IMA 2002 Summer Program
Special Functions in the Digital Age

Sigma: A Summation Package for Discovering and Proving

Carsten.Schneider@risc.uni-linz.ac.at

An Identity from Physics (Essam, Guttmann) - Case 5 -

We eliminate the sum quantifiers in

In[1]:= **mySum** =

$$\sum_{k1=0}^{n}\sum_{k2=0}^{k1}\sum_{k3=0}^{k2}\sum_{k4=0}^{k3}\sum_{k5=0}^{k4}(k1-k2)(k1-k3)(k2-k3)(k1-k4)(k2-k4)(k3-k4)(k1-k5)(k2-k5)(k3-k5)(k4-k5)\binom{n}{k1}\binom{n}{k2}\binom{n}{k3}\binom{n}{k4}\binom{n}{k5}$$

by using the two sums

In[2]:= $\textbf{tower} = \{\sum_{k=0}^{a}\left(\binom{n}{k}^{2}\right), \sum_{k=0}^{a}\binom{n}{k}\};$

We get:

In[3]:= $\textbf{result} = \textbf{SigmaReduce}[\textbf{mySum}, \textbf{Tower} \to \textbf{tower}]$

Out[3]= $$\frac{3\,(-3+n)\,(-2+n)^2(-1+n)^3\,n^5\left(\sum_{\iota_1=0}^{n}\binom{n}{\iota_1}\right)\left(\sum_{\iota_1=0}^{n}\left(\binom{n}{\iota_1}^{2}\right)\right)^2}{256\,(-5+2\,n)\,\left(3-8\,n+4\,n^2\right)^2}$$

By the substitution

In[4]:= $\textbf{subst} = \{\sum_{\iota_1=0}^{n}\binom{n}{\iota_1} \to (2)^{n}, \sum_{\iota_1=0}^{n}\left(\binom{n}{\iota_1}^{2}\right) \to \binom{2\,n}{n}\};$

we obtain the final result:

In[5]:= **result/.subst**

Out[5]= $$\frac{3\,(-3+n)\,(-2+n)^2\,(-1+n)^3\,n^5\,\left(\binom{2\,n}{n}\right)^2\,(2)^{n}}{256\,(-5+2\,n)\,(3-8\,n+4\,n^2)^2}$$

Indefinite Summation in Difference Field

Goal: Find a closed form for

$$\sum_{k=0}^{n} k\,k!$$

A Difference Field for the Problem

Let t_1, t_2 be indeterminates where

$$\begin{aligned} t_1 &\longleftrightarrow k \\ t_2 &\longleftrightarrow k! \end{aligned}$$

Consider the **field automorphism** $\sigma : \mathbb{Q}(t_1, t_2) \to \mathbb{Q}(t_1, t_2)$ canonically defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q} & & \\ \sigma(t_1) &= t_1 + 1 & \qquad \mathrm{S}\,k &= k+1 \\ \sigma(t_2) &= (t_1+1)t_2 & \qquad \mathrm{S}\,k! &= (k+1)! \end{aligned}$$

$(\mathbb{Q}(t_1, t_2), \sigma)$ is our difference field.

The Telescoping Problem

Find $g \in \mathbb{Q}(t_1, t_2)$: $\boxed{\sigma(g) - g = t_1\, t_2}$

$\downarrow$ by Karr

$g = t_2.$

The Closed Form

$$\boxed{(k+1)! - k! = k\,k!}$$

$$\downarrow$$

$$\sum_{k=0}^{n} k\,k! = (n+1)! - 1.$$

From Ramanujan's Notebooks, Bruce C. Berndt

In[6]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{n}} \frac{\binom{\mathbf{n}}{\mathbf{k}} (-\mathbf{1})^{\mathbf{k}}}{(\mathbf{1}+\mathbf{2}\ \mathbf{k})^{\mathbf{2}}}$;

Finding a recurrence

In[7]:= **rec = GenerateRecurrence[mySum]**

Out[7]= $\{-4\ (1+n)\ (2+n)\ \mathtt{SUM}[n] + 8\ (2+n)^2\ \mathtt{SUM}[1+n] - (5+2\ n)^2\ \mathtt{SUM}[2+n] == 0\}$

Solving the recurrence

In[8]:= **tower = FindProductExtensions[rec[[1]], SUM[n]]**

```
I use M. Petkovsek's package Hyper to find product extensions!
```

Out[8]= $\{\prod_{i=1}^{n} \left(\frac{2\ i}{1+2\ i}\right)\}$

In[9]:= **recSol = SolveRecurrence[rec[[1]], SUM[n], Tower− > tower, NestedSumExt− > ∞]**

Out[9]= $\{\{0, \prod_{\iota_1=1}^{n} \left(\frac{2\ \iota_1}{1+2\ \iota_1}\right)\}, \{0, \left(\prod_{\iota_1=1}^{n} \left(\frac{2\ \iota_1}{1+2\ \iota_1}\right)\right) \sum_{\iota_1=0}^{n} \frac{1}{1+2\ \iota_1}\}, \{1, 0\}\}$

Finding the linear combination

In[10]:= **solution = FindLinearCombination[recSol, mySum, 2]**

Out[10]= $\left(\prod_{\iota_1=1}^{n} \left(\frac{2\ \iota_1}{1+2\ \iota_1}\right)\right) \sum_{\iota_1=0}^{n} \frac{1}{1+2\ \iota_1}$

Rewriting in standard objects

In[11]:= **SigmaReduce[solution, n, Tower− > {2^n, n!, (1 + 2 n)!}]**

Out[11]= $\frac{n!^2\ (2^n)^2\ \sum_{\iota_1=0}^{n} \frac{1}{1+2\ \iota_1}}{(1+2\ n)!}$

Calkin's Identity and Variations

Case 1:

$$\sum_{k=0}^{a} x^k \sum_{j=0}^{k} \binom{n}{j} y^j = \frac{x^{a+1} \sum_{j=0}^{a} \binom{n}{j} y^j - \sum_{j=0}^{a} \binom{n}{j} x^j y^j}{x-1}$$

specializes to:

$$\sum_{k=0}^{n} x^k \sum_{j=0}^{k} y^k \binom{n}{j} = \frac{x^{n+1} (1+y)^n - (1+x\,y)^n}{x-1}$$

Case 2, non-alternating:

$$\sum_{k=0}^{a} \left(\sum_{j=0}^{k} \binom{n}{j} \right)^2 = (n-a) \binom{n}{a} \sum_{j=0}^{a} \binom{n}{j} + (1+a-\frac{n}{2}) \left(\sum_{j=0}^{a} \binom{n}{j} \right)^2 - \frac{n}{2}, \sum_{j=0}^{a} \binom{n}{j}^2$$

specializes to:

$$\sum_{k=0}^{n} \left(\sum_{j=0}^{k} \binom{n}{j} \right)^2 = (n+1)\, 4^n - \frac{n}{2}\, 4^n - \frac{n}{2} \binom{2\,n}{n}$$

Case 2, alternating:

$$\sum_{k=0}^{a}(-1)^k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2=2\,(n-a)\binom{n}{a}(-1)^a\sum_{j=0}^{a}\binom{n}{j}+n\,(-1)^a\left(\sum_{j=0}^{a}\binom{n}{j}\right)^2-\sum_{j=0}^{a}(n-2\,j)\binom{n}{j}^2(-1)^j$$

specializes to:

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2=\begin{cases}0 & \text{if } n \text{ is even}\\ -(-1)^{\frac{n-1}{2}}\,n\binom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd}\end{cases}$$

Case 2 for even n, interlaced alternating:

$$\sum_{k=0}^{2\,n}\left(\sum_{j=0}^{k}(-1)^{\frac{1}{2}(j-1)\,j}\binom{2\,n}{j}\right)^2=\frac{2^{2\,n}}{4}\left(4+6\,n-4\,n\,(-1)^n+3\,n\sum_{j=2}^{n}\frac{\binom{4\,j}{2\,j}}{(4\,j-3)\,2^{2\,j}}+3\,n\sum_{j=2}^{n}\frac{\binom{4\,j}{2\,j}}{(4\,j-1)\,2^{2\,j}}\right)$$

Case 3, Calkin's identity:

$$\boxed{\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3=\frac{n}{2}\,8^n+8^n-\frac{3\,n}{4}\,2^n\binom{2\,n}{n}}$$

Case 3 for even n, alternating:

$$\sum_{k=0}^{2\,n}(-1)^k\left(\sum_{j=0}^{k}\binom{2\,n}{j}\right)^3=\frac{64^n}{2}-\frac{(-1)^n\,64^n}{16\,n\,\binom{2\,n}{n}}\sum_{i=0}^{n-1}(3+11\,i)\binom{2\,i}{i}^2\binom{3\,i}{i}64^{-i}$$

Difference Equations and Symbolic Summation

Let $(\mathbb{F}, \sigma)$ be a difference field and

$$\mathbb{K} = \{k \in \mathbb{F} \,|\, \sigma(k) = k\}$$

be the constant field.

Telescoping

- GIVEN $f \in \mathbb{F}$
- FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

Parameterized Telescoping

- GIVEN $f_0, \ldots, f_d \in \mathbb{F}$, $a_0, a_1 \in \mathbb{F}$
- FIND ALL $c_0, \ldots, c_d \in \mathbb{K}$, $h \in \mathbb{F}$:

$$\boxed{a_1\,\sigma(h) - a_0\,h = c_0\,f_0 + \cdots + c_d\,f_d}$$

Remark: Z's "Creative Telescoping"

- GIVEN $f_i = \text{summand}(n+i, k) \in \mathbb{F}$
- FIND ALL $c_0, \ldots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0\,f_0 + \cdots + c_d\,f_d}$$

Linear Difference Equations

- GIVEN $f, a_0, \ldots, a_m \in \mathbb{F}$
- FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

Parameterized Linear Difference Equations

- GIVEN $a_0, \ldots, a_m \in \mathbb{F}$,$f_0, \ldots, f_d \in \mathbb{F}$.
- FIND ALL $g \in \mathbb{F}$, $c_0, \ldots, c_d \in \mathbb{K}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = c_0\,f_0 + \cdots + c_d\,f_d}$$