Padé Approximation, Multisums, and Theorema

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Padé approximation to the logarithm *A. Weideman*

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$\Updownarrow$ *K. Driver, H. Prodinger*

Proving Multisum Identities

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Proving Multisum Identities

$\Updownarrow$ *C. Schneider*

Sigma and Theorema

Padé Approximation to $\log(x)$ at $x = 1$

FIND

$$r_m(x) = \sum_{k=0}^{m} a_k x^k, \qquad s_m(x) = \sum_{k=0}^{m} b_k x^k$$

s.t.

$$\boxed{\frac{s_m(x)}{r_m(x)} \equiv log(x) \mod (x-1)^{2m+1}}$$

"Approximate $\log(x)$ around 1 with a rational function"

Padé Approximation to $\log(x)$ **at** $x = 1$

FIND

$$r_m(x) = \sum_{k=0}^{m} a_k x^k, \qquad s_m(x) = \sum_{k=0}^{m} b_k x^k$$

s.t.

$$-\frac{s_m(x)}{r_m(x)} \equiv log(x) \mod (x-1)^{2m+1}$$

$$\Updownarrow$$

$$\boxed{r_m(x)\,\log(x) + s_m(x) \equiv 0 \mod (x-1)^{2m+1}}$$

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$$\Updownarrow$$

$$\boxed{r_m(x)\,\log(x) + s_m(x) = O((x-1)^{2m+1})}$$

- Linear Padé

FIND $r_m(x) = \sum_{k=0}^{m} a_k x^k, s_m(x) = \sum_{k=0}^{m} b_k x^k$:

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- Quadratic Padé

FIND $r_m(x) = \sum_{k=0}^{m} a_k x^k,\ s_m(x) = \sum_{k=0}^{m} b_k x^k,\ t_m(x) = \sum_{k=0}^{m} c_k x^k$:

$$\boxed{r_m(x)\,(\log x)^2 + s_m(x)\ \log(x) + t_m(x) = O((x-1)^{3m+2})}$$

- Higher Order Padé ($n \geq 1$)

FIND $r_m(x) = \sum_{k=0}^{m} a_k x^k,\ s_m(x) = \sum_{k=0}^{m} b_k x^k, \ldots, t_m(x) = \sum_{k=0}^{m} c_k x^k$:

$$\boxed{r_m(x)\,(\log x)^n + s_m(x)\,(\log x)^{n-1} + \cdots + t_m(x) = O((x-1)^{(n+1)(m+1)-1})}$$

Quadratic Padé

Note: $r_m(x), s_m(x), t_m(x)$ with

$$\boxed{r_m(x)\,(\log x)^2 + s_m(x)\,\log(x) + t_m(x) = O((x-1)^{3m+2})}$$

are uniquely defined (up to a constant factor).

DEFINE residual

$$\boxed{R_m(x) := r_m(x)\,(\log x)^2 + s_m(x)\,\log(x) + t_m(x)}$$

Quadratic Padé

Note: $r_m(x), s_m(x), t_m(x)$ with

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are uniquely defined (up to a constant factor).

DEFINE

$$\boxed{R_m(x) := r_m(x)\,(\log x)^2 + s_m(x)\,\log(x) + t_m(x)}$$

THEN $R_m(x)$ is a solution of

$$\boxed{\left[x(\delta - m)^3 - \delta^3\right] y(x) = 0}$$

for the operator $\delta := x\,\frac{d}{dx}$

$$\boxed{\left[x(\delta - m)^3 - \delta^3\right] y(x) = 0}$$

$$\downarrow$$

Frobenius' method

$$\downarrow$$

General solution:

$$\boxed{y(x) = c_1\, y_1(x) + c_2\, y_2(x) + c_3\, y_3(x)}$$

with

$$y_1(x) = \sum_{k=0}^{m} \binom{m}{k}^3 (-x)^k, \qquad y_2(x) = y_1(x)\log(x) + \sum_{k=0}^{m} \left[\frac{d}{dk}\binom{m}{k}^3\right](-x)^k,$$

$$y_3(x) = y_1(x)\log^2(x) + 2\log(x)\sum_{k=0}^{m} \left[\frac{d}{dk}\binom{m}{k}^3\right](-x)^k + \sum_{k=0}^{m} \left[\frac{d^2}{dk^2}\binom{m}{k}^3\right](-x)^k.$$

Hence

$$\begin{aligned} R_m(x) &= \boxed{r_m(x)}\,(\log x)^2 + \boxed{s_m(x)}\,\log(x) + \boxed{t_m(x)} \\ &= c_1\,y_1(x) + c_2\,y_2(x) + c_3\,y_3(x) \end{aligned}$$

with

$$R_m(1) = 0, \quad R'_m(1) = 0.$$

Hence

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with

$$R_m(1) = 0, \quad R'_m(1) = 0.$$

Computer experiments:

$$c_1 = \pi^2, c_2 = 0, c_3 = 1$$

$$\begin{aligned} R_m(x) = {} & \boxed{\sum_{k=0}^{m} \binom{m}{k}^3 (-x)^k}\,(\log x)^2 + \boxed{2\sum_{k=0}^{m} \Big[\frac{d}{dk}\binom{m}{k}^3\Big](-x)^k}\,\log(x) \\ & + \boxed{\sum_{k=0}^{m} \Big[\frac{d^2}{dk^2}\binom{m}{k}^3 + \pi^2\binom{m}{k}^3\Big](-x)^k} \end{aligned}$$

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$$\Updownarrow \; A.\ Weideman$$

$$\sum_{k=0}^{m} (-1)^k \left(\frac{d^2}{dk^2} + \pi^2 \right) \left[k^\ell \binom{m}{k}^3 \right] = 0, \qquad \ell = 0, 1$$

Computer experiments

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$$\Updownarrow \; H.\ Prodinger$$

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)}\right] = 0,$$

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}^3 \left[k(3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)}) + 2(H_{m-k} - H_k)\right] = 0$$

where

$$H_k = \sum_{i=1}^{k} \frac{1}{i}, \quad H_k^{(2)} = \sum_{i=1}^{k} \frac{1}{i^2}$$

Z's Creative Telescoping Trick

- GIVEN

$$\mathrm{SUM}(m) := \sum_{k=0}^{m} \underbrace{(-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)}\right]}_{=:\, f(m,k)}$$

- FIND $c_0(m)$, $c_1(m)$, $c_2(m)$, and $g(m,k)$ s.t.

$$\boxed{g(m,k+1) - g(m,k)} = \boxed{c_0(m)\, f(m,k) + c_1(m)\, f(m+1,k) + c_2(m)\, f(m+2,k)}$$

for all $0 \le k \le m$ and all $m \ge 0$

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for all $0 \le k \le m$ and all $m \ge 0$

Sigma computes:

$$c_0(m) := 3(3m+2)(3m+4)(3m+8),\;\; c_1(m) := 0,\;\; c_2(m) := (m+2)^2(3m+8)$$

$$g(m,k) := (-1)^k \binom{m}{k}^3 \frac{p_1(k, m, H_k, H^{(2)}_k, H_{m-k}, H^{(2)}_{m-k})}{(m-k+1)^5(m-k+2)^5}$$

$$g(m,k+1) := (-1)^k \binom{m}{k}^3 \frac{p_2(k, m, H_k, H^{(2)}_k, H_{m-k}, H^{(2)}_{m-k})}{(m-k+1)^5}$$

Z's Creative Telescoping Trick

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- GIVEN $c_0(m)$, $c_1(m)$, $c_2(m)$, and $g(m,k)$ s.t.

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for all $0 \le k \le m$ and all $m \ge 0$

Summing this equation over k from 0 to m gives:

$$\boxed{g(m,m+1) - g(m,0)} = \boxed{\begin{array}{l} c_0(m)\,\mathrm{SUM}(m) + \\ c_1(m)\left[\mathrm{SUM}(m+1) - f(m+1,m+1)\right] \\ c_2(m)\left[\mathrm{SUM}(m+2) - f(m+2,m+1) - f(m+2,m+2)\right] \end{array}}$$

- Linear Padé

$$r_m(x)\,\log(x) + s_m(x) = O((x-1)^{2m+1})$$

$$\sum_{k=0}^{m} \binom{m}{k}^2 \left[1 + 2k(\mathrm{H}_{m-k} - \mathrm{H}_k)\right] = 0$$

- Quadratic Padé

$$r_m(x)\,(\log x)^2 + s_m(x)\,\log(x) + t_m(x) = O((x-1)^{3m+2})$$

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k}^3 \left[3(H_{m-k} - H_k)^2 + H^{(2)}_{m-k} + H^{(2)}_k\right] = 0,$$

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k}^3 \left[k(3(H_{m-k} - H_k)^2 + H^{(2)}_{m-k} + H^{(2)}_k) + 2(H_{m-k} - H_k)\right] = 0$$

- Cubic Padé

$$r_m(x)\,(\log x)^3 + s_m(x)\,(\log x)^2 + t_m(x)\,\log(x) + u_m(x) = O((x-1)^{4m+3})$$

$$\sum_{k=0}^{m} \binom{m}{k}^4 \Big[3(H_{m-k} - H_k)^2 + H_{m-k}^{(2)} + H_k^{(2)} + 4k(H_{m-k} - H_k)^3 + 6(H_{m-k} - H_k)(H_{m-k}^{(2)} + H_k^{(2)}) + H_{m-k}^{(3)} - H_k^{(3)}\Big] = 0$$

- Padé of order 4

$$r_m(x)\,(\log x)^4 + s_m(x)\,(\log x)^3 + t_m(x)\,(\log x)^2 + u_m(x)\,\log(x) + v_m = O((x-1)^{5m+4})$$

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k}^5 \Big[125\big(H_k - H_{m-k}^{(1)}\big)^4 + 150\big(H_k - H_{m-k}^{(1)}\big)^2\big(H_k^{(2)} + H_{m-k}^{(2)}\big) + 15\big(H_k^{(2)} + H_{m-k}^{(2)}\big)^2 + 40\big(H_k - H_{m-k}\big)\big(H_k^{(3)} - H_{m-k}^{(3)}\big) + 6H_k^{(4)} + 6H_{m-k}^{(4)}\Big] = 0$$

$$
\begin{aligned}
\sum_{k=0}^{m}(-1)^k\binom{m}{k}^5\Big[& -60\big(H_k-H_{m-k}\big)\big(H_k^{(2)}+H_{m-k}^{(2)}\big)+4\big(25\big(-H_k+H_{m-k}\big)^3-2H_k^{(3)}+2H_{m-k}^{(3)}\big)\\
& +5k\big(-H_k+H_{m-k}\big)\big(25\big(-H_k+H_{m-k}\big)^3-8H_k^{(3)}+8H_{m-k}^{(3)}\big)\\
& +3k\big(5\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big(10\big(H_k-H_{m-k}\big)^2+H_k^{(2)}+H_{m-k}^{(2)}\big)+2\big(H_k^{(4)}+H_{m-k}^{(4)}\big)\big)\Big]=0
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{m}(-1)^k\binom{m}{k}^5\Big[& 125\big(-1+k\big)kH_k^4+100\big(-1+2k\big)H_{m-k}^3+125\big(-1+k\big)kH_{m-k}^4+100H_k^3\big(1-2k\\
& -5\big(-1+k\big)kH_{m-k}\big)+30H_{m-k}^2\big(2+5\big(-1+k\big)k\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big)+30H_k^2\big(2+5\big(\big(-2+4k\big)H_{m-k}\\
& +5\big(-1+k\big)kH_{m-k}^2+\big(-1+k\big)k\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big)\big)+20H_k\big(\big(15-30k\big)H_{m-k}^2-25\big(-1+k\big)kH_{m-k}^3\\
& +\big(3-6k\big)H_k^{(2)}+3H_{m-k}^{(2)}-3H_{m-k}\big(2+5\big(-1+k\big)k\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big)+2k\big(-3H_{m-k}^{(2)}+\big(-1+k\big)\big(H_k^{(3)}\\
& -H_{m-k}^{(3)}\big)\big)\big)+20H_{m-k}\big(\big(-3+6k\big)H_k^{(2)}+\big(-3+6k\big)H_{m-k}^{(2)}-2\big(-1+k\big)k\big(H_k^{(3)}-H_{m-k}^{(3)}\big)\big)\\
& +4\big(3H_k^{(2)}+3H_{m-k}^{(2)}+2H_k^{(3)}-2H_{m-k}^{(3)}\big)+k\big(15\big(-1+k\big){H_k^{(2)}}^2+30\big(-1+k\big)H_k^{(2)}H_{m-k}^{(2)}\\
& +15\big(-1+k\big){H_{m-k}^{(2)}}^2+2\big(-8H_k^{(3)}+8H_{m-k}^{(3)}+3\big(-1+k\big)\big(H_k^{(4)}+H_{m-k}^{(4)}\big)\big)\big)\Big]=0
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{m}(-1)^k\binom{m}{k}^5\Big[& 125\big(-2+k\big)\big(-1+k\big)kH_k^4+100\big(2+3\big(-2+k\big)k\big)H_{m-k}^3+125\big(-2+k\big)\big(-1+k\big)kH_{m-k}^4+\\
& 100H_k^3\big(-2-3\big(-2+k\big)k-5\big(-2+k\big)\big(-1+k\big)kH_{m-k}\big)+30\big(-1+k\big)H_{m-k}^2\big(6+5\big(-2+k\big)k\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big)\\
& +30H_k^2\big(10\big(2+3\big(-2+k\big)k\big)H_{m-k}+25\big(-2+k\big)\big(-1+k\big)kH_{m-k}^2+\big(-1+k\big)\big(6+5\big(-2+k\big)k\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big)\big)\\
& +4H_{m-k}\big(6+15\big(2+3\big(-2+k\big)k\big)H_k^{(2)}+5\big(3\big(2+3\big(-2+k\big)k\big)H_{m-k}^{(2)}-2\big(-2+k\big)\big(-1+k\big)k\big(H_k^{(3)}-H_{m-k}^{(3)}\big)\big)\big)\\
& -4\big(9H_{m-k}^{(2)}+4H_k^{(3)}-4H_{m-k}^{(3)}\big)+4H_k\big(-6\big(1+5H_k^{(2)}+5H_{m-k}^{(2)}\big)-5\big(3+5\big(-1+k\big)H_{m-k}\big)\big(6\big(-1+k\big)H_{m-k}\\
& +5\big(-2+k\big)kH_{m-k}^2+3\big(-2+k\big)k\big(H_k^{(2)}+H_{m-k}^{(2)}\big)\big)+10\big(-2+k\big)\big(-1+k\big)kH_k^{(3)}-10\big(-2+k\big)\big(-1+k\big)kH_{m-k}^{(3)}\big)\\
& +3\big(5\big(-2+k\big)\big(-1+k\big)k{H_k^{(2)}}^2+2\big(-1+k\big)H_k^{(2)}\big(6+5\big(-2+k\big)kH_{m-k}^{(2)}\big)+k\big(12H_{m-k}^{(2)}+5\big(-2+k\big)\big(-1+k\big){H_{m-k}^{(2)}}^2\\
& +2\big(-2+k\big)\big(-4H_k^{(3)}+4H_{m-k}^{(3)}+\big(-1+k\big)\big(H_k^{(4)}+H_{m-k}^{(4)}\big)\big)\big)\big)\Big]=0
\end{aligned}
$$

Ahlgren's Identities (P. Paule, C.S.)

$$\sum_{j=0}^{m}(1 - 1\,j\,\mathrm{H}_j + 1\,j\,\mathrm{H}_{m-j})\binom{m}{j} = 1$$

$$\boxed{\sum_{j=0}^{m}(1 - 2\,j\,\mathrm{H}_j + 2\,j\,\mathrm{H}_{m-j})\binom{m}{j}^2} = 0$$

$$\sum_{j=0}^{m}(1 - 3\,j\,\mathrm{H}_j + 3\,j\,\mathrm{H}_{m-j})\binom{m}{j}^3 = (-1)^m$$

$$\sum_{j=0}^{m}(1 - 4\,j\,\mathrm{H}_j + 4\,j\,\mathrm{H}_{m-j})\binom{m}{j}^4 = (-1)^m \sum_{j=0}^{m}\binom{m}{j}^2 = (-1)^m\binom{2\,m}{m}$$

$$\sum_{j=0}^{m}(1 - 5\,j\,\mathrm{H}_j + 5\,j\,\mathrm{H}_{m-j})\binom{m}{j}^5 = (-1)^m \sum_{j=0}^{m}\binom{m}{j}^2\binom{m+j}{j}$$

Pattern:

$$\sum_{j=0}^{m}(1 - a\ j\ H_j + a\ j\ H_{-j+m})\binom{m}{j}^a,\ a \geq 1$$