $\left\{ \left\{ 0, n \binom{2\ n}{n}^{\cdot} (2)^{n\cdot} \right\}, \left\{ 1, \frac{1}{2}\ (2+n)\ ((2)^{n\cdot})^3 \right\} \right\}$

• Finding the linear combination

In[13]:= $\mathbf{FindLinearCombination[recSol, mySum, 2]}$

Out[13]= $-\frac{3}{4}\ n \binom{2\ n}{n}^{\cdot} (2)^{n\cdot} + \frac{1}{2}\ (2+n)\ ((2)^{n\cdot})^3$

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3 \qquad \left\{c\,n\binom{2\,n}{n}2^n+\frac{1}{2}(2+n)\,2^{3\,n}\mid c\in\mathbb{Q}\right\}$$

find rec ↓ find solutions ↑

$$-4\,(1+2\,n)\,\mathrm{SUM}[n]-(12+7\,n)\mathrm{SUM}[1+n]$$
$$+\,(1+n)\,\mathrm{SUM}[2+n]=2\,2^{3\,n}(-10+9\,n)\quad(1)$$

GOAL: Find $c\in\mathbb{Q}$ such that for all $n\geq 0$:

$$\underbrace{\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3}_{=:\,\mathrm{lhs}[n]}=\underbrace{c\,n\binom{2\,n}{n}2^n+\frac{1}{2}(2+n)\,2^{3\,n}}_{=:\,\mathrm{rhs}[n]}$$

ANSATZ: Find $c\in\mathbb{Q}$ s.t.

$$\begin{array}{l}1=\mathrm{lhs}[0]\overset{!}{=}\mathrm{rhs}[0]=1\\9=\mathrm{lhs}[1]\overset{!}{=}\mathrm{rhs}[1]=12+4\,c\end{array}\quad\rightarrow c=-\frac{3}{4}$$

Any sequence fulfilling (1) is uniquely determined by the first two entries:

$$\mathrm{SUM}[2+n]\leftarrow\frac{1}{n+1}4\,(1+2\,n)\,\mathrm{SUM}[n]$$
$$+\,(12+7\,n)\mathrm{SUM}[1+n]+2\,2^{3\,n}(-10+9\,n)$$

Hence

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3=\underbrace{-\frac{3}{4}}_{c}\,n\binom{2\,n}{n}2^n+\frac{1}{2}(2+n)\,2^{3\,n}$$

Calkin's Identity and Variations

Case 1:

$$\sum_{k=0}^{a} x^k \sum_{j=0}^{k} \binom{n}{j} y^j = \frac{x^{a+1} \sum_{j=0}^{a} \binom{n}{j} y^j - \sum_{j=0}^{a} \binom{n}{j} x^j y^j}{x-1}$$

specializes to:

$$\sum_{k=0}^{n} x^k \sum_{j=0}^{k} y^k \binom{n}{j} = \frac{x^{n+1} (1+y)^n - (1+x\,y)^n}{x-1}$$

Case 2, non-alternating:

$$\sum_{k=0}^{a} \left(\sum_{j=0}^{k} \binom{n}{j} \right)^2 = (n-a) \binom{n}{a} \sum_{j=0}^{a} \binom{n}{j} + (1+a-\frac{n}{2}) \left(\sum_{j=0}^{a} \binom{n}{j} \right)^2 - \frac{n}{2}, \sum_{j=0}^{a} \binom{n}{j}^2$$

specializes to:

$$\sum_{k=0}^{n} \left(\sum_{j=0}^{k} \binom{n}{j} \right)^2 = (n+1)\, 4^n - \frac{n}{2}\, 4^n - \frac{n}{2} \binom{2\,n}{n}$$

Case 2, alternating:

$$\sum_{k=0}^{a}(-1)^k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2 = 2\,(n-a)\binom{n}{a}(-1)^a\sum_{j=0}^{a}\binom{n}{j} + n\,(-1)^a\left(\sum_{j=0}^{a}\binom{n}{j}\right)^2 - \sum_{j=0}^{a}(n-2\,j)\binom{n}{j}^2(-1)^j$$

specializes to:

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2 = \begin{cases} 0 & \text{if } n \text{ is even} \\ -(-1)^{\frac{n-1}{2}}\, n\binom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

Case 2 for even n, interlaced alternating:

$$\sum_{k=0}^{2\,n}\left(\sum_{j=0}^{k}(-1)^{\frac{1}{2}(j-1)\,j}\binom{2\,n}{j}\right)^2 = \frac{2^{2\,n}}{4}\left(4+6\,n-4\,n\,(-1)^n+3\,n\sum_{j=2}^{n}\frac{\binom{4\,j}{2\,j}}{(4\,j-3)\,2^{2\,j}}+3\,n\sum_{j=2}^{n}\frac{\binom{4\,j}{2\,j}}{(4\,j-1)\,2^{2\,j}}\right)$$

Case 3, Calkin's identity:

$$\boxed{\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3 = \frac{n}{2}\,8^n + 8^n - \frac{3\,n}{4}\,2^n\binom{2\,n}{n}}$$

Case 3 for even n, alternating:

$$\sum_{k=0}^{2\,n}(-1)^k\left(\sum_{j=0}^{k}\binom{2\,n}{j}\right)^3 = \frac{64^n}{2} - \frac{(-1)^n\,64^n}{16\,n\,\binom{2\,n}{n}}\sum_{i=0}^{n-1}(3+11\,i)\binom{2\,i}{i}^2\binom{3\,i}{i}64^{-i}$$

Definite Summation

GOAL: Find a closed form for

$$\sum_{k=1}^{n}\left(\frac{H_k\,(3+k+n)!\,(-1)^{k}\,(-1)^{-1+n}}{(1+k)!\,(2+k)!\,(-k+n)!}\right)$$
$$-\frac{(n)!}{(3+n)!}\sum_{k=1}^{n}\left(\frac{(3+k+n)!\,(-1)^{k}\,(1-(2+n)\,(-1)^{n})}{k\,(1+k)!^{2}\,(-k+n)!}\right)$$

(The number of rhombus tilings of a symmetric hexagon, Fulmek & Krattenthaler)

In[14]:= $\mathbf{mySum1} = \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} \left(\frac{\mathbf{H_k}\ (\mathbf{3}+\mathbf{k}+\mathbf{n})!\ (-\mathbf{1})^{\mathbf{k}}\ (-\mathbf{1})^{-\mathbf{1}+\mathbf{n}}}{(\mathbf{1}+\mathbf{k})!\ (\mathbf{2}+\mathbf{k})!\ (-\mathbf{k}+\mathbf{n})!} \right);$

Finding a recurrence

In[15]:= $\mathbf{rec1} = \mathbf{GenerateRecurrence}[\mathbf{mySum1}][[\mathbf{1}]]$

Out[15]= $$\mathtt{n}\ (1+\mathtt{n})\ (2+\mathtt{n})\ (3+\mathtt{n})\ (4+\mathtt{n})\ (-1+\mathtt{n})!\ \Big(-(9+2\,\mathtt{n})\ (8+6\,\mathtt{n}+\mathtt{n}^2)\ \mathtt{SUM}[\mathtt{n}]+ (9+2\,\mathtt{n})\ (13+8\,\mathtt{n}+\mathtt{n}^2)\ \mathtt{SUM}[1+\mathtt{n}]+ (30+42\,\mathtt{n}+17\,\mathtt{n}^2+2\,\mathtt{n}^3)\ \mathtt{SUM}[2+\mathtt{n}]- (3+\mathtt{n})\ (25+15\,\mathtt{n}+2\,\mathtt{n}^2)\ \mathtt{SUM}[3+\mathtt{n}]\Big) == 2\ (-1)^{\mathtt{n}}\ (9+2\,\mathtt{n})\ (35+24\,\mathtt{n}+4\,\mathtt{n}^2)\ (4+\mathtt{n})!$$

Solving the recurrence

In[16]:= $\mathbf{recSol1} = \mathbf{SolveRecurrence}[\mathbf{rec1}, \mathbf{SUM}[\mathbf{n}], \mathbf{Tower} \to \{\mathrm{H}_{\mathbf{n}}\}]$

Out[16]= $$\Big\{\{0,1\}, \Big\{0, \frac{3-\mathtt{n}^2+4\,\mathtt{H_n}+6\,\mathtt{n}\,\mathtt{H_n}+2\,\mathtt{n}^2\,\mathtt{H_n}}{(1+\mathtt{n})\ (2+\mathtt{n})}\Big\}, \Big\{0, \frac{1}{4}\ (2+\mathtt{n})\ (-1)^{\mathtt{n}}\Big\}, \Big\{1, \frac{(16-13\,\mathtt{n}^2-5\,\mathtt{n}^3+32\,\mathtt{H_n}+64\,\mathtt{n}\,\mathtt{H_n}+40\,\mathtt{n}^2\,\mathtt{H_n}+8\,\mathtt{n}^3\,\mathtt{H_n})\ (-1)^{\mathtt{n}}}{4\ (1+\mathtt{n})\ (2+\mathtt{n})}\Big\}\Big\}$$

Finding the linear combination

In[17]:= $\mathbf{solution1} = \mathbf{FindLinearCombination}[\mathbf{recSol1}, \mathbf{mySum1}, \mathbf{3}]$

Out[17]= $$-1-\frac{3-\mathtt{n}^2+4\,\mathtt{H_n}+6\,\mathtt{n}\,\mathtt{H_n}+2\,\mathtt{n}^2\,\mathtt{H_n}}{(1+\mathtt{n})\ (2+\mathtt{n})}+\frac{1}{4}\ (2+\mathtt{n})\ (-1)^{\mathtt{n}}+ \frac{(16-13\,\mathtt{n}^2-5\,\mathtt{n}^3+32\,\mathtt{H_n}+64\,\mathtt{n}\,\mathtt{H_n}+40\,\mathtt{n}^2\,\mathtt{H_n}+8\,\mathtt{n}^3\,\mathtt{H_n})\ (-1)^{\mathtt{n}}}{4\ (1+\mathtt{n})\ (2+\mathtt{n})}$$

In[18]:= $\mathbf{mySum2} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \left(\frac{(\mathbf{3+k+n})!\, (-\mathbf{1})^{\mathbf{k}}\, (\mathbf{1}-(\mathbf{2+n})\, (-\mathbf{1})^{\mathbf{n}})}{\mathbf{k}\, (\mathbf{1+k})!^{\mathbf{2}}\, (-\mathbf{k}+\mathbf{n})!} \right);$

Finding a recurrence

In[19]:= $\mathbf{rec2} = \mathbf{GenerateRecurrence}[\mathbf{mySum2}, \mathbf{RecOrder} \to \mathbf{2}][[\mathbf{1}]]$

Out[19]= $-n\,(1+n)\,(3+n)\,(1+3\,(-1)^n+(-1)^n\,n)\,(-1+4\,(-1)^n+(-1)^n\,n)\,(28+15\,n+2\,n^2)\,(-1+n)!\,\mathrm{SUM}[n]+6\,n\,(1+n)\,(3+n)^2\,(-1+2\,(-1)^n+(-1)^n\,n)\,(-1+4\,(-1)^n+(-1)^n\,n)\,(-1+n)!\,\mathrm{SUM}[1+n]+n\,(1+n)\,(3+n)\,(-1+2\,(-1)^n+(-1)^n\,n)\,(1+3\,(-1)^n+(-1)^n\,n)\,(10+9\,n+2\,n^2)\,(-1+n)!\,\mathrm{SUM}[2+n] == 2\,(-1+2\,(-1)^n+(-1)^n\,n)\,(1+3\,(-1)^n+(-1)^n\,n)\,(-1+4\,(-1)^n+(-1)^n\,n)\,(35+24\,n+4\,n^2)\,(4+n)!$

Solving the recurrence

$((-1)^k)^2 = 1$

In[20]:= $\mathbf{recSol2} = \mathbf{SolveRecurrence}[\mathbf{rec2}, \mathbf{SUM}[\mathbf{n}], \mathbf{Tower} \to \{H_n\}, \boxed{\mathbf{WithMinusPower} \to \mathbf{True}}]$

Out[20]= $\{\{0, 2+n-(-1)^n\}, \{0, 16-6\,n^2-n^3+(-1)^n+28\,n\,(-1)^n+23\,n^2\,(-1)^n+8\,n^3\,(-1)^n+n^4\,(-1)^n\}, \{1, -\frac{1}{28}(260-150\,n^2-39\,n^3+336\,H_n+616\,n\,H_n+336\,n^2\,H_n+56\,n^3\,H_n-325\,(-1)^n+365\,n^2\,(-1)^n+228\,n^3\,(-1)^n+39\,n^4\,(-1)^n-672\,H_n\,(-1)^n-1568\,n\,H_n\,(-1)^n-1288\,n^2\,H_n\,(-1)^n-448\,n^3\,H_n\,(-1)^n-56\,n^4\,H_n\,(-1)^n)\}\}$

Finding the linear combination

In[21]:= $\mathbf{solution2} = \mathbf{FindLinearCombination}[\mathbf{recSol2}, \mathbf{mySum2}, \mathbf{2}]$

Out[21]= $(3+n)\,(-1+3\,n+2\,n^2-(-1+6\,n+7\,n^2+2\,n^3)\,(-1)^n+2\,(2+3\,n+n^2)\,H_n\,(-1+(2+n)\,(-1)^n))$

In[22]:= $\mathbf{solution1 - solution2/((n+1)(n+2)(n+3))//Simplify}$

Out[22]= $-2 + (2 + n)\ (-1)^{n}.$

Difference Equations and Symbolic Summation

Let $(\mathbb{F}, \sigma)$ be a difference field and

$$\mathbb{K} = \{k \in \mathbb{F} \,|\, \sigma(k) = k\}$$

be the constant field. Assume $\mathbb{Q} \subseteq \mathbb{K}$.

Telescoping

- GIVEN $f \in \mathbb{F}$
- FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$$\downarrow \qquad \uparrow$$

Parameterized Telescoping

- GIVEN $f_0, \dots, f_d \in \mathbb{F}$, $a_0, a_1 \in \mathbb{F}$
- FIND ALL $c_0, \dots, c_d \in \mathbb{K}$, $h \in \mathbb{F}$:

$$\boxed{a_1\,\sigma(h) - a_0\,h = c_0\,f_0 + \cdots + c_d\,f_d}$$

Remark: Z's "Creative Telescoping"

- GIVEN $f_i = \text{summand}(n+i, k) \in \mathbb{F}$
- FIND ALL $c_0, \dots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0\,f_0 + \cdots + c_d\,f_d}$$

Linear Difference Equations

- GIVEN $f, a_0, \dots, a_m \in \mathbb{F}$
- FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

Parameterized Linear Difference Equations

- GIVEN $a_0, \dots, a_m \in \mathbb{F}$,$f_0, \dots, f_d \in \mathbb{F}$.
- FIND ALL $g \in \mathbb{F}$, $c_0, \dots, c_d \in \mathbb{K}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = c_0\,f_0 + \cdots + c_d\,f_d}$$

Sum Extensions for Recurrences

In[23]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{N}} \left(\frac{\binom{\mathbf{N}}{\mathbf{k}} (-\mathbf{1})^{\mathbf{k}}}{(\mathbf{k}+\mathbf{1})^{4}} \right);$

Finding a recurrence

In[24]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}]$

Out[24]= $\{(1+N)\ (2+N)\ (3+N)\ (4+N)\ \mathtt{SUM}[N] - 3\ (2+N)\ (3+N)^2\ (4+N)\ \mathtt{SUM}[1+N] + (3+N)\ (4+N)\ (37+21\ N+3\ N^2)\ \mathtt{SUM}[2+N] - (4+N)^4\ \mathtt{SUM}[3+N] == -1\}$

Solving the recurrence (A First Attempt)

In[25]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}[[\mathbf{1}]], \mathbf{SUM}[\mathbf{N}]]$

Out[25]= $\{\{0, \frac{1}{1+N}\}\}$

The underlying difference field is too small!

Solving the recurrence (Step I)

In[26]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}[[\mathbf{1}]], \mathbf{SUM}[\mathbf{N}], \mathbf{NestedSumExt} \to \infty]$

$$\text{Out[26]=}\ \Big\{\Big\{0, \frac{1}{1+N}\Big\}, \Big\{0, \frac{\sum_{\iota_1=1}^{N}\left(\frac{1}{1+\iota_1}\right)}{1+N}\Big\}, \Big\{0, \frac{\sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{1}{1+\iota_2}\right)}{1+\iota_1}\right)}{1+N}\Big\},$$

$$\Big\{1, \frac{\sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{1+\iota_3}\right)}{1+\iota_2}\right)}{1+\iota_1}\right)}{1+N}\Big\}\Big\}$$

- Inspired by Abramov/Petkovšek and Hendrik/Singer
- Theoretical result:

  We can find all sum extensions over a given **ΠΣ-field** which give more solutions of a homogeneous or **inhomogeneous** recurrence!
- Speed up in computation.
- Further simplification by our **indefinite summation algorithm**

Sum Extensions for Indefinite Summation

In[27]:= $$\mathbf{mySum} = \sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{K+\iota_3}\right)}{K+\iota_2}\right)}{K+\iota_1}\right);$$

In[28]:= **SigmaReduce[mySum]**

Out[28]= $$\sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{K+\iota_3}\right)}{K+\iota_2}\right)}{K+\iota_1}\right);$$

In[29]:= **SigmaReduce[mySum, SimplifyByExt → Depth]**

Out[29]= $$\frac{1}{6K^2}\left(6\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)+6K\left(\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)^2+K^2\left(\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)^3+\left(-3-3K\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)\boxed{\sum_{\iota_1=1}^{N}\left(\frac{K+2\iota_1}{(K+\iota_1)^2}\right)}-K\boxed{\sum_{\iota_1=1}^{N}\left(\frac{K+3\iota_1}{(K+\iota_1)^3}\right)}\right)$$

Partial fraction decomposition:

$$\boxed{\frac{K+2i}{(K+i)^2}}=-\frac{K}{(K+i)^2}+\frac{2}{K+i},\quad \boxed{\frac{K+3i}{(K+i)^2}}=-\frac{2K}{(K+i)^3}+\frac{3}{(K+i)^2}$$

In[30]:= **SigmaReduce[mySum, Tower → {{$\mathbf{H_{K+N}}$, N}, {$\mathbf{H^{(2)}_{K+N}}$, N}, {$\mathbf{H^{(3)}_{K+N}}$, N}}]**

Out[30]= $$\frac{1}{6}\Big(-H_K^3-3H_K H_{K+N}^2+H_{K+N}^3+3H_K H_K^{(2)}-3H_K H_{K+N}^{(2)}+H_{K+N}\left(3H_K^2-3H_K^{(2)}+3H_{K+N}^{(2)}\right)-2H_K^{(3)}+2H_{K+N}^{(3)}\Big)$$

Sum Extensions in the Difference Field Setting

$$\sum_{\iota_1=1}^{\mathtt{N}} \left(\frac{\sum_{\iota_2=1}^{\iota_1} \left(\frac{\sum_{\iota_3=1}^{\iota_2} \left(\frac{1}{\mathtt{K}+\iota_3} \right)}{\mathtt{K}+\iota_2} \right)}{\mathtt{K}+\iota_1} \right)$$

The underlying difference field $(\mathbb{Q}(t_1)(t_2)(t_3)(t_4), \sigma)$:

$$\sigma(t_1) = t_1 + 1$$
$$\sigma(t_2) = t_2 + \frac{1}{K + t_1 + 1}$$
$$\sigma(t_3) = t_3 + \sigma(\frac{t_2}{K + t_1})$$
$$\sigma(t_4) = t_4 + \sigma(\frac{t_3}{K + t_1})$$

$$\frac{1}{6}\Big(-\mathtt{H}_{\mathtt{K}}^{3}-3\,\mathtt{H}_{\mathtt{K}}\,\mathtt{H}_{\mathtt{K+N}}^{2}+\mathtt{H}_{\mathtt{K+N}}^{3}+3\,\mathtt{H}_{\mathtt{K}}\,\mathtt{H}_{\mathtt{K}}^{(2)}-3\,\mathtt{H}_{\mathtt{K}}\,\mathtt{H}_{\mathtt{K+N}}^{(2)}$$

$$+\mathtt{H}_{\mathtt{K+N}}\Big(3\,\mathtt{H}_{\mathtt{K}}^{2}-3\,\mathtt{H}_{\mathtt{K}}^{(2)}+3\,\mathtt{H}_{\mathtt{K+N}}^{(2)}\Big)-2\,\mathtt{H}_{\mathtt{K}}^{(3)}+2\,\mathtt{H}_{\mathtt{K+N}}^{(3)}\Big)$$

The underlying difference field $(\mathbb{Q}(t_1)(t_2)(t_3')(t_4'),\sigma)$:

$$\sigma(t_1)=t_1+1$$

$$\sigma(t_2)=t_2+\frac{1}{K+t_1+1}$$

$$\sigma(t_3')=t_3'+\frac{1}{(K+t_1+1)^2}$$

$$\sigma(t_4')=t_4'+\frac{1}{(K+t_1+1)^3}$$

$$\boxed{(\mathbb{Q}(t_1)(t_2)(t_3)(t_4),\sigma)\simeq(\mathbb{Q}(t_1)(t_2)(t_3')(t_4'),\sigma)}$$

Finding the closed form evaluation

We know:

$$\sum_{i=1}^{N}\frac{\displaystyle\sum_{j=1}^{i}\frac{\displaystyle\sum_{k=1}^{j}\frac{1}{K+k}}{K+j}}{K+i} = 3\,\mathrm{H}_K\,\mathrm{H}_{K+N}^{(2)}+\mathrm{H}_{K+N}\left(3\,\mathrm{H}_K^2- 3\,\mathrm{H}_K^{(2)}+3\,\mathrm{H}_{K+N}^{(2)}\right)-2\,\mathrm{H}_K^{(3)}+2\,\mathrm{H}_{K+N}^{(3)}\Big)$$

Solving the recurrence (Step II)

In[31]:= **recSol =**

$$\mathbf{SolveRecurrence[rec[[1]], SUM[N], Tower \to \{H_N, H_N^{(2)}, H_N^{(3)}\}]}$$

Out[31]= $\Big\{\Big\{0, \frac{1}{(1+N)^3}\big(2+2\,H_N+2\,N\,H_N+H_N^2+2\,N\,H_N^2+N^2\,H_N^2+H_N^{(2)}+2\,N\,H_N^{(2)}+N^2\,H_N^{(2)}\big)\Big\},$

$\Big\{0, \frac{1}{(1+N)^3}\big(-4\,N-2\,N^2+2\,H_N+ 2\,N\,H_N+H_N^2+2\,N\,H_N^2+N^2\,H_N^2+H_N^{(2)}+2\,N\,H_N^{(2)}+N^2\,H_N^{(2)}\big)\Big\},$

$\Big\{0, \frac{1}{(1+N)^3}\big(-N+N^2-H_N-4\,N\,H_N- 3\,N^2\,H_N+H_N^2+2\,N\,H_N^2+N^2\,H_N^2+H_N^{(2)}+2\,N\,H_N^{(2)}+N^2\,H_N^{(2)}\big)\Big\},$

$\Big\{1, \frac{1}{6\,(1+N)^4}\big(-6\,N-6\,N\,H_N-6\,N^2\,H_N-3\,N\,H_N^2-6\,N^2\,H_N^2-3\,N^3\,H_N^2+H_N^3+ 3\,N\,H_N^3+3\,N^2\,H_N^3+N^3\,H_N^3-3\,N\,H_N^{(2)}-6\,N^2\,H_N^{(2)}-3\,N^3\,H_N^{(2)}+ 3\,H_N\,H_N^{(2)}+9\,N\,H_N\,H_N^{(2)}+9\,N^2\,H_N\,H_N^{(2)}+3\,N^3\,H_N\,H_N^{(2)}+2\,H_N^{(3)}+ 6\,N\,H_N^{(3)}+6\,N^2\,H_N^{(3)}+2\,N^3\,H_N^{(3)}\big)\Big\}\Big\}$

Finding the linear combination

In[32]:= **FindLinearCombination[recSol, defSum, 3]//Simplify**

Out[32]= $\frac{1}{6\,(1+N)^4}\Big(3\,(1+N)^2\,H_N^2+ (1+N)^3\,H_N^3+3\,(1+N)^2\,H_N^{(2)}+3\,(1+N)\,H_N\,\big(2+(1+N)^2\,H_N^{(2)}\big)+ 2\,\big(3+H_N^{(3)}+3\,N\,H_N^{(3)}+3\,N^2\,H_N^{(3)}+N^3\,H_N^{(3)}\big)\Big)$

Examples of ΠΣ-fields

- The difference field $(\mathbb{Q}(t_1, t_2, t_3, t_4), \sigma)$ with

$$\begin{aligned}\sigma(t_1) &= t_1 + 1\\ \sigma(t_2) &= t_2 + \sigma(\frac{1}{t_1})\\ \sigma(t_3) &= t_3 + \sigma(t_2)\\ \sigma(t_4) &= t_4 + \sigma(t_3)\end{aligned}$$

$$\sum_{i=1}^{N} \frac{\sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{1}{k}}{j}}{i}$$

- The difference $(\mathbb{Q}(n)(t_1, t_2, t_3, t_4), \sigma)$ with

$$\begin{aligned}\sigma(t_1) &= t_1 + 1\\ \sigma(t_2) &= t_2 + \sigma(\frac{n+1-t_1}{t_1})\\ \sigma(t_3) &= t_3 + \sigma(t_2)\\ \sigma(t_4) &= t_4 + \sigma(t_3^3)\end{aligned}$$

$$\sum_{k=0}^{n} \left(\sum_{j=0}^{k} \binom{n}{j} \right)^3$$

In general a ΠΣ-field $(\mathbb{K}(t_1)(t_2)\dots(t_e), \sigma)$ consists of

first order linear extensions,

i.e.,

- $\mathbb{F}(t_1, \dots, t_e)$ is a rational function field,
- we have

$$\boxed{\sigma(t_i) = \alpha\, t_i + \beta},$$

  with $\alpha, \beta \in \mathbb{K}(t_1)(t_2)\dots(t_{i-1})$,
- $\text{const}_\sigma \mathbb{K}(t_1)(t_2)\dots(t_e) = \mathbb{K}$.

General Construction of ΠΣ-fields & Solving Linear Difference Equations

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- M. Bronstein. On solutions of linear ordinary difference equations in their coefficient field. *J. Symbolic Comput.*, 29(6):841–877, June 2000.
- C. Schneider. Symbolic summation in difference fields. Technical Report 01-17, RISC-Linz, J. Kepler University, November 2001. PhD Thesis.
- C. Schneider. Solving parameterized linear difference equations in ΠΣ-fields. Technical Report 02-03, RISC-Linz, J. Kepler University, July 2002. Submitted.
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Simplifying ΠΣ-fields

- C. Schneider. Product representations in ΠΣ-fields. Technical Report 02-24, RISC-Linz, J. Kepler University, 2003. Submitted.
- C. Schneider. Nested sum extensions in ΠΣ-fields. 2003. In preparation.

Higher Order Linear Extensions

- C. Schneider. Higher Order Linear Extensions over ΠΣ-fields 2003. In preparation.

Higher Order Linear Extensions

Given a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ and

$$\boxed{\sigma^m(t) = a_{m-1}\,\sigma^{m-1}(t) + \cdots + a_1\,\sigma(t) + a_0}$$

where $a_i \in \mathbb{F}$.

Fibonacci Numbers

In[33]:= $\mathbf{recFib = Fib[l+2] == \ Fib[l+1] + Fib[l];}$

In[34]:= $\mathbf{BuildEvaluation[recFib, Fib[l], \{1, 1\}, 0]}$

In[35]:= $\mathbf{Table[Fib[i], \{i, 0, 10\}]}$

Out[35]= $\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89\}$

In[36]:= $\mathbf{mySum = \sum_{l=0}^{n} Fib[l]\ x^{l};}$

In[37]:= $\mathbf{SigmaReduce[mySum, \{\{rec, Fib[l]\}\}]}$

Out[37]= $\dfrac{-1 + x^2\ \mathtt{Fib}[n]\ x^{n} + x\ \mathtt{Fib}[1+n]\ x^{n}}{-1 + x + x^2}$

In[38]:= $\mathbf{mySum = \sum_{l=0}^{n} Fib[l]\ \binom{n}{l}}$

In[39]:= $\mathbf{GenerateRecurrence[mySum, n, \{\{rec, Fib[l]\}\}]}$

Out[39]= $\{\mathtt{SUM}[n] - 3\ \mathtt{SUM}[1+n] + \mathtt{SUM}[2+n] == 0\}$

In[40]:= $$\mathbf{mySum} = \sum_{l=1}^{n} \frac{\mathbf{Fib}[l]}{H_{l+n}}$$

In[41]:= $$\mathbf{recSol} = \mathbf{GenerateRecurrence}[\mathbf{mySum}, \mathbf{n}, \{\{\mathbf{rec}, \mathbf{Fib}[l]\}\}]$$

Out[41]= $$\Big\{ -\mathrm{SUM}[n] + \mathrm{SUM}[1+n] + \mathrm{SUM}[2+n] ==$$

$$\Big(-(1+n)\ (5+3\ n)\ (3+4\ n)\ (5+4\ n)$$
$$(5+10\ n+4\ n^2)\ (11+24\ n+12\ n^2) - 2\ (1+n)^2\ (2+n)\ (3+4\ n)$$
$$(5+4\ n)\ (5+10\ n+4\ n^2)\ (11+24\ n+12\ n^2)\ H_n +$$
$$\Big(-(1+n)\ (1+2\ n)\ (5+3\ n)\ \big(2221+13360\ n+$$
$$32774\ n^2+41916\ n^3+29448\ n^4+10768\ n^5+1600\ n^6\big) -$$
$$2\ (1+n)^2\ (2+n)\ (1+2\ n)\ \big(2221+13360\ n+32774\ n^2+$$
$$41916\ n^3+29448\ n^4+10768\ n^5+1600\ n^6\big)\ H_n\Big)$$
$$H_{2\ n} + \ldots$$

$$\big(-2\ (1+n)^2\ (1+2\ n)^2\ (5+3\ n)$$
$$(1088+5071\ n+9162\ n^2+8016\ n^3+3400\ n^4+560\ n^5) -$$
$$4\ (1+n)^3\ (2+n)\ (1+2\ n)^2$$
$$(1088+5071\ n+9162\ n^2+8016\ n^3+3400\ n^4+560\ n^5)\ H_n\big)\ H_{2\ n}^2 +$$
$$\big(-4\ (1+n)^3\ (1+2\ n)^3\ (3+2\ n)$$
$$(5+3\ n)\ (77+200\ n+160\ n^2+40\ n^3) - 8\ (1+n)^4\ (2+n)$$
$$(1+2\ n)^3\ (3+2\ n)\ (77+200\ n+160\ n^2+40\ n^3)\ H_n\big)\ H_{2\ n}^3 +$$
$$\big(-8\ (1+n)^4\ (2+n)\ (1+2\ n)^4\ (3+2\ n)^2\ (5+3\ n) -$$
$$16\ (1+n)^5\ (2+n)^2\ (1+2\ n)^4\ (3+2\ n)^2\ H_n\big)\ H_{2\ n}^4 +$$
$$\mathrm{Fib}[n]\ \big(2\ (1+n)\ (1+2\ n)\ (3+2\ n)\ (11+24\ n+12\ n^2)$$
$$(43+115\ n+94\ n^2+24\ n^3) + 2\ (1+n)^2\ (1+2\ n)\ (5+3\ n)$$
$$(11+24\ n+12\ n^2)\ (43+115\ n+94\ n^2+24\ n^3)\ H_n +$$
$$2\ (1+n)^3\ (2+n)\ (1+2\ n)\ (11+24\ n+12\ n^2)$$
$$(43+115\ n+94\ n^2+24\ n^3)\ H_n^2 + \big(2\ (1+n)\ (1+2\ n)^2\ (3+2\ n)$$
$$(995+4565\ n+8128\ n^2+7020\ n^3+2944\ n^4+480\ n^5) +$$
$$2\ (1+n)^2\ (1+2\ n)^2\ (5+3\ n)$$
$$(995+4565\ n+8128\ n^2+7020\ n^3+2944\ n^4+480\ n^5)\ H_n +$$
$$2\ (1+n)^3\ (2+n)\ (1+2\ n)^2$$
$$(995+4565\ n+8128\ n^2+7020\ n^3+2944\ n^4+480\ n^5)\ H_n^2\big)$$
$$H_{2\ n} + \big(4\ (1+n)^2\ (1+2\ n)^3$$
$$(3+2\ n)^2\ (111+285\ n+226\ n^2+56\ n^3) + 4\ (1+n)^3\ (1+2\ n)^3$$
$$(3+2\ n)\ (5+3\ n)\ (111+285\ n+226\ n^2+56\ n^3)\ H_n +$$
$$4\ (1+n)^4\ (2+n)\ (1+2\ n)^3$$
$$(3+2\ n)\ (111+285\ n+226\ n^2+56\ n^3)\ H_n^2\big)\ H_{2\ n}^2 +$$
$$\big(16\ (1+n)^3\ (2+n)\ (1+2\ n)^4\ (3+2\ n)^3 +$$
$$16\ (1+n)^4\ (2+n)\ (1+2\ n)^4\ (3+2\ n)^2\ (5+3\ n)\ H_n +$$
$$16\ (1+n)^5\ (2+n)^2\ (1+2\ n)^4\ (3+2\ n)^2\ H_n^2\big)\ H_{2\ n}^3\big) +$$

$$\mathrm{Fib}[1+n]\ \big((1+2\ n)\ (3+2\ n)\ (2221+13360\ n+$$
$$32774\ n^2+41916\ n^3+29448\ n^4+10768\ n^5+1600\ n^6) +$$
$$(1+n)\ (1+2\ n)\ (5+3\ n)\ (2221+13360\ n+$$
$$32774\ n^2+41916\ n^3+29448\ n^4+10768\ n^5+1600\ n^6)\ H_n +$$
$$(1+n)^2\ (2+n)\ (1+2\ n)\ (2221+13360\ n+$$
$$32774\ n^2+41916\ n^3+29448\ n^4+10768\ n^5+1600\ n^6)\ H_n^2 +$$
$$(4\ (1+n)\ (1+2\ n)^2\ (3+2\ n)$$
$$(1088+5071\ n+9162\ n^2+8016\ n^3+3400\ n^4+560\ n^5) +$$
$$4\ (1+n)^2\ (1+2\ n)^2\ (5+3\ n)$$
$$(1088+5071\ n+9162\ n^2+8016\ n^3+3400\ n^4+560\ n^5)\ H_n +$$
$$4\ (1+n)^3\ (2+n)\ (1+2\ n)^2\ (1088+5071\ n+$$
$$9162\ n^2+8016\ n^3+3400\ n^4+560\ n^5)\ H_n^2)\ H_{2\ n} +$$
$$(12\ (1+n)^2\ (1+2\ n)^3\ (3+2\ n)^2\ (77+200\ n+160\ n^2+40\ n^3) +$$
$$12\ (1+n)^3\ (1+2\ n)^3\ (3+2\ n)\ (5+3\ n)$$
$$(77+200\ n+160\ n^2+40\ n^3)\ H_n + 12\ (1+n)^4\ (2+n)$$
$$(1+2\ n)^3\ (3+2\ n)\ (77+200\ n+160\ n^2+40\ n^3)\ H_n^2)\ H_{2\ n}^2 +$$
$$(32\ (1+n)^3\ (2+n)\ (1+2\ n)^4\ (3+2\ n)^3 +$$
$$32\ (1+n)^4\ (2+n)\ (1+2\ n)^4\ (3+2\ n)^2\ (5+3\ n)\ H_n +$$
$$32\ (1+n)^5\ (2+n)^2\ (1+2\ n)^4\ (3+2\ n)^2\ H_n^2)\ H_{2\ n}^3\big)\big)/$$
$$((1+(1+n)\ H_n)\ (3+2\ n+(1+n)\ (2+n)\ H_n)$$
$$(1+(1+2\ n)\ H_{2\ n})\ (3+4\ n+2\ (1+n)\ (1+2\ n)\ H_{2\ n})$$
$$(11+24\ n+12\ n^2+2\ (1+n)\ (1+2\ n)\ (3+2\ n)\ H_{2\ n})$$
$$((5+4\ n)\ (5+10\ n+4\ n^2) +$$
$$2\ (1+n)\ (2+n)\ (1+2\ n)\ (3+2\ n)\ H_{2\ n}))\Big\}$$

Hermite Polynomials

In[42]:= $\mathbf{rec = H[l+2] == 2x\ H[l+1] - 2(l+1)H[l]}$

Out[42]= $\mathtt{H[2+l] == -2\ (1+l)\ H[l] + 2\ x\ H[1+l]}$

In[43]:= $\mathbf{initial = \{1, 2x\}}$

Out[43]= $\{1, 2\ x\}$

In[44]:= $\mathbf{BuildEvaluation[rec, H[l], \{1, 2x\}, 0]}$

In[45]:= $\mathbf{Table[H[i], \{i, 0, 6\}]//Simplify}$

Out[45]= $\{1,$

$2\ x,$

$-2 + 4\ x^2,$

$4\ x\ (-3 + 2\ x^2),$

$4\ (3 - 12\ x^2 + 4\ x^4),$

$8\ x\ (15 - 20\ x^2 + 4\ x^4),$

$8\ (-15 + 90\ x^2 - 60\ x^4 + 8\ x^6)$

In[46]:= $\mathbf{mySum = \sum_{l=0}^{n} H[l]\ \binom{n}{l}}$.

In[47]:= $\mathbf{GenerateRecurrence[mySum, n, \{\{rec, H[l]\}\}]}$

Out[47]= $\{-2\ (1+n)\ \mathtt{SUM}[n] + (1 + 2\ x)\ \mathtt{SUM}[1+n] - \mathtt{SUM}[2+n] == 0\}$

In[48]:= $\mathbf{mySum} = \sum_{l=0}^{n} \mathbf{H[l]} \sum_{k=0}^{l} \binom{n}{k}$

In[49]:= $\mathbf{recSol} = \mathbf{GenerateRecurrence[mySum, n, \{\{rec, H[l]\}\}]}$

Out[49]= $\{4\ (1+n)\ \mathrm{SUM}[n] - 2\ (2+n+2\ x)\ \mathrm{SUM}[1+n] + (3+2\ x)\ \mathrm{SUM}[2+n] - \mathrm{SUM}[3+n] ==$

$$\frac{1}{2}\Bigg(\mathrm{H}[n]\ \Bigg(2\ (1+n)\ \left(-4+n+n^2+16\ x+10\ n\ x+2\ n^2\ x\right) - 4\ (1+n)\ (3+2\ x)\ \sum_{k=0}^{n} \frac{(1+n)\ (2+n)\ \binom{n}{k}}{(1-k+n)\ (2-k+n)} + 4\ (1+n)\ (1+2\ x)\ \sum_{k=0}^{n} \frac{(1+n)\ (2+n)\ (3+n)\ \binom{n}{k}}{(1-k+n)\ (2-k+n)\ (3-k+n)}\Bigg) + \mathrm{H}[1+n]\ \Bigg(52+51\ n+17\ n^2+2\ n^3+12\ x+2\ n\ x-2\ n^2\ x-32\ x^2 - 20\ n\ x^2-4\ n^2\ x^2-4\ (2+n+2\ x)\ \sum_{k=0}^{n} \frac{(1+n)\ \binom{n}{k}}{1-k+n} + 2\ (1+2\ x)\ (3+2\ x)\ \sum_{k=0}^{n} \frac{(1+n)\ (2+n)\ \binom{n}{k}}{(1-k+n)\ (2-k+n)} + 2\ \left(3+2\ n-2\ x-4\ x^2\right)\ \sum_{k=0}^{n} \frac{(1+n)\ (2+n)\ (3+n)\ \binom{n}{k}}{(1-k+n)\ (2-k+n)\ (3-k+n)}\Bigg)\Bigg)\}$$

In[50]:= $\mathbf{recSol = recSol/.}$

$$\left\{\sum_{k=0}^{n}\frac{(1+n)\ (2+n)\ (3+n)\ \binom{n}{k}^{.}}{(1-k+n)\ (2-k+n)\ (3-k+n)}\to\frac{1}{2}\left(-14-7\ n-n^2+16\ 2^{n.}\right),\right.$$

$$\sum_{k=0}^{n}\frac{(1+n)\ (2+n)\ \binom{n}{k}^{.}}{(1-k+n)\ (2-k+n)}\to -3-n+4\ 2^{n.},$$

$$\sum_{k=0}^{n}\frac{(1+n)\ \binom{n}{k}^{.}}{1-k+n}\to -1+2\ 2^{n.},$$

$$\left.\sum_{\iota_1=0}^{n}\binom{n}{\iota_1}^{.}\to 2^{n.}\right\}$$

Out[50]= $\{4\ (1+n)\ \mathrm{SUM}[n]-$
$2\ (2+n+2\ x)\ \mathrm{SUM}[1+n]+(3+2\ x)\ \mathrm{SUM}[2+n]-\mathrm{SUM}[3+n]==$
$4\ \left(2\ (1+n)\ (-1+2\ x)\ \mathrm{H}[n]+\left(7+3\ n+2\ x-4\ x^2\right)\ \mathrm{H}[1+n]\right)\ 2^{n.}\}$

A Series of Identities (S. Ahlgren)

PROBLEM: Find a closed form for

$$\sum_{j=0}^{n} (1 - a\ j\ H_j + a\ j\ H_{-j+n}) \binom{n}{j}^a, \ a \geq 1$$

$$\sum_{j=0}^{n} (1 - j\,\mathrm{H}_j + j\,\mathrm{H}_{n-j}) \binom{n}{j} = 1$$

$$\sum_{j=0}^{n} (1 - 2\,j\,\mathrm{H}_j + 2\,j\,\mathrm{H}_{n-j}) \binom{n}{j}^2 = 0$$

$$\sum_{j=0}^{n} (1 - 3\,j\,\mathrm{H}_j + 3\,j\,\mathrm{H}_{n-j}) \binom{n}{j}^3 = (-1)^n$$

$$\sum_{j=0}^{n} (1 - 4\,j\,\mathrm{H}_j + 4\,j\,\mathrm{H}_{n-j}) \binom{n}{j}^4 = (-1)^n \sum_{j=0}^{n} \binom{n}{j}^2 = (-1)^n \binom{2\,n}{n}$$

$$\sum_{j=0}^{n} (1 - 5\,j\,\mathrm{H}_j + 5\,j\,\mathrm{H}_{n-j}) \binom{n}{j}^5 = (-1)^n \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j}$$

- Case 4 -

Naive summation

In[51]:= $\mathbf{mySum4} = \sum_{\mathbf{j=0}}^{\mathbf{n}} (\mathbf{1 - 4\ j\ H_j + 4\ j\ H_{-j+n}}) \left(\left(\binom{\mathbf{n}}{\mathbf{j}}\right)\right)_{\mathbf{j}}^{\mathbf{4}}$;

In[52]:= **GenerateRecurrence[mySum4]**

Out[52]= $\{ -8\ (1+n)\ (1+2\ n)\ (3+4\ n)\ (5+4\ n)\ (129+193\ n+94\ n^2+15\ n^3)\ \mathrm{SUM}[n] - 4\ (7560+39369\ n+82597\ n^2+92434\ n^3+60256\ n^4+23024\ n^5+4792\ n^6+420\ n^7)\ \mathrm{SUM}[1+n] - 2\ (2+n)\ (1425+6187\ n+9949\ n^2+7891\ n^3+3314\ n^4+706\ n^5+60\ n^6)\ \mathrm{SUM}[2+n] + (2+n)^2\ (3+n)^2\ (15+50\ n+49\ n^2+15\ n^3)\ \mathrm{SUM}[3+n] == 0\}$

Creative summation

By

$$\sum_{j=0}^{n} j\,\mathrm{H}_{n-j}\binom{n}{j}^4 = \sum_{j=0}^{n}(n-j)\,\mathrm{H}_j\binom{n}{n-j}^4 = \sum_{j=0}^{n}(n-j)\,\mathrm{H}_j\binom{n}{j}^4,$$

we obtain

$$\sum_{j=0}^{n}(1-4\,j\,\mathrm{H}_j+4\,j\,\mathrm{H}_{n-j})\binom{n}{j}^4 = \sum_{j=0}^{n}(1-4\,(n-2\,j)\mathrm{H}_j)\binom{n}{j}^4.$$

In[53]:= $\mathbf{mySum4} = \sum_{\mathbf{j=0}}^{\mathbf{n}}(\mathbf{1} - \mathbf{4\ j\ H_j} + \mathbf{4}\ (-\mathbf{j} + \mathbf{n})\ \mathbf{H_j})\ ((\binom{\mathbf{n}}{\mathbf{j}}^{\mathbf{4}}))_{\mathbf{j}}$;

In[54]:= **GenerateRecurrence[mySum4]**

```
Order:  1
Order:  2
Solution!
```

Out[54]= {4 (1 + 2 n)^2 (11 + 8 n) SUM[n] + 2 (29 + 110 n + 108 n^2 + 32 n^3) SUM[1 + n] + (2 + n)^2 (3 + 8 n) SUM[2 + n] == 0}

A Recurrence with minimal order

In[55]:= **rec = GenerateRecurrence[mySum4, SimplifyByExt → DepthNumber]**

```
Order:  1
Solution!
```

Out[55]= $\{2\ (1+\mathtt{n})\ (1+2\ \mathtt{n})\ \mathtt{SUM}[\mathtt{n}]+(1+\mathtt{n})^2\ \mathtt{SUM}[1+\mathtt{n}]==$

$$2\left(\mathtt{n}\ (3+8\ \mathtt{n})+(-3-8\ \mathtt{n})\ \sum_{\iota_1=0}^{\mathtt{n}}\frac{(2+\mathtt{n}-2\ \iota_1)\ \iota_1^4\ (((\binom{\mathtt{n}}{\iota_1})^{.\ 4})_{\iota_1.})}{(1+\mathtt{n}-\iota_1)^4}\right)\}$$

The rhs vanishes (computer proof!), hence

$$2\ (1+n)\ (1+2\ n)\ \mathbf{SUM}[n]+(1+n)^2\ \mathbf{SUM}[1+n]=0.$$

Remember:

$$\sum_{j=0}^{n}(1-4\,j\,\mathrm{H}_j+4\,j\,\mathrm{H}_{n-j})\binom{n}{j}^4=(-1)^n\binom{2\,n}{n}$$

Challenging Sums (A. Weideman)

Case 3

In[56]:= $\mathbf{Sum3} = \sum_{\mathbf{k=0}}^{\mathbf{m}} \left(\mathbf{3}\ (-\mathbf{H_k} + \mathbf{H_{-k+m}})^{\mathbf{2}} + \mathbf{H_k^{(2)}} + \mathbf{H_{-k+m}^{(2)}}\right) \left(\left(\binom{\mathbf{m}}{\mathbf{k}}\right)^{\mathbf{3}} (-\mathbf{1})^{\mathbf{k}}\right)_{\mathbf{k}};$

In[57]:= $\mathbf{GenerateRecurrence[Sum3a, RecOrder -> 2]}$

```
50.42 Second
```

Out[57]= $\{3\ (2+3\ m)\ (4+3\ m)\ \mathrm{SUM}[m] + (2+m)^2\ \mathrm{SUM}[2+m] == 0\}$

$$\sum_{k=0}^{2m} (-1)^k \binom{2m}{k}^3 \left[3(H_{2m-k} - H_k)^2 + H_{2m-k}^{(2)} + H_k^{(2)}\right] = 0$$

Case 4

In[58]:= $\mathbf{Sum4} = \sum_{\mathbf{k=0}}^{\mathbf{m}} \Big(\mathbf{3}\ \big(\mathbf{4}\ (\mathbf{H_k} - \mathbf{H_{-k+m}})^{\mathbf{2}} + \mathbf{H_k^{(2)}} + \mathbf{H_{-k+m}^{(2)}}\big) + \mathbf{2\ k}\ \big(\mathbf{8}\ (-\mathbf{H_k} + \mathbf{H_{-k+m}})^{\mathbf{3}} + \mathbf{6}\ (-\mathbf{H_k} + \mathbf{H_{-k+m}})\ \big(\mathbf{H_k^{(2)}} + \mathbf{H_{-k+m}^{(2)}}\big) - \mathbf{H_k^{(3)}} + \mathbf{H_{-k+m}^{(3)}}\big)\Big) \left(\left(\binom{\mathbf{m}}{\mathbf{k}}\right)^{\mathbf{4}}\right)_{\mathbf{k}};$

In[59]:= $\mathbf{GenerateRecurrence[Sum4, RecOrder -> 3]}$

```
1190.07 Second
```

Out[59]= $\{-8\ (1+m)\ (1+2\ m)\ (3+4\ m)\ (5+4\ m)\ (129+193\ m+94\ m^2+15\ m^3)\ \mathrm{SUM}[m] - 4\ (7560+39369\ m+82597\ m^2+92434\ m^3+60256\ m^4+23024\ m^5+4792\ m^6+420\ m^7)\ \mathrm{SUM}[1+m] - 2\ (2+m)\ (1425+6187\ m+9949\ m^2+7891\ m^3+3314\ m^4+706\ m^5+60\ m^6)\ \mathrm{SUM}[2+m] + (2+m)^2\ (3+m)^2\ (15+50\ m+49\ m^2+15\ m^3)\ \mathrm{SUM}[3+m] == 0\}$

Case 5 (the simplest of 4)

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}^5 \Big[125\big(H_k - H_{m-k}^{(1)}\big)^4 + 150\big(H_k - H_{m-k}^{(1)}\big)^2\big(H_k^{(2)} + H_{m-k}^{(2)}\big) + 15\big(H_k^{(2)} + H_{m-k}^{(2)}\big)^2 + 40\big(H_k - H_{m-k}\big)\big(H_k^{(3)} - H_{m-k}^{(3)}\big) + 6H_k^{(4)} + 6H_{m-k}^{(4)}\Big] = 0$$

Discovery of $$\boxed{\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\Big[3(H_{2m-k}-H_k)^2+H_{2m-k}^{(2)}+H_k^{(2)}\Big]}$$

Sigma finds: $$S_n^{(1)}:=\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^kH_k^{(2)}=\frac{(3m)!(-1)^m}{2m!m!m!}\Big[H_m^{(2)}+H_{2m}^{(2)}\Big]$$

$$S_n^{(2)}:=\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^kH_k^2=\frac{(3m)!(-1)^m}{m!m!m!}\frac{1}{12}\Big(3H_m^2+12H_{2m}(H_{2m}+H_m-H_{3m})-6H_mH_{3m}+3H_{3m}^2-H_m^{(2)}+2H_{2m}^{(2)}-3H_{3m}^{(2)}\Big)$$

$$S_n^{(3)}:=\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^kH_kH_{2m-k}=\frac{(3m)!(-1)^m}{m!m!m!}\frac{1}{12}\Big(3H_m^2+12H_{2m}(H_{2m}+H_m-H_{3m})-6H_mH_{3m}+3H_{3m}^2+H_m^{(2)}+4H_{2m}^{(2)}-3H_{3m}^{(2)}\Big)$$

The right combination delivers:

$$0=3\,S_n^{(3)}-3\,S_n^{(2)}+S_n^{(1)}=\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^k\Big[3H_k^2-3H_kH_{2m-k}+H_k^{(2)}\Big]$$

Applying symmetries we obtain

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}^3\Big[3(H_{2m-k}-H_k)^2+H_{2m-k}^{(2)}+H_k^{(2)}\Big]=\sum_{k=0}^{2m}\binom{2m}{k}^3(-1)^k\Big[3H_k^2-3H_kH_{2m-k}+H_k^{(2)}\Big]=0$$

By symmetries it follows that

$$\sum_{k=0}^{2m-1}(-1)^k\binom{2m-1}{k}^3\Big[3(H_{2m-1-k}-H_k)^2+H_{2m-1-k}^{(2)}+H_k^{(2)}\Big]=0$$