Karr's Indefinite Summation Algorithm

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Gosper's Hypergeometric Summation Algorithm (1978)

Chyzak's Holonomic Indefinite Summation Algorithm (1998)

(1981)

1 A Bonus Problem in "Concrete Mathematics"

Chapter 6. Special Numbers, Bonus problem 69:

Find a closed form for

$$\sum_{k=1}^{n} k^2 H_{n+k},$$
where $H_n := \sum_{k=1}^{n} \frac{1}{k}.$

Knuth's answer to the problem is

$$\frac{1}{3}n\left(n+\frac{1}{2}\right)(n+1)\left(2H_{2n}-H_n\right) - \frac{1}{36}n\left(10n^2+9n-1\right)$$

with the remark

"It would be nice to automate the derivation of formulas such as this."

2 Karr's Method and an Example

Goal: Find a closed form for

$$\sum_{k=0}^{n} k \cdot k!.$$

A Difference Field for the Problem

Let t_1, t_2 be indeterminants where

$$\begin{array}{rccc} k & \longleftrightarrow & t_1 \\ k! & \longleftrightarrow & t_2. \end{array}$$

Consider the field automorphism $\sigma : \mathbb{Q}(t_1, t_2) \to \mathbb{Q}(t_1, t_2)$ induced by

$$\sigma(c) = c \quad \forall c \in \mathbb{Q}$$

$$\sigma(t_1) = t_1 + 1$$

$$\sigma(t_2) = (t_1 + 1)t_2.$$

 $(\mathbb{Q}(t_1, t_2), \sigma)$ is our difference field.

The Telescoping Problem

Find
$$g \in \mathbb{Q}(t_1, t_2)$$
: $\sigma(g) - g = t_1 t_2$
 \downarrow by Karr
 $g = t_2$.

The Closed Form

$$(k+1)! - k! = k \cdot k!$$

$$\downarrow$$
$$\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1.$$

3 Karr's Solution Space

Let

 \mathbb{K} be a field of characteristic 0 (\mathbb{Q} or $\mathbb{Q}(n)$ $\mathbb{F} = \mathbb{K}(t_1)(t_2)\dots(t_n)$ and $f, a, f_i, g \in \mathbb{F}$.

 \bullet Given f

Find g:

$$\sigma(g) - g = f$$

$$\downarrow$$

 \bullet Given f,a

Find g:

$$\sigma(g) - \boxed{a} \cdot g = f$$

$$\downarrow$$

• Given f, a

Find $c \in \mathbb{K}$ and $g \in \mathbb{F}$:

$$\sigma(g) - a \cdot g = \boxed{f \cdot c}$$

 \downarrow

• Given $\langle f_1, \ldots, f_d \rangle$ and a and $W \leq \mathbb{F}$ specified by Karr's reduction process

Find all
$$\langle c_1, \dots, c_d \rangle \in \mathbb{K}^d$$
 and all $g \in W$:

$$\sigma(g) - a \cdot g = f_1 \cdot c_1 + \cdots f_d \cdot c_d$$

Remark: $f_i = N^{i-1} f(n, k)$.

3 KARR'S SOLUTION SPACE

\downarrow

• Given $\mathbf{f} = \langle f_1, \dots, f_d \rangle$ and a and $W \leq \mathbb{F}$ specified by Karr's reduction process

Find basis of

$$V(a, \mathbf{f}, W) := \{ \overline{\langle c_1, \dots, c_d, g \rangle} \in \mathbb{K}^d \times W \mid \sigma(g) - ag = f_1 \cdot c_1 + \dots + f_d \cdot c_d \}$$

4 Back To Our Example: The Reduction Process

Consider the vector spaces

$$\begin{aligned} \mathbb{Q}(t_1, t_2) &:= \{\underbrace{p}_{poly.part} + \underbrace{\frac{q}{r}}_{fract.part} \mid p, q, r \in \boxed{\mathbb{Q}(t_1)[t_2]}, \deg q < \deg r \} \\ \mathbb{Q}(t_1, t_2)_{\boxed{d}} &:= \{u \in \mathbb{Q}(t_1, t_2) \mid \text{the polynomial part of } u \text{ has } \deg \leq \boxed{d} \} \\ \mathbb{Q}(t_1, t_2)_{\boxed{-1}} &:= \{u \in \mathbb{Q}(t_1, t_2) \mid \text{the polynomial part is } 0 \}. \end{aligned}$$

Find
$$g \in \mathbb{Q}(t_1, t_2) : \overline{\sigma(g) - g = t_1 \cdot t_2}$$

 \downarrow

 \downarrow

DEGREE ANALYSIS: The polynomial part of g has deg ≤ 1 .

Find basis of
$$V(1, \langle t_1 t_2 \rangle, \mathbb{Q}(t_1, t_2))$$

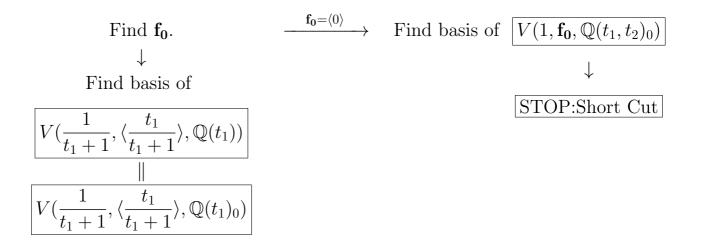
 $\|$
 $V(1, \langle t_1 t_2 \rangle, \mathbb{Q}(t_1, t_2)_1)$

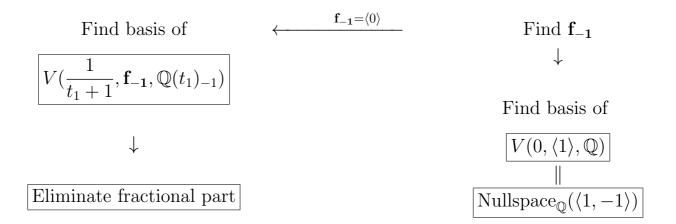
REDUCTION-POLY

Find
$$\mathbf{f}_0$$
. \longrightarrow Find basis of $V(1, \mathbf{f}_0, \mathbb{Q}(t_1, t_2)_0)$
 \downarrow
 \vdots
 \downarrow
Find basis of $V(1, \mathbf{f}_{-1}, \mathbb{Q}(t_1, t_2)_{-1})$
 \downarrow
REDUCTION-FRAC \checkmark

5 The Recursion Process in Our Example

Find
$$g \in \mathbb{Q}(t_1, t_2)$$
, s.t. $\sigma(g) - g = t_1 \cdot t_2$
 \downarrow
Find basis of $V(1, \langle t_1 \cdot t_2 \rangle, \mathbb{Q}(t_1, t_2))$
 \parallel
 $V(1, \langle t_1 \cdot t_2 \rangle, \mathbb{Q}(t_1, t_2)_1)$





6 My Main Tasks/Topics of My PhD-Work

- 1. Extension and Optimization of Karr's Algorithm:
 - Avoidance of factorizations by using greatest factorial factorization.
 - Are algebraic extensions of the constant field possible in Karr's approach?
 - Automatic computation of "good" difference field extensions.
 - Embedding of Karr's theory in difference rings: E.g. we want to use

$$(-1)^n := \prod_{i=1}^n (-1)$$

as an extension element.

- Which classes of definite summation can be handled by Karr?
- 2. Karr's approach & related algorithms:
 - Karr \longleftrightarrow Gosper
 - Karr \longleftrightarrow Risch