Karr's Indefinite Summation Algorithm

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Gosper's Hypergeometric
Summation Algorithm (1978)

Chyzak's Holonomic
Indefinite Summation Algorithm (1998)

(1981)

1 A Bonus Problem in "Concrete Mathematics"

Chapter 6. Special Numbers, Bonus problem 69:

Find a closed form for

$$\sum_{k=1}^{n} k^2 H_{n+k},$$

where $H_n := \sum_{k=1}^{n} \frac{1}{k}$.

Knuth's answer to the problem is

$$\frac{1}{3}n\left(n+\frac{1}{2}\right)(n+1)(2H_{2n}-H_n)-\frac{1}{36}n\left(10n^2+9n-1\right)$$

with the remark

"It would be nice to automate the derivation of formulas such as this."

2 Karr's Method and an Example

Goal: Find a closed form for

$$\sum_{k=0}^{n} k \cdot k!.$$

A Difference Field for the Problem

Let t_1, t_2 be indeterminants where

$$\begin{array}{lcl} k & \longleftrightarrow & t_1 \\ k! & \longleftrightarrow & t_2. \end{array}$$

Consider the field automorphism $\sigma : \mathbb{Q}(t_1, t_2) \to \mathbb{Q}(t_1, t_2)$ induced by

$$\begin{array}{rcl} \sigma(c) & = & c \quad \forall c \in \mathbb{Q} \\ \sigma(t_1) & = & t_1 + 1 \\ \sigma(t_2) & = & (t_1 + 1)t_2. \end{array}$$

$(\mathbb{Q}(t_1, t_2), \sigma)$ is our difference field.

The Telescoping Problem

$$\text{Find } g \in \mathbb{Q}(t_1, t_2) : \quad \boxed{\sigma(g) - g = t_1 t_2}$$

$$\downarrow \text{ by Karr}$$

$$g = t_2.$$

The Closed Form

$$\boxed{(k + 1)! - k! = k \cdot k!}$$

$$\downarrow$$

$$\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1.$$

3 Karr's Solution Space

Let

$\mathbb{K}$ be a field of characteristic 0 ($\mathbb{Q}$ or $\mathbb{Q}(n)$

$\mathbb{F} = \mathbb{K}(t_1)(t_2)\dots(t_n)$ and $f, a, f_i, g \in \mathbb{F}$.

- Given f

  `Find g:`

$$\boxed{\sigma(g) - g = f}$$

$$\downarrow$$

- Given f, a

  `Find g:`

$$\boxed{\sigma(g) - \boxed{a} \cdot g = f}$$

$$\downarrow$$

- Given f, a

  `Find` $\boxed{c \in \mathbb{K}}$ `and` $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - a \cdot g = \boxed{f \cdot c}}$$

$$\downarrow$$

- Given $\langle f_1, \dots, f_d \rangle$ and a and $W \leq \mathbb{F}$ specified by Karr's reduction process

  `Find all` $\langle c_1, \dots, c_d \rangle \in \mathbb{K}^d$ `and all` $\boxed{g \in W}$:

$$\boxed{\sigma(g) - a \cdot g = \boxed{f_1 \cdot c_1 + \cdots f_d \cdot c_d}}$$

Remark: $f_i = \mathrm{N}^{i-1} f(n, k)$.

$$\downarrow$$

- Given $\mathbf{f} = \langle f_1, \ldots, f_d \rangle$ and a and $W \leq \mathbb{F}$ specified by Karr's reduction process

  Find basis of

  $$\boxed{V(a, \mathbf{f}, W) := \{\boxed{\langle c_1, \ldots, c_d, g \rangle} \in \mathbb{K}^d \times W \mid \sigma(g) - ag = f_1 \cdot c_1 + \cdots f_d \cdot c_d\}}$$

4 Back To Our Example: The Reduction Process

Consider the vectorspaces

$$\begin{aligned}
\mathbb{Q}(t_1,t_2) &:= \{ \underbrace{p}_{poly.part} + \underbrace{\frac{q}{r}}_{fract.part} \mid p,q,r \in \boxed{\mathbb{Q}(t_1)[t_2]}, \deg q < \deg r\} \\
\mathbb{Q}(t_1,t_2)_{\boxed{d}} &:= \{u \in \mathbb{Q}(t_1,t_2) \mid \text{the polynomial part of } u \text{ has } \deg \leq \boxed{d}\} \\
\mathbb{Q}(t_1,t_2)_{\boxed{-1}} &:= \{u \in \mathbb{Q}(t_1,t_2) \mid \text{the polynomial part is } 0\}.
\end{aligned}$$

Find $g \in \mathbb{Q}(t_1,t_2)$: $\boxed{\sigma(g) - g = t_1 \cdot t_2}$

$\downarrow$

DEGREE ANALYSIS: The polynomial part of g has deg ≤ 1.

$\downarrow$

Find basis of $\boxed{V(1, \langle t_1 t_2\rangle, \mathbb{Q}(t_1,t_2))}$

$\|$

$\boxed{V(1, \overbrace{\langle t_1 t_2\rangle}^{\mathbf{f_1}}, \mathbb{Q}(t_1,t_2)_1)}$

REDUCTION-POLY

Find $\mathbf{f}_0$. $\longrightarrow$ Find basis of $\boxed{V(1, \mathbf{f_0}, \mathbb{Q}(t_1,t_2)_0)}$

$\downarrow$

$\vdots$

$\downarrow$

Find basis of $\boxed{V(1, \mathbf{f_{-1}}, \mathbb{Q}(t_1,t_2)_{-1})}$

$\downarrow$

REDUCTION-FRAC $\boxed{\cdots}$

5 The Recursion Process in Our Example

Find $g \in \mathbb{Q}(t_1, t_2)$, s.t. $\boxed{\sigma(g) - g = t_1 \cdot t_2}$

$\downarrow$

Find basis of $\boxed{V(1, \langle t_1 \cdot t_2 \rangle, \mathbb{Q}(t_1, t_2))}$

$\|$

$\boxed{V(1, \langle t_1 \cdot t_2 \rangle, \mathbb{Q}(t_1, t_2)_1)}$

Find $\mathbf{f_0}$.

$\downarrow$

Find basis of

$$\boxed{V(\frac{1}{t_1+1}, \langle \frac{t_1}{t_1+1} \rangle, \mathbb{Q}(t_1))}$$

$\|$

$$\boxed{V(\frac{1}{t_1+1}, \langle \frac{t_1}{t_1+1} \rangle, \mathbb{Q}(t_1)_0)}$$

$\xrightarrow{\mathbf{f_0} = \langle 0 \rangle}$ Find basis of $\boxed{V(1, \mathbf{f_0}, \mathbb{Q}(t_1, t_2)_0)}$

$\downarrow$

STOP:Short Cut

Find $\mathbf{f_{-1}}$

$\downarrow$

Find basis of

$\boxed{V(0, \langle 1 \rangle, \mathbb{Q})}$

$\|$

$\boxed{\text{Nullspace}_{\mathbb{Q}}(\langle 1, -1 \rangle)}$

$\xleftarrow{\mathbf{f_{-1}} = \langle 0 \rangle}$ Find basis of

$$\boxed{V(\frac{1}{t_1+1}, \mathbf{f_{-1}}, \mathbb{Q}(t_1)_{-1})}$$

$\downarrow$

Eliminate fractional part

6 My Main Tasks/Topics of My PhD-Work

1. Extension and Optimization of Karr's Algorithm:
   - Avoidance of factorizations by using greatest factorial factorization.
   - Are algebraic extensions of the constant field possible in Karr's approach?
   - Automatic computation of "good" difference field extensions.
   - Embedding of Karr's theory in difference rings:
     E.g. we want to use
     $$(-1)^n := \prod_{i=1}^{n}(-1)$$
     as an extension element.
   - Which classes of definite summation can be handled by Karr?
2. Karr's approach & related algorithms:
   - Karr $\longleftrightarrow$ Gosper
   - Karr $\longleftrightarrow$ Risch