Symbolic Summation in Difference Fields

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1 A Bonus Problem in "Concrete Mathematics"

Chapter 6. Special Numbers, Bonus problem 69:

> Find a closed form for
> $$\sum_{k=1}^{n} k^2 \mathrm{H}_{n+k},$$
> where $\mathrm{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

Knuth's answer to the problem is

$$\frac{1}{3}n\left(n+\frac{1}{2}\right)(n+1)\left(2\mathrm{H}_{2n}-\mathrm{H}_n\right)-\frac{1}{36}n\left(10n^2+9n-1\right)$$

with the remark

> **"It would be nice to automate the derivation of formulas such as this."**

In[1]:= $\mathbf{Problem69 = DefineSum[k\hat{}2\ DefineHNumber[n+k],\{k,1,n\}]}$

Out[1]= $\sum_{k=1}^{n}\left(k^2\ H_{k+n}\right)$

In[2]:= $\mathbf{KReduce[Problem69]//Simplify}$

Out[2]= $-\frac{1}{36}\ n\ (1+n)\ (-1+10\ n+6\ (1+2\ n)\ H_n-12\ (1+2\ n)\ H_{2n})$

2 Calkin's Identity

Find a closed form for

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3$$

Case 1:

In[3]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{a}}\left(\sum_{\mathbf{j=0}}^{\mathbf{k}}\left(\binom{\mathbf{n}}{\mathbf{j}}^{\cdot}\right)\right);$

In[4]:= **KReduce[mySum]//Simplify**

Out[4]= $\frac{1}{2}\left((-\mathtt{a}+\mathtt{n})\binom{\mathtt{n}}{\mathtt{a}}^{\cdot}+(2+2\,\mathtt{a}-\mathtt{n})\sum_{\iota_1=0}^{\mathtt{a}}\left(\binom{\mathtt{n}}{\iota_1}^{\cdot}\right)\right)$

Case 2:

In[5]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{a}}\left(\left(\sum_{\mathbf{j=0}}^{\mathbf{k}}\left(\binom{\mathbf{n}}{\mathbf{j}}^{\cdot}\right)\right)^{\mathbf{2}}\right);$

In[6]:= **KReduce[mySum]**

Out[6]= $\sum_{\iota_1=0}^{\mathtt{a}}\left(\left(\sum_{\iota_2=0}^{\iota_1}\left(\binom{\mathtt{n}}{\iota_2}^{\cdot}\right)\right)^{2}\right)$

In[7]:= **KReduce[mySum, TowerSuggestion → True]**

Out[7]= $(-\mathtt{a}+\mathtt{n})\binom{\mathtt{n}}{\mathtt{a}}^{\cdot}\sum_{\iota_1=0}^{\mathtt{a}}\left(\binom{\mathtt{n}}{\iota_1}^{\cdot}\right)+\left(1+\mathtt{a}-\frac{\mathtt{n}}{2}\right)\left(\sum_{\iota_1=0}^{\mathtt{a}}\left(\binom{\mathtt{n}}{\iota_1}^{\cdot}\right)\right)^2+$

$\sum_{\iota_1=0}^{\mathtt{a}}\left(-\frac{1}{2}\,\mathtt{n}\left(\binom{\mathtt{n}}{\iota_1}^{\cdot}\right)^{2}\right)$

Case 3:

In[8]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{n}}\left(\left(\sum_{\mathbf{j=0}}^{\mathbf{k}}\left(\binom{\mathbf{n}}{\mathbf{j}}^{\cdot}\right)\right)^{\mathbf{3}}\right)$

Finding a recurrence

In[9]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}][[\mathbf{1}]]$

Out[9]= $-16\ (1+2\ \mathtt{n})\ \mathtt{SUM}[\mathtt{n}] - 4\ (12+7\ \mathtt{n})\ \mathtt{SUM}[1+\mathtt{n}]$
$+4\ (1+\mathtt{n})\ \mathtt{SUM}[2+\mathtt{n}]$
$== 8\left(-10\left(\sum_{\iota_1=0}^{\mathtt{n}}\left(\binom{\mathtt{n}}{\iota_1}^{\cdot}\right)\right)^{3} + 9\ \mathtt{n}\left(\sum_{\iota_1=0}^{\mathtt{n}}\left(\binom{\mathtt{n}}{\iota_1}^{\cdot}\right)\right)^{3}\right)$

In[10]:= $\mathbf{rec} = \mathbf{rec}/.\{\sum_{\iota_1=0}^{\mathbf{n}}\left(\binom{\mathbf{n}}{\iota_1}^{\cdot}\right) \to (\mathbf{2})^{\mathbf{n}.}\}$

Out[10]= $-16\ (1+2\ \mathtt{n})\ \mathtt{SUM}[\mathtt{n}] - 4\ (12+7\ \mathtt{n})\mathtt{SUM}[1+\mathtt{n}]$
$+4\ (1+\mathtt{n})\ \mathtt{SUM}[2+\mathtt{n}] == 8\left(-10\ ((2)^{\mathtt{n}.})^{3} + 9\ \mathtt{n}\ ((2)^{\mathtt{n}.})^{3}\right)$

Solving the recurrence

In[11]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[\mathbf{n}],$
$\mathbf{Tower} \to \{\binom{\mathbf{2n}}{\mathbf{n}}^{\cdot}\}]$

Out[11]= $\left\{\left\{0, \mathtt{n}\binom{2\ \mathtt{n}}{\mathtt{n}}^{\cdot}\ (2)^{\mathtt{n}.}\right\}, \left\{1, \frac{1}{2}\ (2+\mathtt{n})\ ((2)^{\mathtt{n}.})^{3}\right\}\right\}$

Finding the linear combination

In[12]:= $\mathbf{FindLinearCombination}[\mathbf{recSol}, \mathbf{mySum}]//\mathbf{Simplify}$

Out[12]= $-\frac{3}{4}\ \mathtt{n}\binom{2\ \mathtt{n}}{\mathtt{n}}^{\cdot}\ (2)^{\mathtt{n}.} + \frac{1}{2}\ (2+\mathtt{n})\ ((2)^{\mathtt{n}.})^{3}$

3 An Alternating Version of Calkin's Identity

(Zhizheng Zhang)

Case 1:

In[13]:= $$\textbf{mySum} = \sum_{k=0}^{a}\left((-1)^{k}\ \sum_{j=0}^{k}\left(\binom{n}{j}\right)\right);$$

In[14]:= **KReduce[mySum]//Simplify**

Out[14]= $$\frac{(-1)^{a}\left((-a+n)\binom{n}{a} + n\sum_{\iota_1=0}^{a}\left(\binom{n}{\iota_1}\right)\right)}{2\,n}$$

Case 2:

In[15]:= $$\textbf{mySum} = \sum_{k=0}^{n}\left((-1)^{k}\left(\sum_{j=0}^{k}\left(\binom{n}{j}\right)\right)^{2}\right);$$

In[16]:= **KReduce[mySum, TowerSuggestion → True]**

Out[16]= $$\frac{1}{2}(-1)^{n}\left(\sum_{\iota_1=0}^{n}\left(\binom{n}{\iota_1}\right)\right)^{2} + \frac{1}{n}\sum_{\iota_1=0}^{n}\left(-\frac{1}{2}(n-2\,\iota_1)\left(\binom{n}{\iota_1}\right)^{2}(-1)^{\iota_1}\right)$$

where

$$\sum_{\iota_1=0}^{n}\left(-\frac{1}{2}(n-2\,\iota_1)\left(\binom{n}{\iota_1}\right)^{2}(-1)^{\iota_1}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -(-1)^{\frac{n-1}{2}} n \binom{n-1}{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

Case 3: Find a closed form for

$$\sum_{k=0}^{n}(-1)^k\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3 \qquad (n \text{ odd})$$

In[17]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{-1+2\ n}}\left((-\mathbf{1})^{\mathbf{k}}\left(\sum_{\mathbf{j=0}}^{\mathbf{k}}\left(\binom{\mathbf{-1+2\ n}}{\mathbf{j}}\right)\right)^{\mathbf{3}}\right);$

Finding a recurrence

In[18]:= **rec = GenerateRecurrence[mySum]//Simplify**

Out[18]= $\{24\ (17-12\ n-656\ n^2+432\ n^3+1584\ n^4)\ \mathrm{SUM}[n]+ (-40-115\ n+510\ n^2+3628\ n^3+3784\ n^4)\ \mathrm{SUM}[1+n]+ (1+2\ n)^2\ (-5+17\ n+22\ n^2)\ \mathrm{SUM}[2+n] == 4\ (-2829-2492\ n+37952\ n^2+110192\ n^3+80080\ n^4)\ (-1)^{-1+2\ n}\left(\sum_{\iota_1=0}^{-1+2\ n}\left(\binom{-1+2\ n}{\iota_1}\right)\right)^3\}$

In[19]:= $\mathbf{rec} = \mathbf{rec}[[\mathbf{1}]]/.\{\sum_{\iota_1=\mathbf{0}}^{\mathbf{2\ n-1}}\left(\binom{\mathbf{2\ n-1}}{\iota_1}\right) \to (\mathbf{2})^{\mathbf{-1+2\ n}}, (-\mathbf{1})^{\mathbf{2\ n-1}} \to -\mathbf{1}\}$

Out[19]= $24\ (17-12\ n-656\ n^2+432\ n^3+1584\ n^4)\ \mathrm{SUM}[n]+ (-40-115\ n+510\ n^2+3628\ n^3+3784\ n^4)\ \mathrm{SUM}[1+n]+ (1+2\ n)^2\ (-5+17\ n+22\ n^2)\ \mathrm{SUM}[2+n] == -4\ (-2829-2492\ n+37952\ n^2+110192\ n^3+80080\ n^4)\ ((2)^{-1+2\ n})^3$

Solving the recurrence

In[20]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[\mathbf{n}], \mathbf{Tower} \to \{\mathbf{DefinePower}[-\mathbf{1}, \mathbf{k}], \mathbf{DefineBinomial}[\mathbf{2n}, \mathbf{n}]\}]$

Out[20]= $$\left\{\left\{0, \frac{2\,n \binom{2\,n}{n}^{\cdot} (-1)^{n.} (2)^{-1+2\,n.}}{-1+2\,n}\right\}, \left\{1, -\frac{1}{2}\left((2)^{-1+2\,n.}\right)^{3}\right\}\right\}$$

Finding the linear combination

In[21]:= $\mathbf{FindLinearCombination}[\mathbf{recSol}, \mathbf{mySum}, \mathbf{n}, \mathbf{1}, \{\}]$

Out[21]= $$\frac{3\,n \binom{2\,n}{n}^{\cdot} (-1)^{n.} (2)^{-1+2\,n.}}{4\,(-1+2\,n)} - \frac{1}{2}\left((2)^{-1+2\,n.}\right)^{3}$$

4 Karr's Method and an Example

Goal: Find a closed form for

$$\sum_{k=0}^{n} k \cdot k!$$

A Difference Field for the Problem

Let t_1, t_2 be indeterminates where

$$\begin{aligned} t_1 &\longleftrightarrow k \\ t_2 &\longleftrightarrow k! \end{aligned}$$

Consider the field automorphism $\sigma : \mathbb{Q}(t_1, t_2) \to \mathbb{Q}(t_1, t_2)$ induced by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q} \\ \sigma(t_1) &= t_1 + 1 \\ \sigma(t_2) &= (t_1 + 1)t_2 \end{aligned}$$

$(\mathbb{Q}(t_1, t_2), \sigma)$ is our difference field.

The Telescoping Problem

Find $g \in \mathbb{Q}(t_1, t_2)$: $\boxed{\sigma(g) - g = t_1 t_2}$

$$\downarrow \text{ by Karr}$$

$$g = t_2.$$

The Closed Form

$$\boxed{(k+1)! - k! = k \cdot k!}$$

$$\downarrow$$

$$\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1.$$

5 Difference Equations and Symbolic Summation

Let $(\mathbb{F}, \sigma)$ be a difference field and

$$\mathbb{K} = \{k \in \mathbb{F} \,|\, \sigma(k) = k\}$$

be the constant field.

Telescoping

- GIVEN $f \in \mathbb{F}$
- FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = f}$$

$$\downarrow \qquad \uparrow$$

Extended Telescoping

- GIVEN $f_0, \ldots, f_d \in \mathbb{F}$, $a_0, a_1 \in \mathbb{F}$
- FIND ALL $c_0, \ldots, c_d \in \mathbb{K}$, $h \in \mathbb{F}$:

$$\boxed{a_1\,\sigma(h) - a_0\,h = c_0\,f_0 + \cdots + c_d\,f_d}$$

Remark: Z's "Creative Telescoping"

- GIVEN $f_i = \text{summand}(n+i, k) \in \mathbb{F}$
- FIND ALL $c_0, \ldots, c_d \in \mathbb{K}$, $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = c_0\,f_0 + \cdots + c_d\,f_d}$$

m-th Order Linear Difference Equations

- GIVEN $f, a_0, \dots, a_m \in \mathbb{F}$
- FIND ALL $g \in \mathbb{F}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots + a_0\,g = f}$$

$$\downarrow \qquad\qquad \uparrow$$

The General Problem

- GIVEN $a_0, \dots, a_m \in \mathbb{F}$,$f_0, \dots, f_d \in \mathbb{F}$.
- FIND ALL $g \in \mathbb{F}$, $c_0, \dots, c_d \in \mathbb{K}$:

$$\boxed{a_m\,\sigma^m(g) + \cdots a_0\,g = c_0\,f_0 + \cdots c_d\,f_d}$$

6 Definite Summation

GOAL: Find a closed form for

$$\sum_{k=1}^{n}\left(\frac{H_k\,(3+k+n)!\,(-1)^{k}\,(-1)^{-1+n}}{(1+k)!\,(2+k)!\,(-k+n)!}\right) - \frac{(n)!}{(3+n)!}\sum_{k=1}^{n}\left(\frac{(3+k+n)!\,(-1)^{k}\,(1-(2+n)\,(-1)^{n})}{k\,(1+k)!^{2}\,(-k+n)!}\right)$$

(The number of rhombus tilings of a symmetric hexagon, Fulmek & Krattenthaler)

In[22]:= $\mathbf{mySum1} = \sum_{\mathbf{k=1}}^{\mathbf{n}} \left(\frac{\mathbf{H_k\ (3+k+n)!\ (-1)^{k}\ (-1)^{-1+n}}}{\mathbf{(1+k)!\ (2+k)!\ (-k+n)!}} \right);$

Finding a recurrence

In[23]:= $\mathbf{rec1 = GenerateRecurrence[mySum1][[1]]//Simplify}$

Out[23]= $n\ (1+n)\ (2+n)\ (3+n)\ (4+n)\ (-1+n)!\ \big(-(9+2n)\ (8+6n+n^2)\ \mathrm{SUM}[n] + (9+2n)\ (13+8n+n^2)\ \mathrm{SUM}[1+n] + (30+42n+17n^2+2n^3)\ \mathrm{SUM}[2+n] - (3+n)\ (25+15n+2n^2)\ \mathrm{SUM}[3+n]\big) == 2\ (-1)^n\ (9+2n)\ (35+24n+4n^2)\ (4+n)!$

Solving the recurrence

In[24]:= $\mathbf{recSol1 = SolveRecurrence[rec1, SUM[n], Tower \to \{H_n\}]}$

Out[24]= $\Big\{\{0,1\}, \Big\{0, \frac{3-n^2+4H_n+6nH_n+2n^2H_n}{(1+n)\ (2+n)}\Big\}, \Big\{0, \frac{1}{4}\ (2+n)\ (-1)^{n}\Big\}, \Big\{1, \frac{(16-13n^2-5n^3+32H_n+64nH_n+40n^2H_n+8n^3H_n)\ (-1)^{n}}{4\ (1+n)\ (2+n)}\Big\}\Big\}$

Finding the linear combination

In[25]:= $\mathbf{solution1 = FindLinearCombination[recSol1, mySum1]}$

Out[25]= $-1 - \frac{3-n^2+4H_n+6nH_n+2n^2H_n}{(1+n)\ (2+n)} + \frac{1}{4}\ (2+n)\ (-1)^{n} + \frac{(16-13n^2-5n^3+32H_n+64nH_n+40n^2H_n+8n^3H_n)\ (-1)^{n}}{4\ (1+n)\ (2+n)}$

In[26]:= $$\mathbf{mySum2} = \sum_{k=1}^{n}\left(\frac{(3+k+n)!\,(-1)^{k}\,(1-(2+n)\,(-1)^{n})}{k\,(1+k)!^{2}\,(-k+n)!}\right);$$

Finding a recurrence

In[27]:= **rec2 = GenerateRecurrence[mySum2, RecOrder → 2][[1]]**

Out[27]= $$-n\,(1+n)\,(3+n)\,(1+3\,(-1)^n+(-1)^n\,n)\,(-1+4\,(-1)^n+(-1)^n\,n)\,(28+15\,n+2\,n^2)\,(-1+n)!\,\mathtt{SUM}[n]+6\,n\,(1+n)\,(3+n)^2\,(-1+2\,(-1)^n+(-1)^n\,n)\,(-1+4\,(-1)^n+(-1)^n\,n)\,(-1+n)!\,\mathtt{SUM}[1+n]+n\,(1+n)\,(3+n)\,(-1+2\,(-1)^n+(-1)^n\,n)\,(1+3\,(-1)^n+(-1)^n\,n)\,(10+9\,n+2\,n^2)\,(-1+n)!\,\mathtt{SUM}[2+n]==2\,(-1+2\,(-1)^n+(-1)^n\,n)\,(1+3\,(-1)^n+(-1)^n\,n)\,(-1+4\,(-1)^n+(-1)^n\,n)\,(35+24\,n+4\,n^2)\,(4+n)!$$

Solving the recurrence

In[28]:= **recSol2 = SolveRecurrence[rec2, SUM[n], Tower → {H_n}, WithMinusPower → True]**

Out[28]= $$\{\{0,2+n-(-1)^{n}\},\{0,16-6\,n^2-n^3+(-1)^{n}+28\,n\,(-1)^{n}+23\,n^2\,(-1)^{n}+8\,n^3\,(-1)^{n}+n^4\,(-1)^{n}\},\{1,-\frac{1}{28}(260-150\,n^2-39\,n^3+336\,H_n+616\,n\,H_n+336\,n^2\,H_n+56\,n^3\,H_n-325\,(-1)^{n}+365\,n^2\,(-1)^{n}+228\,n^3\,(-1)^{n}+39\,n^4\,(-1)^{n}-672\,H_n\,(-1)^{n}-1568\,n\,H_n\,(-1)^{n}-1288\,n^2\,H_n\,(-1)^{n}-448\,n^3\,H_n\,(-1)^{n}-56\,n^4\,H_n\,(-1)^{n})\}\}$$

Finding the linear combination

In[29]:= **solution2 = FindLinearCombination[recSol2, mySum2]**

Out[29]= $$(3+n)\,\left(-1+3\,n+2\,n^2-\left(-1+6\,n+7\,n^2+2\,n^3\right)\,(-1)^{n}+2\,\left(2+3\,n+n^2\right)\,H_n\,\left(-1+(2+n)\,(-1)^{n}\right)\right)$$

In[30]:= $\mathbf{solution1 - solution2/((n+1)(n+2)(n+3))//Simplify}$

Out[30]= $-2 + (2 + \mathtt{n})\ (-1)^{\mathtt{n}}.$

7 An Alternating Sum (P. Kirschenhofer)

In[31]:= $\mathbf{mySum} = \sum_{\mathbf{k=0}}^{\mathbf{N}} \left(\frac{\binom{\mathbf{N}}{\mathbf{k}} (-\mathbf{1})^{\mathbf{k}}}{(\mathbf{k}+\mathbf{K})^{\mathbf{3}}} \right);$

Finding a recurrence

In[32]:= $\mathbf{rec} = \mathbf{GenerateRecurrence}[\mathbf{mySum}][[\mathbf{1}]]$

Out[32]= $-(1+N)\,(2+N)\,(3+N)\,\mathrm{SUM}[N]+ 3\,\left(12+6\,K+16\,N+5\,K\,N+7\,N^2+K\,N^2+N^3\right)\,\mathrm{SUM}[1+N]+ \left(-57-45\,K-9\,K^2-64\,N-33\,K\,N-3\,K^2\,N-24\,N^2-6\,K\,N^2-3\,N^3\right) \mathrm{SUM}[2+N]+(3+K+N)^3\,\mathrm{SUM}[3+N]==0$

Solving the recurrence

- Manual product extension:

In[33]:= $\mathbf{tower} = \{\{\binom{\mathbf{K+N}}{\mathbf{K}}, \mathbf{N}\}\};$

In[34]:= $\mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[\mathbf{N}], \mathbf{Tower} \to \mathbf{tower}]$

Out[34]= $\left\{\left\{0, \frac{1}{\binom{K+N}{K}}\right\}, \{1, 0\}\right\}$

- Automatic sum extension:

In[35]:= $\mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[\mathbf{N}], \mathbf{Tower} \to \mathbf{tower}, \mathbf{TowerSuggestion} \to \mathbf{True}]$

Out[35]= $\left\{\left\{0, \frac{1}{\binom{K+N}{K}}\right\}, \left\{0, -\frac{\sum_{\iota_1=1}^{N} \left(\frac{1}{K+\iota_1}\right)}{\binom{K+N}{K}}\right\}, \{1, 0\}\right\}$

In[36]:= $\mathbf{tower} = \mathbf{Join}\left[\left\{\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right\}, \mathbf{tower}\right]$

Out[36]= $\left\{\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right), \left\{\binom{K+N}{K}^{\cdot}, N\right\}\right\}$

- Automatic Sum Extension:

In[37]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{SUM}[N],$
$\mathbf{Tower} \to \mathbf{tower}, \mathbf{TowerSuggestion} \to \mathbf{True}]$

Out[37]= $\left\{\left\{0, \frac{1+K}{\binom{K+N}{K}^{\cdot}}\right\}, \left\{0, \frac{(1+K)\ \sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)}{\binom{K+N}{K}^{\cdot}}\right\},\right.$

$\left\{0, \frac{-\sum_{\iota_1=1}^{N}\left(\frac{-1+\iota_1}{(K+\iota_1)^2}\right) + \left(\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)^2 + K\left(\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)^2}{\binom{K+N}{K}^{\cdot}}\right\},$

$\left.\{1, 0\}\right\}$

Finding the linear combination

In[38]:= $\mathbf{FindLinearCombination}[\mathbf{recSol}, \mathbf{mySum}]//\mathbf{Simplify}$

Out[38]= $\left(2\ (1+K) - K^2\ \sum_{\iota_1=1}^{N}\left(\frac{-1+\iota_1}{(K+\iota_1)^2}\right) +\right.$

$\left. K\ (2+3\ K)\ \sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right) + K^2\ (1+K)\ \left(\sum_{\iota_1=1}^{N}\left(\frac{1}{K+\iota_1}\right)\right)^2\right)\Big/$

$\left(2\ K^3\ (1+K)\ \binom{K+N}{K}^{\cdot}\right)$

8 An Identity from Physics (Essam, Guttmann) - Case 2 -

Goal: Eliminate sum quantifiers in

$$\sum_{\mathbf{k1}=0}^{\mathbf{a}}\left(\sum_{\mathbf{k2}=0}^{\mathbf{k1}}\left((\mathbf{k1}-\mathbf{k2})\binom{\mathbf{n}}{\mathbf{k1}}^{\cdot}\binom{\mathbf{n}}{\mathbf{k2}}^{\cdot}\right)\right)$$

by using the two sums

In[39]:= $\mathbf{tower} = \{\sum_{\mathbf{k}=0}^{\mathbf{a}}((\binom{\mathbf{n}}{\mathbf{k}}^{\cdot})^{2}), \sum_{\mathbf{k}=0}^{\mathbf{a}}(\binom{\mathbf{n}}{\mathbf{k}}^{\cdot})\};$

- Step 1:

In[40]:= $\mathbf{Sum1} = \sum_{\mathbf{k2}=0}^{\mathbf{k1}}\left((\mathbf{k1}-\mathbf{k2})\binom{\mathbf{n}}{\mathbf{k1}}^{\cdot}\binom{\mathbf{n}}{\mathbf{k2}}^{\cdot}\right);$

In[41]:= $\mathbf{summand} = \mathbf{KReduce}[\mathbf{Sum1}, \mathbf{Tower} \to \mathbf{tower}]$

Out[41]= $-\frac{1}{2}\binom{n}{k1}^{\cdot}\left((k1-n)\binom{n}{k1}^{\cdot}+(-2\,k1+n)\sum_{\iota_1=0}^{k1}(\binom{n}{\iota_1}^{\cdot})\right)$

- Step 2:

In[42]:= $\mathbf{Sum2} = \mathbf{DefineSum}[\mathbf{summand}, \{\mathbf{k1}, \mathbf{0}, \mathbf{a}\}]$

Out[42]= $\sum_{k1=0}^{a}\left(-\frac{1}{2}\binom{n}{k1}^{\cdot}\left((k1-n)\binom{n}{k1}^{\cdot}+(-2\,k1+n)\sum_{\iota_1=0}^{k1}(\binom{n}{\iota_1}^{\cdot})\right)\right)$

In[43]:= $\mathbf{KReduce}[\mathbf{Sum2}, \mathbf{Tower} \to \mathbf{tower}]//\mathbf{Simplify}$

Out[43]= $\frac{1}{2}\left((a-n)\binom{n}{a}^{\cdot}\sum_{\iota_1=0}^{a}(\binom{n}{\iota_1}^{\cdot})+n\sum_{\iota_1=0}^{a}((\binom{n}{\iota_1}^{\cdot})^{2})\right)$

- Case 3 -

Goal: Eliminate sum quantifiers in

$$\sum_{k1=0}^{n}\left(\sum_{k2=0}^{k1}\left(\sum_{k3=0}^{k2}\left((k1-k2)\,(k1-k3)\,(k2-k3)\binom{n}{k2}\binom{n}{k3}\binom{n}{k1}\right)\right)\right)$$

- Step 1:

In[44]:= **Sum1 =**

$$\sum_{k3=0}^{k2}\left((k1-k2)\,(k1-k3)\,(k2-k3)\binom{n}{k2}\binom{n}{k3}\binom{n}{k1}\right);$$

In[45]:= $\mathbf{summand = KReduce[Sum1, Tower \to tower]}$

Out[45]= $-\frac{1}{4}\,(k1-k2)\binom{n}{k1}\binom{n}{k2}\left((-1+2\,k1-n)\,(k2-n)\binom{n}{k2}-\left(4\,k1\,k2+n-2\,k1\,n-2\,k2\,n+n^2\right)\sum_{\iota_1=0}^{k2}\left(\binom{n}{\iota_1}\right)\right)$

• Step 2:

In[46]:= **Sum2 = DefineSum[summand, {k2, 0, k1}];**

In[47]:= **summand = KReduce[Sum2, Tower → tower]**

Out[47]= $$\frac{1}{4\,n\,(-1+2\,n)}\left(\binom{n}{k1}^{\cdot}\left((k1-n)^2\,\left(k1-2\,k1\,n+n^2+2\,k1\,n^2-n^3\right)\,\left(\binom{n}{k1}^{\cdot}\right)^2-k1\,(k1-n)\,n\,(-1+2\,n)\,\binom{n}{k1}^{\cdot}\,\sum_{\iota_1=0}^{k1}\left(\binom{n}{\iota_1}^{\cdot}\right)+n^2\left(2\,k1\,(1-2\,n)\,n+(-1+n)\,n^2+k1^2\,(-2+4\,n)\right)\,\sum_{\iota_1=0}^{k1}\left(\left(\binom{n}{\iota_1}^{\cdot}\right)^2\right)\right)\right)$$

• Step 3:

In[48]:= **Sum3 = DefineSum[summand, {k1, 0, n}];**

In[49]:= **KReduce[Sum3]//Simplify**

Out[49]= $$\frac{(-1+n)\,n^2\,\left(\sum_{\iota_1=0}^{n}\binom{n}{\iota_1}^{\cdot}\right)\,\sum_{\iota_1=0}^{n}\left(\binom{n}{\iota_1}^{\cdot}\right)^2}{-8+16\,n}$$

- Case 5 -

We eliminate the sum quantifiers in

In[50]:= $\mathbf{mySum} =$

$$\sum_{k1=0}^{n}\left(\sum_{k2=0}^{k1}\left(\sum_{k3=0}^{k2}\left(\sum_{k4=0}^{k3}\left(\sum_{k5=0}^{k4}\left((k1-k2)(k1-k3)(k2-k3)(k1-k4)(k2-k4)(k3-k4)(k1-k5)(k2-k5)(k3-k5)(k4-k5)\binom{n}{k1}\binom{n}{k2}\binom{n}{k3}\binom{n}{k4}\binom{n}{k5}\right)\right)\right)\right)\right)$$

by using the two sums

In[51]:= $\mathbf{tower} = \{\sum_{k=0}^{a}\left(\binom{n}{k}^{2}\right), \sum_{k=0}^{a}\binom{n}{k}\};$

We get:

In[52]:= $\mathbf{result} = \mathbf{KReduce}[\mathbf{mySum}, \mathbf{Tower} \to \mathbf{tower}]$

Out[52]=

$$\frac{3\,(-3+n)\,(-2+n)^2(-1+n)^3\,n^5\left(\sum_{\iota_1=0}^{n}\binom{n}{\iota_1}\right)\left(\sum_{\iota_1=0}^{n}\binom{n}{\iota_1}^2\right)^2}{256\,(-5+2\,n)\left(3-8\,n+4\,n^2\right)^2}$$

By the substitution

In[53]:= $$\mathbf{subst} = \{\sum_{\iota_1=0}^{\mathbf{n}}(\binom{\mathbf{n}}{\iota_1}^{\cdot}) \to (\mathbf{2})^{\mathbf{n}\cdot}, \sum_{\iota_1=0}^{\mathbf{n}}((\binom{\mathbf{n}}{\iota_1}^{\cdot})^{\mathbf{2}}) \to \binom{\mathbf{2\ n}}{\mathbf{n}}^{\cdot}\};$$

we get the final result:

In[54]:= **result/.subst**

Out[54]= $$\frac{3\ (-3+n)\ (-2+n)^2\ (-1+n)^3\ n^5\ ((\binom{2\ n}{n}^{\cdot})^2\ (2)^{n\cdot}}{256\ (-5+2\ n)\ (3-8\ n+4\ n^2)^2}$$

The Corresponding q-Identity - Case 2

We eliminate the sum quantifiers in

In[55]:= $\mathbf{mySum} = \sum_{k1=0}^{a}\left(\sum_{k2=0}^{k1}\left(\left((q)^{\frac{1}{2}(-1+k1)k1}\begin{bmatrix}n\\k1\end{bmatrix}_q\right)_{k1}\left((q)^{\frac{1}{2}(-1+k2)k2}\begin{bmatrix}n\\k2\end{bmatrix}_q\right)_{k2}\left(-(q)^{k1}+(q)^{k2}\right)\right)\right);$

by using the elements

In[56]:= $\mathbf{tower} = \left\{\{(q)^{k}, k\}, \left((q)^{\frac{1}{2}(-1+k)k}\begin{bmatrix}n\\k\end{bmatrix}_q\right)_{k}, \sum_{k=0}^{a}\left(\left((q)^{\frac{1}{2}(-1+k)k}\begin{bmatrix}n\\k\end{bmatrix}_q\right)^2_{k}\right), \sum_{k=0}^{a}\left(\left((q)^{\frac{1}{2}(-1+k)k}\begin{bmatrix}n\\k\end{bmatrix}_q\right)_{k}\right)\right\};$

We get:

In[57]:= $\mathbf{result} = \mathbf{KReduce}[\mathbf{mySum}, \mathbf{Tower}->\mathbf{tower}]//\mathbf{Simplify}$

Out[57]= $\frac{1}{2}\left(\left((q)^{\frac{1}{2}(-1+a)a}\begin{bmatrix}n\\a\end{bmatrix}_q\right)_{a}(q^n-(q)^{a})\sum_{\iota_1=0}^{a}\left(\left((q)^{\frac{1}{2}(-1+\iota_1)\iota_1}\begin{bmatrix}n\\\iota_1\end{bmatrix}_q\right)_{\iota_1}\right) - (-1+q^n)\sum_{\iota_1=0}^{a}\left(\left((q)^{\frac{1}{2}(-1+\iota_1)\iota_1}\begin{bmatrix}n\\\iota_1\end{bmatrix}_q\right)^2_{\iota_1}\right)\right)$

9 A Series of Identities by S. Ahlgren

-Work in Progress-

Case 3: Prove

$$\sum_{j=0}^{n}(1-3\ j\ H_j+3\ j\ H_{-j+n})\binom{n}{j}^3=0$$

Case 3a - Straight forward

In[58]:= $\mathbf{mySum3a}=\sum_{\mathbf{j=0}}^{\mathbf{n}}\left((\mathbf{1}-\mathbf{3\ j\ H_j}+\mathbf{3\ j\ H_{-j+n}})\ (((\binom{\mathbf{n}}{\mathbf{j}})^{\mathbf{3}})\ )_{\mathbf{j}}\right);$

In[59]:= $\mathbf{GenerateRecurrence[mySum3a, RecOrder \to 3]}$

103.06 Second

Out[59]= $\{8\ (1+n)^2\ (7+3\ n)\ \mathtt{SUM}[n]+(256+432\ n+243\ n^2+45\ n^3)\ \mathtt{SUM}[1+n]+2\ (88+129\ n+60\ n^2+9\ n^3)\ \mathtt{SUM}[2+n]-(24+38\ n+19\ n^2+3\ n^3)\ \mathtt{SUM}[3+n])==0\}$

Case 3b - Creative Summation

In[60]:= $\mathbf{mySum3b} = \sum_{j=0}^{n}\left((1 - 3\,j\,H_j + 3\,(-j+n)\,H_j)\,\left(\left(\binom{n}{j}\right)^{3}\right)_{j}\right);$

In[61]:= **creativeSol =**
CreativeTelescoping[mySum3b, RecOrder → 2]//Simplify

```
16.85 Second
```

Out[61]= $\Big\{\Big\{-1, \frac{-3-2\,n}{1+n}, -\frac{2+n}{1+n}, \frac{1}{(1-j+n)^3\,(2-j+n)^3}$
$\Big(j^2\,\big(2\,j^4 - 6\,j^3\,(3+2\,n) + 6\,(2+n)^2\,\big(3+5\,n+2\,n^2\big) +$
$3\,j^2\,\big(21+28\,n+9\,n^2\big) - 2\,j\,\big(52+104\,n+67\,n^2+14\,n^3\big) -$
$3\,j\,\big(j^4 - j^3\,(8+5\,n) + 2\,(2+n)^2\,\big(3+5\,n+2\,n^2\big) +$
$3\,j^2\,\big(8+10\,n+3\,n^2\big) - j\,\big(34+65\,n+40\,n^2+8\,n^3\big)\big)\,H_j\big)$
$\left(\left(\binom{n}{j}\right)^{3}\right)_{j}\Big)\Big\},$
$\{0, 0, 0, 1\}\Big\}$

In[62]:= **rec = TransformToRecurrence[creativeSol, mySum3b]**

```
5.66 Second
```

Out[62]= $(-1-n)\,\mathtt{SUM}[n] + (-3-2\,n)\,\mathtt{SUM}[1+n]$
$+(-2-n)\,\mathtt{SUM}[2+n] == 0\}$

Case 4: Prove

$$\sum_{j=0}^{n}(1-4\ j\ H_j+4\ (-j+n)\ H_j)\binom{n}{j}^4=(-1)^n\binom{2n}{n}$$

In[63]:= $\mathbf{mySum4} = \sum_{\mathbf{j=0}}^{\mathbf{n}} \left((\mathbf{1} - \mathbf{4}\ \mathbf{j}\ \mathbf{H_j} + \mathbf{4}\ (-\mathbf{j} + \mathbf{n})\ \mathbf{H_j})\ ((\binom{\mathbf{n}}{\mathbf{j}}^{\mathbf{4}}))_{\mathbf{j}} \right)$

In[64]:= $\mathbf{GenerateRecurrence}[\mathbf{mySum4}, \mathbf{RecOrder} \to \mathbf{2}]$

37.78 Second

Out[64]= $\{4\ (1+2\ n)^2\ (11+8\ n)\ \mathtt{SUM}[n]$
$+2\ (29+110\ n+108\ n^2+32\ n^3)\ \mathtt{SUM}[1+n]$
$+(2+n)^2\ (3+8\ n)\ \mathtt{SUM}[2+n] == 0\}$

10 Difference Field Extensions

In[65]:= $\mathbf{tripleSum = DefineSum[DefineSum[DefineHNumber[i]/(i\hat{}2-1), \{i,2,k\}], \{k,1,n\}]}$

Out[65]= $$\sum_{k=1}^{n}\Bigl(\sum_{i=2}^{k}\Bigl(\frac{H_i}{-1+i^2}\Bigr)\Bigr)$$

In[66]:= $\mathbf{KReduce[tripleSum]//Simplify}$

Out[66]= $$\frac{1}{4n(1+n)}\Bigl(2H_n-2n(1+n)H_n^2+ (1+n)\Bigl(2-n+4n(1+n)\sum_{\iota_1=2}^{n}\Bigl(\frac{H_{\iota_1}}{-1+\iota_1^2}\Bigr)\Bigr)\Bigr)$$

Manual Extension

In[67]:= $\mathbf{KReduce\bigl[tripleSum, Tower\to\{H_n^{(2)}\}\bigr]//Simplify}$

Out[67]= $$\frac{1}{4(1+n)}\Bigl(-2(3+2n)H_n-2(1+n)H_n^{(2)}+(1+n)\bigl(3n+2(1+n)H_n^{(2)}\bigr)\Bigr)$$

Automatic Extension

In[68]:= $\mathbf{KReduce[tripleSum, TowerSuggestion\to Clever]}$

Out[68]= $$\frac{1}{4n(1+n)}\Biggl(-2n(3+2n)H_n-2n(1+n)H_n^2+ (1+n)\Biggl(2+2n+3n^2+2n(1+n)\sum_{\iota_1=2}^{n}\Biggl(\frac{-1+2\iota_1}{(-1+\iota_1)\iota_1^2}\Biggr)\Biggr)\Biggr)$$

where

$$\sum_{i=2}^{n}\frac{2i-1}{2(i-1)i^2}=\frac{1}{2}\left(\sum_{i=2}^{n}\frac{1}{i-1}+\sum_{i=2}^{n}\frac{1}{i^2}-\sum_{i=2}^{n}\frac{1}{i}\right)$$