RWCA'02

# How One Can Play With Sums

## Symbolic Summation in Difference Fields

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# 1 A Bonus Problem in "Concrete Mathematics"

Chapter 6. Special Numbers, Bonus problem 69:

Find a closed form for

$$\sum_{k=1}^{n} k^2 \mathrm{H}_{n+k},$$

where $\mathrm{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

Knuth's answer to the problem is

$$\frac{1}{3}n\left(n+\frac{1}{2}\right)(n+1)\left(2\mathrm{H}_{2n}-\mathrm{H}_n\right)-\frac{1}{36}n\left(10n^2+9n-1\right)$$

with the remark

> *"It would be nice to automate the derivation of formulas such as this."*

In[1]:= << **Sigma`**

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Sigma -A summation package by Carsten Schneider
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In[2]:= **Problem69 = SigmaSum[k^2 SigmaHNumber[n + k], {k, 1, n}]**

Out[2]= $\sum_{k=1}^{n}\left(k^2\ H_{k+n}\right)$

In[3]:= **SigmaReduce[Problem69]//Simplify**

Out[3]= $-\frac{1}{36}\ n\ (1+n)\ (-1+10\ n+6\ (1+2\ n)\ H_n-12\ (1+2\ n)\ H_{2n})$

- Streamlining and generalizations of Karr's ideas (by Bronstein's denominator bounding)
- New aspects in symbolic summation

## 2 A Strategy to Handle Definite Sums

In[4]:= $\mathbf{mySum} = \sum_{i=0}^{2n} \left( \frac{(-1+i)\, i^3 \left(\binom{2n}{i}\right)^3 (-1)^i}{(1-i+2n)^3} \right)$

### Finding a recurrence

In[5]:= **rec = GenerateRecurrence[mySum]**

Out[5]= $\{18\,(1+3n)\,(2+3n)\,(1+6n)\,(5+6n)\,(13+18n+6n^2)\,\mathrm{SUM}[n] + 12\,(90+814n+2543n^2+3864n^3+3126n^4+1296n^5+216n^6)\,\mathrm{SUM}[1+n] + 2\,(1+n)\,(2+n)\,(3+2n)^2\,(1+6n+6n^2)\,\mathrm{SUM}[2+n] == 8\,(288+3465n+18272n^2+55648n^3+98548n^4+97016n^5+48384n^6+9408n^7)\}$

### Solving the recurrence (first attempt)

In[6]:= **SolveRecurrence[rec[[1]], SUM[n]]**

Out[6]= $\{\{1, 2n\}\}$

### Second attempt: product extensions

In[7]:= **<< Hyper`;**
In[8]:= **FindProductExtensions[rec, SUM[n], Solutions → All]**

    I use M. Petkovšek's package Hyper to find product extensions.

Out[8]= $\{\prod_{i=2}^{n}\left(-\frac{3\,(-2+3i)\,(-1+3i)}{(-1+i)\,i}\right), \prod_{i=1}^{n}\left(-\frac{3\,(-5+6i)\,(-1+6i)}{(-1+2i)^2}\right)\}$

In[9]:= **recSol = SolveRecurrence[rec, SUM[n], Tower →** $\{\left(\frac{(3n)!\,(-1)^n}{((n)!)^3}\right)_n, \left(\frac{((n)!)^3\,(6n)!\,(-1)^n}{((2n)!)^3\,(3n)!}\right)_n\}$**]**

Out[9]= $\{\{0, n\left(\frac{(3n)!\,(-1)^n}{(n)!^3}\right)_n\}, \{0, \left(\frac{(n)!^3\,(6n)!\,(-1)^n}{(2n)!^3\,(3n)!}\right)_n\}, \{1, 2n\}\}$

### Finding the linear combination

In[10]:= **FindLinearCombination[recSol, mySum, 2]**

Out[10]= $2n - n\left(\frac{(3n)!\,(-1)^n}{(n)!^3}\right)_n$

## 3 How One Can Play with Sums

**Problem 1.** *Find a closed form for*

$$\sum_{k=0}^{2n} (-1)^k \, k \, H_k \binom{2n}{k}^3 = \quad \textbf{?}$$

*where* $H_k = \sum_{i=1}^{k} \frac{1}{i}$ *are the* $k$*-th harmonic numbers.*

In[11]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{2\ n}} \left( \mathbf{k\ H_k} \left( \left( \binom{\mathbf{2\ n}}{\mathbf{k}} \right)^{\mathbf{3}} (-\mathbf{1})^{\mathbf{k}} \right)_{\mathbf{k}} \right);$

# Creative telescoping

In[12]:= **rec = GenerateRecurrence[mySum, RecOrder → 4]**

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1841.48 Second
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Out[12]=
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{81 (1 + n) (10 + 117 n + 441 n^2 + 648 n^3 + 324 n^4)^2 (579023679111696+
        6203096595284292 n + 30574972749055508 n^2+
        92475481987210701 n^3 + 192864735750636284 n^4+
        295166120513347017 n^5 + 344113220933469194 n^6+
        312890401572444600 n^7 + 225181229898272112 n^8+
        129339961859979540 n^9 + 59474372437202472 n^10+
        21854565707771808 n^11 + 6372893337871680 n^12+
        1455288215784768 n^13 + 254598040577664 n^14+
        32934777209856 n^15 + 2967155877888 n^16+
        166161051648 n^17 + 4353564672 n^18) SUM[n]+
  108 ( - 8911086594732000+
        595686855250231800 n + 16380227867435099780 n^2+
        185672492904312930710 n^3 + 1271723758536088957353 n^4+
        6026151073985872712073 n^5 + 21197749937538020891079 n^6+
        57793321639546981142298 n^7 + 125693551925945528389705 n^8+
        222521457681141044963341 n^9+
        325368258856450491542511 n^10+
        397108616509050749048718 n^11+
        407622807225028518763356 n^12+
        353729663174629500044400 n^13+
        260330393614389288503220 n^14+
        162709980603775713128520 n^15+
        86335405854765454150272 n^16 + 38809363531072919958144 n^17+
        14720133478715210657664 n^18 + 4681642828665855843072 n^19+
        1237296059054356451328 n^20 + 268300933294762027008 n^21+
        46890597952821408768 n^22 + 6437495043769780224 n^23+
        668002856934260736 n^24 + 49220844925353984 n^25+
        2293562354761728 n^26 + 50779978334208 n^27) SUM[1 + n]+
  18 (2 + n) (3 + 2 n) ( - 8228295571986000 - 29467353203684820 n+
        1381518393267116428 n^2 + 19978139922191293573 n^3+
        139144387971971638219 n^4 + 625542630805627460455 n^5+
        2017285686440215860490 n^6 + 4933055970124372861135 n^7+
        9465689765373655917267 n^8 + 14579998008141370748253 n^9+
        18312629998410321364656 n^10 + 18961209332586432771048 n^11+
        16304139557770332127212 n^12 + 11695416700671314908740 n^13+
        7013537868185350191792 n^14 + 3515617464514069708512 n^15+
        1469465760759532649280 n^16 + 509652781805658910464 n^17+
        145518011266651170048 n^18 + 33806212169624059392 n^19+
        6282436535103246336 n^20 + 910948598145469440 n^21+
        99231835717287936 n^22 + 7633845045411840 n^23+
        369565397876736 n^24 + 8463329722368 n^25)
    SUM[2 + n] + 12 (2 + n) (3 + n)
    (3 + 2 n) (5 + 2 n) ( - 64001714143920 - 503422860673228 n+
        4002975025720952 n^2 + 79747990756043705 n^3+
        565678480977551301 n^4 + 2447100392628223047 n^5+
        7404218627394040182 n^6 + 16709317348234374364 n^7+
        29191436701822318447 n^8 + 40425384732611573230 n^9+
        45074461215631426464 n^10 + 40878463232569911732 n^11+
        30338483534960452020 n^12 + 18477110572629289128 n^13+
        9232514580951306000 n^14 + 3772738135947714336 n^15+
        1252587607610477760 n^16 + 334329670014178176 n^17+
        70597472266909440 n^18 + 11513259270314496 n^19+
        1397288190984192 n^20 + 118711550287872 n^21+
        6295254515712 n^22 + 156728328192 n^23) SUM[3 + n]+
  (2 + n) (3 + n)^2 (4 + n)^2 (3 + 2 n) (5 + 2 n)
    (7 + 2 n)^3 ( - 945554940 - 7607976456 n + 35254988575 n^2+
        756814949687 n^3 + 4816720182041 n^4 + 17947420546069 n^5+
        45372683784936 n^6 + 83005099177032 n^7 + 113701841575020 n^8+
        118788006388788 n^9 + 95405698339488 n^10+
        58876332512544 n^11 + 27669385543104 n^12+
        9716847158592 n^13 + 2466213765120 n^14 + 426750114816 n^15+
        44986834944 n^16 + 2176782336 n^17) SUM[4 + n] ==
 0}
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## Creative telescoping with sum extensions

In[13]:= **rec = GenerateRecurrence[mySum, SimplifyByExt → DepthNumber]**

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71.52 Second
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Out[13]= $\{-36\,(1+n)\,(2+n)\,(1+2\,n)\,(3+2\,n)\,(1+3\,n)\,(2+3\,n)\,(1+6\,n)\,(5+6\,n)\,(1+6\,n+6\,n^2)\,(13+18\,n+6\,n^2)\,\mathtt{SUM}[n]-24\,(1+n)\,(2+n)\,(1+2\,n)\,(3+2\,n)\,(1+6\,n+6\,n^2)\,(90+814\,n+2543\,n^2+3864\,n^3+3126\,n^4+1296\,n^5+216\,n^6)\,\mathtt{SUM}[1+n]-4\,(1+n)^2\,(2+n)^2\,(1+2\,n)\,(3+2\,n)^3\,(1+6\,n+6\,n^2)^2\,\mathtt{SUM}[2+n]==$

$$2\Big(\big(-19512-448728\,n-4422462\,n^2-24996138\,n^3-91349700\,n^4-227427644\,n^5-376226464\,n^6-308925516\,n^7+319086320\,n^8+1617697256\,n^9+3088351728\,n^{10}+3851758512\,n^{11}+3453843392\,n^{12}+2288224320\,n^{13}+1119909888\,n^{14}+396032256\,n^{15}+96095232\,n^{16}+14349312\,n^{17}+995328\,n^{18}\big)\,(-1)^{2\,n.}+$$

$$\big(-14802-376587\,n-3834063\,n^2-21159534\,n^3-71496792\,n^4-157297032\,n^5-232167060\,n^6-231571656\,n^7-153801504\,n^8-65046240\,n^9-15816384\,n^{10}-1679616\,n^{11}\big)\sum_{\iota_1=1}^{2\,n}\left(\frac{(-1+\iota_1)\,\iota_1^3\,\left(\left(\binom{2\,n}{\iota_1}^{.}\right)^3\,(-1)^{\iota_1.}\right)_{\iota_1.}}{(1+2\,n-\iota_1)^3}\right)+$$

$$\big(-26016-715824\,n-8970272\,n^2-68124912\,n^3-352009200\,n^4-1316397856\,n^5-3697583664\,n^6-7984118976\,n^7-13441452832\,n^8-17772262080\,n^9-18480846528\,n^{10}-15046225664\,n^{11}-9482866944\,n^{12}-4533055488\,n^{13}-1588110336\,n^{14}-384380928\,n^{15}-57397248\,n^{16}-3981312\,n^{17}\big)$$

$$\sum_{\iota_1=1}^{2\,n}\left(\frac{(-1+\iota_1)\,\iota_1^3\,\left(\left(\binom{2\,n}{\iota_1}^{.}\right)^3\,(-1)^{\iota_1.}\right)_{\iota_1.}}{(1+2\,n-\iota_1)^3\,(2+2\,n-\iota_1)^3\,(3+2\,n-\iota_1)^3}\right)\Big)\}$$

# What is going on?

In[14]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{n}} \left(\mathbf{H_k} \binom{\mathbf{n}}{\mathbf{k}}\right);$

## "Creative telescoping"

In[15]:= $\mathbf{GenerateRecurrence[mySum]}$

Out[15]= $\{4\ (1+n)\ \mathtt{SUM}[n] - 2\ (3+2\ n)\ \mathtt{SUM}[1+n] + (2+n)\ \mathtt{SUM}[2+n] == 1\}$

## Indefinite summation

In[16]:= $\mathbf{SigmaReduce}\left[\mathbf{mySum}, \mathbf{Tower} \to \left\{\sum_{\mathbf{k=0}}^{\mathbf{a}} \left(\mathbf{H_k} \binom{\mathbf{n+1}}{\mathbf{k}}\right), \sum_{\mathbf{k=0}}^{\mathbf{a}} \left(\mathbf{H_k} \binom{\mathbf{2+n}}{\mathbf{k}}\right)\right\}\right]$

Out[16]= $$\frac{1}{4\ (1+n)}\left(-n - n\ (1+n)\ H_n + 2\ (3+2\ n)\ \sum_{\iota_1=0}^{n}\left(H_{\iota_1}\binom{1+n}{\iota_1}\right) + (-2-n)\ \sum_{\iota_1=0}^{n}\left(\frac{(2+n)\ H_{\iota_1}\binom{1+n}{\iota_1}}{2+n-\iota_1}\right)\right)$$

$$
\begin{aligned}
\mathrm{SUM}[n] &= \sum_{k=0}^{n} \mathrm{H}_k \binom{n}{k} \\
\mathrm{SUM}[n+1] &= \sum_{k=0}^{n+1} \mathrm{H}_k \binom{n+1}{k} = \sum_{k=0}^{n} \mathrm{H}_k \binom{n+1}{k} + \mathrm{H}_{n+1} \\
\mathrm{SUM}[n+2] &= \sum_{k=0}^{n+2} \mathrm{H}_k \binom{n+2}{k} \\
&= \sum_{k=0}^{n} \mathrm{H}_k \binom{n+2}{k} + (n+2)\,\mathrm{H}_{n+1} + \mathrm{H}_{n+2}
\end{aligned}
$$

## Indefinite summation

In[17]:= **SigmaReduce[mySum, Tower → {}]**

Out[17]= $\sum_{\iota_1}^{n}\left(H_{\iota_1}\binom{n}{\iota_1}\right)$

## Indefinite summation (+ 1 sum)

In[18]:= **SigmaReduce[mySum, Tower → {$\sum_{k=0}^{a}\left(H_k\binom{n+1}{k}\right)$}]**

Out[18]= $\sum_{\iota_1}^{n}\left(H_{\iota_1}\binom{n}{\iota_1}\right)$

## Indefinite summation (+ 1 sum + "Simple Sum")

In[19]:= **SigmaReduce[mySum, Tower → {$\sum_{k=0}^{a}\left(H_k\binom{n+1}{k}\right)$},**
**SimplifyByExt → DepthNumber]**

Out[19]= $\frac{1}{4\,(1+n)}\left(1-n+2\,(1+n)\,H_n+2\,(1+n)\sum_{\iota_1=0}^{n}\left(H_{\iota_1}\binom{1+n}{\iota_1}\right)+(-2-n)\sum_{\iota_1=0}^{n}\left(\frac{\binom{1+n}{\iota_1}}{2+n-\iota_1}\right)\right)$

## Indefinite summation $(+\ 1$ sum $+$ "Simple Sum"$)$

In[20]:= $\mathbf{SigmaReduce}\Big[\mathbf{mySum}, \mathbf{Tower} \to \{\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{a}} \Big(\mathbf{H_k}\ \binom{\mathbf{n}+\mathbf{1}}{\mathbf{k}}\Big)\},$
$\mathbf{SimplifyByExt} \to \mathbf{DepthNumber}\Big]$

$$\text{Out[20]=}\ \frac{1}{4\ (1+n)}\Bigg(1-n+2\ (1+n)\ H_n+2\ (1+n)\ \sum_{\iota_1=0}^{n}\Big(H_{\iota_1}\ \binom{1+n}{\iota_1}\Big)+(-2-n)\ \sum_{\iota_1=0}^{n}\left(\frac{\binom{1+n}{\iota_1}}{2+n-\iota_1}\right)\Bigg)$$

## "Creative telescoping" with sum extensions

In[21]:= $\mathbf{GenerateRecurrence}[\mathbf{mySum}, \mathbf{SimplifyByExt} \to \mathbf{DepthNumber}]$

$$\text{Out[21]=}\ \Big\{-2\ \mathrm{SUM}[n]+\mathrm{SUM}[1+n]==\sum_{\iota_1=0}^{n}\left(\frac{\binom{n}{\iota_1}}{1+n-\iota_1}\right)\Big\}$$

$$\sum_{k=0}^{n} \mathrm{H}_k \binom{n}{k} = 2^n \left( \mathrm{H}_n - \sum_{i=1}^{n} \frac{1}{i\,2^i} \right)$$

## Simplification of the recurrence

$$\sum_{i=0}^{2n} \frac{(i-1)i^3(-1)^i\binom{2n}{i}^3}{(2n-i+1)^3} = n(2-(-1)^n\frac{(3n)!}{n!^3})$$

$$\sum_{i=0}^{2n} \frac{(i-1)i^3(-1)^i\binom{2n}{i}^3}{(2n-i+1)^3(2n-i+2)^3(2n-i+3)^3} = (-1)^n\frac{(3n)!}{n!^3}\frac{3\ (1+3\ n)\ (2+3\ n)}{8\ (1+n)^4\ (1+2\ n)^3} + \frac{-3-15\ n-12\ n^2+17\ n^3+38\ n^4+28\ n^5+8\ n^6}{4\ (1+n)^2\ (1+2\ n)^3}$$

In[22]:= $\mathbf{rec} = \{4\ (1+n)\ (2+n)\ (1+2\ n)\ (3+2\ n)\ (1+6\ n+6\ n^2)\ (-9\ (1+3\ n)\ (2+3\ n)\ (1+6\ n)\ (5+6\ n)\ (13+18\ n+6\ n^2)\ \mathrm{SUM}[n] - 6\ (90+814\ n+2543\ n^2+3864\ n^3+3126\ n^4+1296\ n^5+216\ n^6)\ \mathrm{SUM}[1+n] - (1+n)\ (2+n)\ (3+2\ n)^2\ (1+6\ n+6\ n^2)\ \mathrm{SUM}[2+n]) ==$

$$-\frac{1}{1+n}\Big(6\ (6504+144754\ n+1384851\ n^2+7537254\ n^3+26070977\ n^4+60620448\ n^5+97542252\ n^6+109802520\ n^7+86051628\ n^8+45881424\ n^9+15822864\ n^{10}+3172608\ n^{11}+279936\ n^{12})\ \left(\frac{(3\ n)!\ (-1)^n}{((n)!)^3}\right)_n\Big)\}$$

## Solving the recurrence with product extensions

In[23]:= $\textbf{SolveRecurrence}\Big[\textbf{rec}[[\textbf{1}]], \textbf{Tower} \to \{\Big(\frac{((\textbf{n})!_{.})^{\textbf{3}}\ (\textbf{6 n})!_{.}\ (-\textbf{1})^{\textbf{n}.}}{((\textbf{2 n})!_{.})^{\textbf{3}}\ (\textbf{3 n})!_{.}}\Big)_{\textbf{n}.}\}\Big]$

13.73 Second

Out[23]= $\Big\{\Big\{0, \Big(\frac{(n)!_{.}^{3}\ (6\ n)!_{.}\ (-1)^{n.}}{(2\ n)!_{.}^{3}\ (3\ n)!_{.}}\Big)_{n.}\Big\}, \quad \Big\{0, n\ \Big(\frac{(3\ n)!_{.}\ (-1)^{n.}}{(n)!_{.}^{3}}\Big)_{n.}\Big\}$

## Solving the recurrence with nested sum extensions

In[24]:= $\textbf{SolveRecurrence}\Big[\textbf{rec}[[\textbf{1}]], \textbf{SUM}[\textbf{n}], \textbf{NestedSumExt} \to \infty,$

$\textbf{Tower} \to \{\Big(\frac{((\textbf{n})!_{.})^{\textbf{3}}\ (\textbf{6 n})!_{.}\ (-\textbf{1})^{\textbf{n}.}}{((\textbf{2 n})!_{.})^{\textbf{3}}\ (\textbf{3 n})!_{.}}\Big)_{\textbf{n}.}\},$

$\textbf{AlgebraicRelationInSumSolutions} \to \textbf{True}\Big]$

13.73 Second

Out[24]= $\Big\{\Big\{0, \Big(\frac{(n)!_{.}^{3}\ (6\ n)!_{.}\ (-1)^{n.}}{(2\ n)!_{.}^{3}\ (3\ n)!_{.}}\Big)_{n.}\Big\},$

$\Big\{0, n\ \Big(\frac{(3\ n)!_{.}\ (-1)^{n.}}{(n)!_{.}^{3}}\Big)_{n.}\Big\}, \Big\{1, \frac{1}{6}\ \Big(\frac{(n)!_{.}^{3}\ (6\ n)!_{.}\ (-1)^{n.}}{(2\ n)!_{.}^{3}\ (3\ n)!_{.}}\Big)_{n.}$

$\sum_{\iota_1=1}^{n} \Bigg(\Bigg(\iota_1\ (1 - 6\ \iota_1 + 6\ \iota_1^2)\ \Big(\frac{(3\ \iota_1)!_{.}\ (-1)^{\iota_1.}}{(\iota_1)!_{.}^{3}}\Big)_{\iota_1.}$

$\sum_{\iota_2=2}^{\iota_1} \Big(\big(9360 - 64710\ \iota_2 + 63189\ \iota_2^2 + 413410\ \iota_2^3 -$

$1436799\ \iota_2^4 + 2117172\ \iota_2^5 - 1737846\ \iota_2^6 +$

$826740\ \iota_2^7 - 213840\ \iota_2^8 + 23328\ \iota_2^9\big)/$

$\big((-1 + \iota_2)\ \iota_2\ (-3 + 2\ \iota_2)\ (-5 + 3\ \iota_2)$

$(-4 + 3\ \iota_2)\ (13 - 18\ \iota_2 + 6\ \iota_2^2)\ (1 - 6\ \iota_2 + 6\ \iota_2^2)\big)\Big)\Bigg) \Big/$

$\Bigg((-1 + 2\ \iota_1)^2\ (-2 + 3\ \iota_1)\ (-1 + 3\ \iota_1)$

$\Big(\frac{(\iota_1)!_{.}^{3}\ (6\ \iota_1)!_{.}\ (-1)^{\iota_1.}}{(2\ \iota_1)!_{.}^{3}\ (3\ \iota_1)!_{.}}\Big)_{\iota_1.}\Bigg)\Bigg)\Big\}\Big\}$

# Solving the recurrence with simplification

In[25]:= **recSol =**

$$\mathbf{SolveRecurrence[rec[[1]], SUM[n], NestedSumExt \to \infty,}$$
$$\mathbf{Tower \to \{\left(\frac{((n)!.)^3\ (6\ n)!.\ (-1)^{n.}}{((2\ n)!.)^3\ (3\ n)!.}\right)_{n.}\}]}$$

`27.19 Second`

Out[25]=

$$\Bigg\{\Big\{0, \left(\frac{((n)!.)^3\ (6\ n)!.\ (-1)^{n.}}{((2\ n)!.)^3\ (3\ n)!.}\right)_{n.}\Big\}, \Big\{0, n\left(\frac{(3\ n)!.\ (-1)^{n.}}{((n)!.)^3}\right)_{n.}\Big\}, \Big\{1, -\Bigg(n\left(\frac{(3\ n)!.\ (-1)^{n.}}{((n)!.)^3}\right)_{n.}$$
$$\Bigg(60 - 887\ n + 4948\ n^2 - 13599\ n^3 + 19512\ n^4 - 13932\ n^5 +$$
$$3888\ n^6 + \left(2 - 25\ n + 117\ n^2 - 258\ n^3 + 270\ n^4 - 108\ n^5\right)$$
$$\sum_{\iota_1=2}^{n}\Big(\big(9360 - 64710\ \iota_1 + 63189\ \iota_1^2 +$$
$$413410\ \iota_1^3 - 1436799\ \iota_1^4 + 2117172\ \iota_1^5 -$$
$$1737846\ \iota_1^6 + 826740\ \iota_1^7 - 213840\ \iota_1^8 + 23328\ \iota_1^9\big)/$$
$$\big((-1+\iota_1)\ \iota_1\ (-3 + 2\ \iota_1)\ (-5 + 3\ \iota_1)\ (-4 + 3\ \iota_1)$$
$$\left(13 - 18\ \iota_1 + 6\ \iota_1^2\right)\ \left(1 - 6\ \iota_1 + 6\ \iota_1^2\right)\big)\Big)\Bigg)\Bigg)\Big/$$
$$\left(12\ (-1 + 2\ n)\ (-2 + 3\ n)\ (-1 + 3\ n)\ \left(1 - 6\ n + 6\ n^2\right)\right)\Big\}\Bigg\}$$

# Solving the recurrence with simple sums

In[26]:= $\mathbf{SolveRecurrence[rec, SUM[n], NestedSumExt \to \infty,}$
$\mathbf{Tower \to tower, SimpleSumRepresentation \to True]}$

`67.95 Second`

Out[26]=

$$\Bigg\{\Big\{0, \left(\frac{(n)!.^3\ (6\ n)!.\ (-1)^{n.}}{(2\ n)!.^3\ (3\ n)!.}\right)_{n.}\Big\}, \Big\{0, n\left(\frac{(3\ n)!.\ (-1)^{n.}}{(n)!.^3}\right)_{n.}\Big\}, \Big\{1, \frac{1}{6\,(-1+2\ n)(-2+3\ n)(-1+3\ n)}$$
$$\Bigg(\left(\frac{(3\ n)!.\ (-1)^{n.}}{(n)!.^3}\right)_{n.}\ \Bigg(-12 + 91\ n - 245\ n^2 + 261\ n^3 -$$
$$90\ n^4 + \left(-10\ n + 65\ n^2 - 135\ n^3 + 90\ n^4\right)\ \sum_{\iota_1=2}^{n}\left(\frac{1}{-1+\iota_1}\right) +$$
$$\left(-12\ n + 78\ n^2 - 162\ n^3 + 108\ n^4\right)\ \sum_{\iota_1=2}^{n}\left(\frac{1}{-3+2\ \iota_1}\right) +$$
$$\left(6\ n - 39\ n^2 + 81\ n^3 - 54\ n^4\right)\ \sum_{\iota_1=2}^{n}\left(\frac{1}{-5+3\ \iota_1}\right) +$$
$$\left(6\ n - 39\ n^2 + 81\ n^3 - 54\ n^4\right)\ \sum_{\iota_1=2}^{n}\left(\frac{1}{-4+3\ \iota_1}\right)\Bigg)\Bigg)\Big\}\Bigg\}$$

## Solving the recurrence with standard objects

In[27]:= $\mathbf{tower} = \{\left(\frac{(\mathbf{3\ n})!_.\ (-\mathbf{1})^{\mathbf{n}_.}}{((\mathbf{n})!_.)^{\mathbf{3}}}\right)_{\mathbf{n}_.}, \left(\frac{((\mathbf{n})!_.)^{\mathbf{3}}\ (\mathbf{6\ n})!_.\ (-\mathbf{1})^{\mathbf{n}_.}}{((\mathbf{2\ n})!_.)^{\mathbf{3}}\ (\mathbf{3\ n})!_.}\right)_{\mathbf{n}_.}, \mathbf{H_n}, \mathbf{H_{2\ n}}, \mathbf{H_{3\ n}}\};$

In[28]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[\mathbf{n}],$
$\mathbf{NestedSumExt} \to \infty, \mathbf{Tower} \to \mathbf{tower}]$

```
31.19 Second
```

Out[28]= $\{\{0, \left(\frac{(n)!_.^{3}\ (6\ n)!_.\ (-1)^{n_.}}{(2\ n)!_.^{3}\ (3\ n)!_.}\right)_{n_.}\}, \{0, n\left(\frac{(3\ n)!_.\ (-1)^{n_.}}{(n)!_.^{3}}\right)_{n_.}\},$
$\{1, \frac{1}{6}\ (1 + 3\ n\ H_n + 6\ n\ H_{2\ n} - 3\ n\ H_{3\ n})\ \left(\frac{(3\ n)!_.\ (-1)^{n_.}}{(n)!_.^{3}}\right)_{n_.}\}\}$

## The closed form

In[29]:= $\mathbf{FindLinearCombination}[\mathbf{recSol}, \mathbf{mySum}, \mathbf{2}]$

Out[29]= $\frac{1}{6}\left((1 + 3\ n\ H_n + 6\ n\ H_{2\ n} - 3\ n\ H_{3\ n})\ \left(\frac{(3\ n)!_.\ (-1)^{n_.}}{((n)!_.)^{3}}\right)_{n_.} - \right.$
$\left.\left(\frac{((n)!_.)^{3}\ (6\ n)!_.\ (-1)^{n_.}}{((2\ n)!_.)^{3}\ (3\ n)!_.}\right)_{n_.}\right)$

**Theorem 1.** *For nonnegative integers n we have*

$$\sum_{k=0}^{2n} (-1)^k \, k \, H_k \binom{2n}{k}^3 = \frac{1}{6} \left(1 + 3n\,H_n + 6n\,H_{2n} - 3n\,H_{3n}\right)(-1)^n \frac{(3n)!}{n!^3} - \frac{1}{6}(-1)^n \frac{n!^3\,(6n)!}{(2n)!^3\,(3n)!}$$

*where* $H_k = \sum_{i=1}^{k} \frac{1}{i}$ *are the k-th harmonic numbers.*

# 4 How `SimpleSumRepresentation` works

In[30]:= $\mathbf{mySum} = \sum_{\iota_1=2}^{n} \left(\left(9360 - 64710\,\iota_1 + 63189\,\iota_1^2 + 413410\,\iota_1^3 - 1436799\,\iota_1^4 + 2117172\,\iota_1^5 - 1737846\,\iota_1^6 + 826740\,\iota_1^7 - 213840\,\iota_1^8 + 23328\,\iota_1^9\right) / \left((-1+\iota_1)\,\iota_1\,(-3+2\,\iota_1)\,(-5+3\,\iota_1)\,(-4+3\,\iota_1)\,\left(13-18\,\iota_1+6\,\iota_1^2\right)\left(1-6\,\iota_1+6\,\iota_1^2\right)\right)\right);$

## Take the summand

In[31]:= $\mathbf{summand} = \mathbf{mySum}[[1]]$

Out[31]= $\left(9360 - 64710\,\iota_1 + 63189\,\iota_1^2 + 413410\,\iota_1^3 - 1436799\,\iota_1^4 + 2117172\,\iota_1^5 - 1737846\,\iota_1^6 + 826740\,\iota_1^7 - 213840\,\iota_1^8 + 23328\,\iota_1^9\right) / \left((-1+\iota_1)\,\iota_1\,(-3+2\,\iota_1)\,(-5+3\,\iota_1)\,(-4+3\,\iota_1)\,\left(13-18\,\iota_1+6\,\iota_1^2\right)\left(1-6\,\iota_1+6\,\iota_1^2\right)\right)$

## Apply partial fraction decomposition

In[32]:= $\mathbf{pfd} = \mathbf{Apart}[\mathbf{summand}]$

Out[32]= $36 - \frac{2}{-1+\iota_1} + \frac{12}{\iota_1} + \frac{12}{-3+2\,\iota_1} - \frac{6}{-5+3\,\iota_1} - \frac{6}{-4+3\,\iota_1} + \frac{-11+5\,\iota_1}{13-18\,\iota_1+6\,\iota_1^2} + \frac{6-5\,\iota_1}{1-6\,\iota_1+6\,\iota_1^2}$

## Expand numerator

In[33]:= $\mathbf{pfd} = \mathbf{Apart}[\mathbf{summand}]$

Out[33]= $36 - \frac{2}{-1+\iota_1} + \frac{12}{\iota_1} + \frac{12}{-3+2\,\iota_1} - \frac{6}{-5+3\,\iota_1} - \frac{6}{-4+3\,\iota_1} - \frac{11}{13-18\,\iota_1+6\,\iota_1^2} + \frac{5\,\iota_1}{13-18\,\iota_1+6\,\iota_1^2} + \frac{6}{1-6\,\iota_1+6\,\iota_1^2} - \frac{5\,\iota_1}{1-6\,\iota_1+6\,\iota_1^2}$

## Build sum over each fraction

In[34]:= $\mathbf{sumNew} = \mathbf{SigmaSum}[\mathbf{pfd}, \{\iota_1, 2, n\}]//\mathbf{SplitSum}$

$$\text{Out[34]}= \sum_{\iota_1=2}^{n}(36) - 2\sum_{\iota_1=2}^{n}\left(\frac{1}{-1+\iota_1}\right) + 12\sum_{\iota_1=2}^{n}\left(\frac{1}{\iota_1}\right) + 12\sum_{\iota_1=2}^{n}\left(\frac{1}{-3+2\iota_1}\right) - 6\sum_{\iota_1=2}^{n}\left(\frac{1}{-5+3\iota_1}\right) - 6\sum_{\iota_1=2}^{n}\left(\frac{1}{-4+3\iota_1}\right) - 11\sum_{\iota_1=2}^{n}\left(\frac{1}{13-18\iota_1+6\iota_1^2}\right) + 5\sum_{\iota_1=2}^{n}\left(\frac{\iota_1}{13-18\iota_1+6\iota_1^2}\right) + 6\sum_{\iota_1=2}^{n}\left(\frac{1}{1-6\iota_1+6\iota_1^2}\right) - 5\sum_{\iota_1=2}^{n}\left(\frac{\iota_1}{1-6\iota_1+6\iota_1^2}\right)$$

## Represent expression by transcendental sum extensions

In[35]:= $\mathbf{SigmaReduce}[\mathbf{sumNew}, \mathbf{n}]$

$$\text{Out[35]}= \frac{1}{n(1-6n+6n^2)}\Bigg(12 - 115n + 397n^2 - 510n^3 + 216n^4 + (10n - 60n^2 + 60n^3)\sum_{\iota_1=2}^{n}\left(\frac{1}{-1+\iota_1}\right) + (12n - 72n^2 + 72n^3)\sum_{\iota_1=2}^{n}\left(\frac{1}{-3+2\iota_1}\right) + (-6n + 36n^2 - 36n^3)\sum_{\iota_1=2}^{n}\left(\frac{1}{-5+3\iota_1}\right) + (-6n + 36n^2 - 36n^3)\sum_{\iota_1=2}^{n}\left(\frac{1}{-4+3\iota_1}\right)\Bigg)$$

# 5 Difference Equations and Symbolic Summation

Let $(\mathbb{F}, \sigma)$ be a difference field and

$$\mathbb{K} = \{k \in \mathbb{F} \,|\, \sigma(k) = k\}$$

be the constant field.

## Telescoping

- GIVEN $f \in \mathbb{F}$
- FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

## Parameterized Telescoping

- GIVEN $f_0, \dots, f_d \in \mathbb{F}$, $a_0, a_1 \in \mathbb{F}$
- FIND ALL $c_0, \dots, c_d \in \mathbb{K}$, $h \in \mathbb{F}$:

$$\boxed{a_1\, \sigma(h) - a_0\, h = c_0\, f_0 + \cdots + c_d\, f_d}$$

## Remark: Z's "Creative Telescoping"

- GIVEN $f_i = \text{summand}(n+i, k) \in \mathbb{F}$
- FIND ALL $c_0, \dots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0\, f_0 + \cdots + c_d\, f_d}$$

## Linear Difference Equations

- GIVEN $f, a_0, \dots, a_m \in \mathbb{F}$
- FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

## Parameterized Linear Difference Equations

- GIVEN $a_0, \dots, a_m \in \mathbb{F}$,$f_0, \dots, f_d \in \mathbb{F}$.
- FIND ALL $g \in \mathbb{F}$, $c_0, \dots, c_d \in \mathbb{K}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = c_0\,f_0 + \cdots + c_d\,f_d}$$

## My results

- Streamlining of Karr's ideas result in a simpler algorithm
- Generalization of Karr's algorithm:

  first order $\longrightarrow$ $m$-th order

- New connections:

  indefinite-$\Sigma$ $\longleftrightarrow$ definite-$\Sigma$

# 6 Sum Extensions for Indefinite Summation

In[36]:= $\mathbf{mySum} = \sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{K+\iota_3}\right)}{K+\iota_2}\right)}{K+\iota_1}\right);$

In[37]:= **SigmaReduce[mySum]**

Out[37]= $\sum_{\iota_1=1}^{N}\left(\frac{\sum_{\iota_2=1}^{\iota_1}\left(\frac{\sum_{\iota_3=1}^{\iota_2}\left(\frac{1}{K+\iota_3}\right)}{K+\iota_2}\right)}{K+\iota_1}\right);$

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In[38]:= **SigmaReduce[mySum, SimplifyByExt → Depth]**

Out[38]= $\frac{1}{6K^2}\left(6\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)+6K\left(\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)^2+K^2\left(\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)^3+\left(-3-3K\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)\boxed{\sum_{\iota_1=1}^{N}\left(\frac{K+2\iota_1}{(K+\iota_1)^2}\right)}-K\boxed{\sum_{\iota_1=1}^{N}\left(\frac{K+3\iota_1}{(K+\iota_1)^3}\right)}\right)$

Partial fraction decomposition:

$$\boxed{\frac{K+2i}{(K+i)^2}}=-\frac{K}{(K+i)^2}+\frac{2}{K+i},\quad \boxed{\frac{K+3i}{(K+i)^2}}=-\frac{2K}{(K+i)^3}+\frac{3}{(K+i)^2}$$

In[39]:= **SigmaReduce[mySum,**
**Tower → {{$H_{K+N}$, N}, {$H^{(2)}_{K+N}$, N}, {$H^{(3)}_{K+N}$, N}}]**

Out[39]= $\frac{1}{6}\left(-H_K^3-3H_KH_{K+N}^2+H_{K+N}^3+3H_KH_K^{(2)}-3H_KH_{K+N}^{(2)}+H_{K+N}\left(3H_K^2-3H_K^{(2)}+3H_{K+N}^{(2)}\right)-2H_K^{(3)}+2H_{K+N}^{(3)}\right)$

# New Insights in Indefinite Summation

$$\sum_{\iota_1=1}^{\mathtt{N}} \left( \frac{\sum_{\iota_2=1}^{\iota_1} \left( \frac{\sum_{\iota_3=1}^{\iota_2} \left( \frac{\mathtt{1}}{\mathtt{K} + \iota_3} \right)}{\mathtt{K} + \iota_2} \right)}{\mathtt{K} + \iota_1} \right)$$

The underlying difference field $(\mathbb{Q}(t_1)(t_2)(t_3)(t_4), \sigma)$:

$$\sigma(t_1) = t_1 + 1$$
$$\sigma(t_2) = t_2 + \frac{1}{K + t_1 + 1}$$
$$\sigma(t_3) = t_3 + \sigma(\frac{t_2}{K + t_1})$$
$$\sigma(t_4) = t_4 + \sigma(\frac{t_3}{K + t_1})$$

$$\frac{1}{6}\Big( - \mathtt{H}_{\mathtt{K}}^{3} - 3\,\mathtt{H}_{\mathtt{K}}\,\mathtt{H}_{\mathtt{K+N}}^{2} + \mathtt{H}_{\mathtt{K+N}}^{3} + 3\,\mathtt{H}_{\mathtt{K}}\,\mathtt{H}_{\mathtt{K}}^{(2)} - 3\,\mathtt{H}_{\mathtt{K}}\,\mathtt{H}_{\mathtt{K+N}}^{(2)}$$

$$+\mathtt{H}_{\mathtt{K+N}}\Big(3\,\mathtt{H}_{\mathtt{K}}^{2} - 3\,\mathtt{H}_{\mathtt{K}}^{(2)} + 3\,\mathtt{H}_{\mathtt{K+N}}^{(2)}\Big) - 2\,\mathtt{H}_{\mathtt{K}}^{(3)} + 2\,\mathtt{H}_{\mathtt{K+N}}^{(3)}\Big)$$

The underlying difference field $(\mathbb{Q}(t_1)(t_2)(t'_3)(t'_4),\sigma)$:

$$\begin{aligned}
\sigma(t_1) &= t_1 + 1\\
\sigma(t_2) &= t_2 + \frac{1}{K + t_1 + 1}\\
\sigma(t'_3) &= t'_3 + \frac{1}{(K + t_1 + 1)^2}\\
\sigma(t'_4) &= t'_4 + \frac{1}{(K + t_1 + 1)^3}
\end{aligned}$$

$$\boxed{(\mathbb{Q}(t_1)(t_2)(t_3)(t_4),\sigma)\simeq(\mathbb{Q}(t_1)(t_2)(t'_3)(t'_4),\sigma)}$$