PLANE PARTITIONS VI: STEMBRIDGE'S TSPP THEOREM — A DETAILED ALGORITHMIC PROOF

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Dedicated to the memory of David Robbins

Abstract. We provide a new proof of Stembridge's theorem which validated the Totally Symmetric Plane Partitions (TSPP) Conjecture. The overall strategy of our proof follows the same general pattern of determinant evaluation as discussed by the first named author in a series of papers. The resulting hypergeometric multiple sum identities turn out to be quite complicated. Their correctness is proved by applying new algorithmic methods from symbolic summation.

1. Introduction

The object of this paper is to provide a new proof of John Stembridge's theorem [Ste95] which validated the Totally Symmetric Plane Partitions (TSPP) Conjecture. We begin with a description of the objects in question.

A plane partition π is an array of nonnegative integers $(\pi_{ij})_{i\geq 1, j\geq 1}$ wherein $\pi_{ij} \geq \pi_{i'j'}$ if $i \geq i'$ and $j \geq j'$. We say this is a plane partition of n if

$$n = |\pi| := \sum_{i,j\geq 1} \pi_{ij}.$$

Obviously this means that all but finitely many of the π_{ij} are zero.

Often a plane partition is displayed in the fourth quadrant (rather than the expected first quadrant). So if $\pi_{1,1} = \pi_{1,2} = 4$, $\pi_{1,3} = 3$, $\pi_{1,4} = 1$, $\pi_{2,1} = 3$, $\pi_{2,2} = \pi_{2,3} = 2$, $\pi_{3,1} = \pi_{3,2} = 1$ and all other $\pi_{ij} = 0$, then π is represented by the diagram

$$\begin{matrix} 4 & 4 & 3 & 1 \\ 3 & 2 & 2 & \\ 1 & 1 & & \end{matrix}$$

and π is a plane partition of 21 with 3 rows and 4 columns.

A plane partition is called *symmetric* if $\pi_{ij} = \pi_{ji}$. A plane partition is called *cyclically symmetric* if the i^{th} row π (considered as an ordinary partition) is the conjugate of the i^{th} column of π.

A plane partition is called *totally symmetric* if it is both symmetric and cyclically symmetric.

In [Sta86a], R. Stanley provides an account of many conjectures on plane partitions (many of which have been settled); of these, we are interested in

Conjecture 1 (cf. [Sta86b, Case 4]). *Let T_n equal the number of totally symmetric plane partitions with largest part $\leq n$. Then for $n \geq 1$*

$$T_n = \prod_{1\leq i\leq j\leq k\leq n} \frac{i+j+k-1}{i+j+k-2}. \tag{1}$$

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This assertion is now known as Stembridge's theorem: Stembridge's proof [Ste95] combines a variety of masterful steps involving the combinatorics of Pfaffians and reduction of such to known determinant representations from which (1) follows.

Prior to Stembridge's work, S. Okada had obtained an evaluation of T_n which is easily seen to be equivalent to the following:

Okada's Theorem ([Oka89, Sec. 4])**.** For $n \geq 3$,

$$T_{n-2}^2 = \begin{cases} \det\big(M(n)\big)x^{-1} & \text{if } n \text{ is odd,} \\ \det\big(M(n)\big) & \text{if } n \text{ is even,} \end{cases} \tag{2}$$

where

$$M(n) = \big(\mu_1(i,j)\big)_{0 \leq i,j \leq n-1}, \tag{3}$$

$$\mu(i,j) = \begin{cases} 0 & \text{if } j \leq i, \\ 2^{j-1} + (-1)^{j-1} & \text{if } i = 0,\ i < j, \\ (-1)^{j-i-1} + \sum_{s=i}^{j-1} \binom{i+j-2}{s} & \text{if } 0 < i < j, \end{cases} \tag{4}$$

and

$$\mu_1(i,j) = \begin{cases} x & \text{if } i = j = 0, \\ (-1)^{j-1} & \text{if } i = 0,\ j > 0, \\ (-1)^i & \text{if } j = 0,\ i > 0, \\ 0 & \text{if } i = j > 0, \\ \mu(i-1, j-1) & \text{if } j > i \geq 1, \\ -\mu(j-1, i-1) & \text{if } 1 \leq j < i. \end{cases} \tag{5}$$

In this paper, we shall deduce Stembridge's theorem by exhibiting an upper triangular matrix $W(n)$ with 1's on the main diagonal for which we prove that

$$M(n)W(n) = \begin{pmatrix} A_{00} & 0 & 0 & \dots & 0 \\ A_{10} & A_{11} & 0 & \dots & 0 \\ A_{20} & A_{21} & A_{22} & \dots & 0 \\ \vdots & & & \ddots & \\ A_{n-1,0} & A_{n-1,1} & A_{n-1,2} & \dots & A_{n-1,n-1} \end{pmatrix} \tag{6}$$

is a lower triangular matrix with the A_{ij} explicitly determined. From the form of the A_{ii} and the immediate fact that

$$\det\big(M(n)\big) = A_{00}A_{11}\dots A_{n-1,n-1}, \tag{7}$$

we obtain Stembridge's theorem.

Remark: We remark that Okada's Theorem 5 [Oka89] is a much more simply stated version of the above result. However, to our surprise our techniques were unsuccessful in studying this apparently simpler formulation of the problem.

The entries of the matrix $W(n)$ will be presented in the next section. They are complicated to say the least. This then naturally suggests the question: Why provide a new proof of Stembridge's theorem under these circumstances?

We believe there are two compelling reasons. First, the five previous entries [And79a, And77, And79b, And87, And94] in this series have been devoted to determinant evaluations precisely along the lines of (6), and it is of interest to see that Stembridge's theorem fits this general pattern. Second, the proof of (6) requires the development of really new results concerning Zeilberger's telescoping and its extensions. The full treatment of these discoveries will be presented in [Sch04a] and [PS04]. Consequently the effort to prove (6) has yielded substantial improvement in the methods of symbolic summation.

The remainder of this paper is structured as follows. Section 2 introduces notation and basic definitions together with Theorem 1 which implies Stembridge's theorem as a corollary. Section 3 presents all the identities we need to verify in order to prove Theorem 1. In Section 4 we briefly describe the general proof method which we used to prove all the identities in question. The details of all the proof steps are given in the Sections 5 to 8. In Section 9 concluding remarks are made.

2. Notation and Definitions

We begin with the standard rising factorial

$$(x)_n = \begin{cases} 1 & \text{if } n = 0, \\ x(x+1)\dots(x+n-1) & \text{if } n > 0, \end{cases}$$

and the binomial coefficient

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!}, \qquad \text{for } n \geq 0.$$

The remaining symbols are all specific to this paper and are not to be confused with other meanings in other settings:

$$\left\{ \begin{matrix} x \\ n \end{matrix} \right\} = \frac{1}{2}\left(\binom{x}{n} + \binom{x-1}{n}\right);$$

$$t_1(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{(n+1)(n+3)\dots(3n-1)}{(n)_n} & \text{if } n > 0; \end{cases}$$

$$t(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{t_1(n)}{t_1(n-1)} & \text{if } n > 0. \end{cases}$$

We remark in passing that if the right-hand side of (1) is denoted by τ_n, then for $n \geq 1$

$$\frac{\tau_n}{\tau_{n-1}} = \prod_{1 \leq i \leq j \leq n} \frac{i+j+n-1}{i+j+n-2} = \prod_{j=1}^{n} \frac{2j+n-1}{j+n-1} = t_1(n).$$

Therefore to prove (1) we need only prove equivalently

$$T_n^2 = \prod_{j=0}^{n} t_1(j)^2 = \tau_n^2$$

in light of the fact that each of the T_n and τ_n is positive.

Now we come to the more intricate components of $W(n)$:

$$r_3(s,j) = 4^{-s} \sum_{k=0}^{s} \frac{(j-k)(j)_k(-3j-1)_k}{jk!(-2j+\frac{1}{2})_k},$$

$$f_1(c,j) = (-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} \frac{(-1)^s \binom{j-1-s}{c-2s} (j)_s (-3j+1)_s (3j-3s-1)}{4^s s! (-2j+\frac{3}{2})_s (3j-1)},$$

$$f_2(c,j) = (-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} (-1)^s \left\{ \begin{matrix} j-s \\ c-2s \end{matrix} \right\} r_3(s,j),$$

$$r_2(j) = \begin{cases} \frac{t_1(j-1)}{2} & \text{if } j \text{ even}, \\ \frac{t_1(j-1)}{2} + \frac{f_2(j-2,\frac{j-1}{2})}{2} & \text{if } j \text{ odd}, \end{cases}$$

$$r_1(j) = \begin{cases} t_1(j-1) & \text{if } j \text{ even,} \\ 0 & \text{if } j \text{ odd,} \end{cases}$$

$$e_1(i,j) = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ r_1(j) & \text{if } i = 0,\ i < j, \\ r_2(j) & \text{if } i = 1,\ i < j, \\ f_1(j-i, \frac{j}{2}) & \text{if } 2 \le i < j,\ j \text{ even,} \\ f_2(j-i, \frac{j-1}{2}) & \text{if } 2 \le i < j,\ j \text{ odd,} \end{cases}$$

$$e(i,j) = \begin{cases} 0 & \text{if } i > j, \\ 1 & \text{if } i = j, \\ e_1(i,j) - t(j-1)x^{(-1)^j} e_1(i,j-1) & \text{if } i < j, \end{cases}$$

$$W(n) = \big(e(i,j)\big)_{0 \le i,j \le n-1}.$$

Given all of the above, our object in the remainder of the paper will be to prove the following

Theorem 1. *For each $n \ge 1$,*

$$M(n)W(n) = \begin{pmatrix} x & 0 & 0 & 0 & \dots & 0 \\ A_{10} & \frac{t_1(0)^2}{x} & 0 & 0 & \dots & 0 \\ A_{20} & A_{21} & t_1(1)^2 x & 0 & \dots & 0 \\ A_{30} & A_{31} & A_{32} & \frac{t_1(2)^2}{x} & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ A_{n-1,0} & A_{n-1,1} & \dots\dots & & & t_1(n-2)^2 x^{\pm 1} \end{pmatrix} \tag{8}$$

where the entry $x^{\pm 1}$ in the lower triangular matrix has to be interpreted as x, if n is odd, and as x^{-1}, if n is even.

We note before proceeding that in light of the fact that

$$\det\big(W(n)\big) = 1,$$

and (2) together with (8) we have the following

Corollary 1 (Stembridge's theorem [Ste95, Th. 0.2])**.** *For each $n \ge 2$,*

$$T_{n-2}^2 = \det\big(M(n)_{x=1}\big) = \prod_{j=0}^{n-2} t_1(j)^2.$$

3. The Identities To Prove

3.1. **The odd case $j = 2m+1$.** The following definitions will be convenient for proving Theorem 1.

Definition 1. For $i, j \ge 0$ denote the entry of the ith row and the jth column of the matrix $M(n)W(n)$ by $G(i,j)$; i.e.,

$$G(i,j) = \sum_{k=0}^{j} \mu_1(i,k) e(k,j).$$

Definition 2. Extend the definition of μ by

$$\mu(-1,k) := \begin{cases} 0, & \text{if } k = 0, \\ (-1)^k, & \text{if } k \ge 1. \end{cases}$$

For this subsection we fix $j = 2m+1$ with $m \ge 1$.

Definition 3. For $i \geq 0$ and $m \geq 1$ define

$$a_1(i,m) := \sum_{k=1}^{i-2} \mu(k,i-1) f_2(2m-k,m), \qquad a_2(i,m) := \sum_{k=i}^{2m} \mu(i-1,k) f_2(2m-k,m),$$
$$b_1(i,m) := \sum_{k=1}^{i-2} \mu(k,i-1) f_1(2m-k-1,m), \quad b_2(i,m) := \sum_{k=i}^{2m} \mu(i-1,k) f_1(2m-k-1,m).$$

The next fact is immediate from the definitions.

Lemma 1. *We have $f_1(0,j) = 1$ for all $j \geq 0$ and $f_2(0,j) = 1$ for all $j \geq 1$.*

From Lemma 1 and the definitions the following representation is straightforward.

Proposition 1. *For $m \geq 1$ and $0 \leq i \leq 2m+1$,*

$$G(i,2m+1) = \mu_1(i,1)\frac{t_1(2m)+f_2(2m-1,m)}{2} - a_1(i,m) + a_2(i,m)$$
$$- \frac{1}{x}\frac{t_1(2m)}{t_1(2m-1)}\Big(\mu_1(i,0)t_1(2m-1) + \mu_1(i,1)\frac{t_1(2m-1)}{2} - b_1(i,m) + b_2(i,m)\Big).$$

Since Proposition 1 involves $\mu_1(i,0)$ and $\mu_1(i,1)$ we have to distinguish three cases: $i = 0$ (case A), $i = 1$ (case B), and $i \geq 2$ (case C).

3.1.1. *Case A: $i = 0$.* In this case, Proposition 1 turns into the following form.

Proposition 2. *For $m \geq 1$,*

$$G(0,2m+1) = \frac{-t_1(2m)+f_2(2m-1,m)}{2} + a_2(0,m) - \frac{1}{x}\frac{t_1(2m)}{t_1(2m-1)}\Big(\frac{t_1(2m-1)}{2} + b_2(0,m)\Big).$$

Considering the "$\frac{1}{x}$–part" and the "$\frac{1}{x}$–free part" of Proposition 2, it is immediate that showing $G(0,2m+1) = 0$ for all $m \geq 1$ is equivalent to proving the following identities for $m \geq 1$; namely,

$$\sum_{k=1}^{2m}(-1)^k f_2(2m-k,m) = \frac{t_1(2m)-f_2(2m-1,m)}{2} \tag{9}$$

and

$$\sum_{k=1}^{2m-1}(-1)^k f_1(2m-k-1,m) = -\frac{t_1(2m-1)}{2}. \tag{10}$$

Proposition 3. *For $m \geq 1$,*

$$\sum_{k=0}^{2m}(-1)^k f_2(2m-k,m) = \frac{1}{2}t_1(2m) \tag{11}$$

and

$$\sum_{k=0}^{2m-1}(-1)^k f_1(2m-k-1,m) = -\frac{1}{2}t_1(2m-1). \tag{12}$$

Proof: See Section 8. □

Lemma 2. *For $m \geq 1$,*

$$f_1(2m-1,m) = 0 \qquad \text{and} \qquad f_2(2m-1,m) = 2f_2(2m,m).$$

Proof: The first assertion is immediate. The proof of the second statement is elementary and is left to the reader. □

From Lemma 2 it is easily seen that (11) and (12) imply (9) and (10), respectively. These steps conclude the proof of

$$G(0,2m+1)=0 \text{ for } m\geq 1. \tag{13}$$

3.1.2. *Case B: $i=1$.* In this case, Proposition 1 turns into the following form.

Proposition 4. *For $m\geq 1$,*

$$G(1,2m+1)=a_2(1,m)-\frac{1}{x}\frac{t_1(2m)}{t_1(2m-1)}\Big(-t_1(2m-1)+b_2(1,m)\Big).$$

Considering the "$\frac{1}{x}$–part" and the "$\frac{1}{x}$–free part" of Proposition 4, it is immediate that showing $G(1,2m+1)=0$ $(m\geq 1)$ is equivalent to proving the following identities for $m\geq 1$; namely,

$$\sum_{k=1}^{2m}(-1)^k f_2(2m-k,m)=\frac{1}{2}\sum_{k=1}^{2m}2^k f_2(2m-k,m) \tag{14}$$

and

$$\sum_{k=1}^{2m-1}(2^{k-1}-(-1)^k)f_1(2m-k-1,m)=t_1(2m-1). \tag{15}$$

Proposition 5. *For $m\geq 1$,*

$$\sum_{k=0}^{2m}2^k f_2(2m-k,m)=t_1(2m)-f_2(2m,m) \tag{16}$$

and

$$\sum_{k=0}^{2m-1}2^k f_1(2m-k-1,m)=t_1(2m-1). \tag{17}$$

Proof: See Section 8. □

It is easily seen that (16) and (17) together with (9), (10) and Lemma 2 imply (14) and (15), respectively. These steps conclude the proof of

$$G(1,2m+1)=0 \text{ for } m\geq 1. \tag{18}$$

3.1.3. *Case C: $i\geq 2$.* In this case, Proposition 1 turns into the following form.

Proposition 6. *For $m\geq 1$ and $2\leq i\leq 2m+1$,*

$$\begin{aligned}G(i,2m+1)=-\big(2^{i-2}+(-1)^i\big)\frac{t_1(2m)+f_2(2m-1,m)}{2}-a_1(i,m)+a_2(i,m)\\ -\frac{1}{x}\frac{t_1(2m)}{t_1(2m-1)}\Big(-(2^{i-2}-(-1)^i)\frac{t_1(2m-1)}{2}-b_1(i,m)+b_2(i,m)\Big).\end{aligned}$$

Considering the "$\frac{1}{x}$–part" and the "$\frac{1}{x}$–free part" of Proposition 6, it is immediate that showing

$$\forall m\geq 1: \qquad G(i,2m+1)=\begin{cases}0, & \text{if } 2\leq i\leq 2m,\\ \frac{1}{x}t_1(2m)^2, & \text{if } i=2m+1\end{cases} \tag{19}$$

is equivalent to proving the following identities for $m\geq 1$; namely[1],

$$a_2(i,m)-a_1(i,m)=\big(2^{i-2}+(-1)^i\big)\frac{t_1(2m)+f_2(2m-1,m)}{2} \quad \text{for all } i \text{ s.t. } 2\leq i\leq 2m+1, \tag{20}$$

[1]Note: These bounds for i are sharp!

and

$$b_2(i,m) - b_1(i,m) = \left(2^{i-2} - (-1)^i\right)\frac{t_1(2m-1)}{2} \quad \text{for all } i \text{ s.t. } 2 \le i \le 2m, \tag{21}$$

and

$$b_2(2m+1,m) - b_1(2m+1,m) = -t_1(2m)t_1(2m-1) + \left(2^{2m-1}+1\right)\frac{t_1(2m-1)}{2}. \tag{22}$$

Ideally one would prove (20) to (22) directly with the Sigma package [Sch01, PS03, Sch04b, Sch04a]. However, it turns out that the computational complexity in this formulation of the problem is still too high. More precisely, the left hand sides of (20) and (21) are differences of two 4-fold sums, whereas the left hand side of (22) is a difference of two 3-fold sums. So, in order to be able to use Sigma, we transform the problem by an induction argument which is presented in the next subsection.

3.2. **Transforming the problem by induction.** The following transformation is based on an induction argument. It reduces the 4-fold sums on the left hand sides of (20) and (21) to 3-fold sums, and the 3-fold sums on the left hand side of (22) to 2-fold sums.

3.2.1. *Recurrences for the induction step.*

Definition 4. For $i, j \ge 1$ define

$$\alpha(i,j) := (-1)^{j-i-1} \qquad \text{and} \qquad \beta(i,j) := \sum_{s=i}^{j-1} \binom{i+j-2}{s}. \tag{23}$$

It is convenient to introduce the following lemma. Its proof is elementary.

Lemma 3. *For integers $i, j \ge 1$:*

$$\alpha(i+1,j) = -\alpha(i,j) \qquad \textit{and} \qquad \alpha(i,j+1) = -\alpha(i,j); \tag{24}$$

$$\beta(i+1,j) = 2\beta(i,j) - \binom{i+j-1}{i} \quad (1 \le i \le j-1); \tag{25}$$

$$\beta(i,j+1) = 2\beta(i,j) - \binom{i+j-1}{j} \quad (1 \le i \le j). \tag{26}$$

Lemma 3 implies the following recurrences for μ.

Proposition 7. *For integers*[2] *i, j:*

$$\mu(i+1,j) = 2\mu(i,j) + 3(-1)^{j-i} - \binom{i+j-1}{i} \qquad (1 \le i \le j-2); \tag{27}$$

$$\mu(i,j+1) = 2\mu(i,j) + 3(-1)^{j-i} + \binom{i+j-1}{j} \qquad (0 \le i \le j-1). \tag{28}$$

Proof: The definition of μ implies

$$\mu(i,j) = \alpha(i,j) + \beta(i,j) \qquad (1 \le i < j).$$

Hence Lemma 3 for $1 \le i \le j-2$ gives

$$\mu(i+1,j) = \alpha(i+1,j) + \beta(i+1,j) = -\alpha(i,j) + 2\beta(i,j) - \binom{i+j-1}{i}$$
$$= 2(\alpha(i,j) + \beta(i,j)) - 3\alpha(i,j) - \binom{i+j-1}{i},$$

which proves (27). Recurrence (28) is proved analogously. □

[2]Note that the ranges for i and j in both recurrences are sharp.

With Proposition 7 in hand one finds the relations of Proposition 8 and Proposition 9 which will be used in the induction step and which involve the following sums.

Definition 5. For $2 \leq i \leq 2m+1$ define

$$A_1(i,m) := \sum_{k=1}^{i-3} (-1)^k f_2(2m-k,m), \qquad B_1(i,m) := \sum_{k=1}^{i-3} (-1)^k f_1(2m-k-1,m),$$

$$A_2(i,m) := \sum_{k=i}^{2m} (-1)^k f_2(2m-k,m), \qquad B_2(i,m) := \sum_{k=i}^{2m-1} (-1)^k f_1(2m-k-1,m),$$

and

$$A_0(i,m) := \sum_{k=0}^{2m} \binom{i+k-3}{i-2} f_2(2m-k,m), \quad B_0(i,m) := \sum_{k=0}^{2m-1} \binom{i+k-3}{i-2} f_1(2m-k-1,m).$$

Proposition 8. *For* $3 \leq i \leq 2m+1$,

$$\begin{aligned} &a_2(i,m) - a_1(i,m) = 2(a_2(i-1,m) - a_1(i-1,m) \\ &\quad - \Big(f_2(2m-i+2,m) + 2f_2(2m-i+1,m)\Big) + 3(-1)^i\Big(A_2(i,m) - A_1(i,m)\Big) - A_0(i,m). \end{aligned} \tag{29}$$

Proof: Applying (27) and (28), respectively, yields

$$\begin{aligned} a_2(i,m) - a_1(i,m) &= \sum_{k=i}^{2m} \mu(i-1,k) f_2(2m-k,m) \\ &\qquad - \Big(\sum_{k=1}^{i-3} \mu(k,i-1) f_2(2m-k,m) + \mu(i-2,i-1) f_2(2m-i+2,m)\Big) \\ &= 2\Big(a_2(i-1,m) - \mu(i-2,i-1) f_2(2m-i+1,m)\Big) \\ &\quad + 3(-1)^i A_2(i,m) - \sum_{k=i}^{2m} \binom{i+k-3}{i-2} f_2(2m-k,m) - \Big[2a_1(i-1,m) + 3(-1)^i A_1(i,m) \\ &\qquad + \sum_{k=1}^{i-3} \binom{i+k-3}{i-2} f_2(2m-k,m) + \mu(i-2,i-1) f_2(2m-i+2,m)\Big] \\ &= 2\big(a_2(i-1,m) - a_1(i-1,m)\big) + 3(-1)^i\big(A_2(i,m) - A_1(i,m)\big) - A_0(i,m) \\ &\qquad + \Big(\binom{2i-5}{i-2} - \mu(i-2,i-1)\Big) f_2(2m-i+2,m) \\ &\qquad + \Big(\binom{2i-4}{i-2} - 2\mu(i-2,i-1)\Big) f_2(2m-i+1,m) \end{aligned}$$

which gives the right side of (29) by invoking the fact that

$$\mu(i-2,i-1) = 1 + \binom{2i-5}{i-2} \qquad (i \geq 3). \tag{30}$$

□

Proposition 9. *For* $3 \leq i \leq 2m+1$,

$$\begin{aligned} &b_2(i,m) - b_1(i,m) = 2\big(b_2(i-1,m) - b_1(i-1,m)\big) \\ &\qquad - \big(f_1(2m-i+1,m) + 2f_1(2m-i,m)\big) + 3(-1)^i\big(B_2(i,m) - B_1(i,m)\big) - B_0(i,m). \end{aligned} \tag{31}$$

Proof: The proof is completely analogous to that of Proposition 8. □

3.2.2. *Identities for the induction step.* Suppose we have proved the case $i = 2$ of (20) and (21). Then proving (20) and (21) in their full generality can be done by induction on i for fixed $m \geq 1$. Namely, due to Proposition 8, identity (20) is equivalent to showing the case $i = 2$ and that for any i with[3] $3 \leq i \leq 2m + 1$ the equality

$$\begin{aligned}&\frac{3}{2}(-1)^i\big(t_1(2m) + f_2(2m-1,m)\big)\\&= -\big(f_2(2m-i+2,m) + 2f_2(2m-i+1,m)\big) + 3(-1)^i\big(A_2(i,m) - A_1(i,m)\big) - A_0(i,m)\end{aligned} \tag{32}$$

holds. Analogously, because of Proposition 9, identity (21) is equivalent to showing the case $i = 2$ and that for any i with $3 \leq i \leq 2m$ the equality

$$\begin{aligned}&-\frac{3}{2}(-1)^i t_1(2m-1)\\&\quad= -\big(f_1(2m-i+1,m) + 2f_1(2m-i,m)\big) + 3(-1)^i\big(B_2(i,m) - B_1(i,m)\big) - B_0(i,m)\end{aligned} \tag{33}$$

holds.

The base case $i = 2$ of (20) and (21), respectively, simplifies as explained in Subsection 3.2.3 below. Identities (32) and (33) are proved in Sections 7 and 6, respectively.

Finally, the proof of (19) is completed by showing (22). Since (21) will be proved independently from it, we can proceed as follows. Applying (31) and then (21) gives

$$b_2(2m+1,m) - b_1(2m+1,m) = \big(2^{2m-2} - 1\big)t_1(2m-1) - 1 + 3B_1(2m+1,m) - B_0(2m+1,m).$$

By (12) and Lemma 1 one has

$$B_1(2m+1,m) = -\sum_{k=1}^{2m-2}(-1)^k f_1(k,m) = 1 - \frac{t_1(2m-1)}{2}. \tag{34}$$

Consequently the proof of (22) is equivalent to showing that for $m \geq 1$,

$$B_0(2m+1,m) = t_1(2m)t_1(2m-1) - 3t_1(2m-1) + 2. \tag{35}$$

Proof: This will be done in Sections 5 and 8. □

3.2.3. *The base case for $i = 2$.* The case $i = 2$ and $m \geq 1$ of identity (20) reads as

$$\sum_{k=2}^{2m}\mu(1,k)f_2(2m-k,m) = t_1(2m) + f_2(2m-1,m). \tag{36}$$

Because of

$$\mu(1,k) = 2^{k-1} + (-1)^k - 1 \qquad (k \geq 2), \tag{37}$$

and then (11) and (16), the left side of (36) turns into $-\sum_{k=0}^{2m-2} f_2(k,m) + t_1(2m) - 2f_2(2m,m)$. Consequently, identity (36) is implied by the following lemma.

Lemma 4. *For $m \geq 1$,*

$$\sum_{k=0}^{2m} f_2(k,m) = -f_2(2m,m). \tag{38}$$

Proof: This follows by identity (118) and Lemma 2. □

The case $i = 2$ and $m \geq 1$ of identity (21) reads as

$$\sum_{k=2}^{2m-1}\mu(1,k)f_1(2m-k-1,m) = 0. \tag{39}$$

[3] Note that these bounds for identity (32) are sharp; similarly for identity (33).

The case $m = 1$ is trivial. Because of (37) and then (12) and (16), the left side of (39) turns into $-\sum_{k=2}^{2m-1} f_1(2m-k-1, m)$. Consequently, identity (39) is implied by Lemma 2 and

Lemma 5. *For $m \geq 1$,*

$$\sum_{k=0}^{2m-1} f_1(2m-k-1, m) = f_1(2m-2, m). \tag{40}$$

Proof: See identity (97). □

3.3. **The even case: $j = 2m$.** Analogously to Proposition 1, from all the definitions we can derive the following representation.

Proposition 10. *For $m \geq 2$ and $0 \leq i \leq 2m$,*

$$\begin{aligned} G(i, 2m) &= \big(2\mu_1(i,0) + \mu_1(i,1)\big)\frac{t_1(2m-1)}{2} - b_1(i,m) + b_2(i,m) \\ &- x\frac{t_1(2m-1)}{t_1(2m-2)}\Big(\frac{\mu_1(i,1)}{2}\big(t_1(2m-2) + f_2(2m-3, m-1)\big) - a_1(i, m-1) + a_2(i, m-1)\Big). \end{aligned} \tag{41}$$

Suppose we have proved $G(i, 2m+1) = 0$ for $0 \leq i \leq 2m+1$. From Proposition 1 we know that the "x–free part" of the right side of (41) is 0 for $0 \leq i \leq 2m$, and also that the "x–part" of the right side of (41) is 0 for $0 \leq i \leq 2m-1$. Consequently, to complete the proof of

$$\forall m \geq 2: \qquad G(i, 2m) = \begin{cases} 0, & \text{if } 0 \leq i \leq 2m-1, \\ x t_1(2m-1)^2, & \text{if } i = 2m, \end{cases} \tag{42}$$

it remains to show that

$$\begin{aligned} - x\frac{t_1(2m-1)}{t_1(2m-2)}\Big[\frac{\mu_1(2m,1)}{2}\big(t_1(2m-2) + f_2(2m-3, m-1)\big) \\ + a_2(2m, m-1) - a_1(2m, m-1)\Big] = x t_1(2m-1)^2 \end{aligned} \tag{43}$$

for all $m \geq 2$.

If we define

$$a(m) := a_2(2m, m-1) - a_1(2m, m-1) \qquad (m \geq 2) \tag{44}$$

then (43) is equivalent to

$$a(m) = \big(2^{2m-3} + \frac{1}{2}\big)\big(t_1(2m-2) + f_2(2m-3, m-1)\big) - t_1(2m-1)t_1(2m-2) \qquad (m \geq 2). \tag{45}$$

On the other hand, from (29) with $m \to m-1$ and $i = 2m$ we obtain that

$$\begin{aligned} a(m) = 2\big(a_2(2m-1, m-1) - a_1(2m-1, m-1)\big) \\ - 1 - A_0(2m, m-1) - 3\sum_{k=1}^{2m-3}(-1)^k f_2(2m-k-2, m-1), \end{aligned}$$

which by (20) [with $m \to m-1$ and $i = 2m-1$] and by (11) can be simplified further to

$$\begin{aligned} a(m) = 2 - A_0(2m, m-1) + 3f_2(2m-2, m-1) \\ - \frac{3}{2}t_1(2m-2) + \big(2^{2m-3} - 1\big)\big(t_1(2m-2) + f_2(2m-3, m-1)\big). \end{aligned} \tag{46}$$

Comparing this to (45) and using the elementary property of f_2 described in Lemma 2, to prove identity (43) it suffices to show that for all $m \geq 2$,

$$\boxed{t_1(2m-1)t_1(2m-2) - 3t_1(2m-2) + 2 = A_0(2m, m-1).} \tag{47}$$

Proof: This will be done in Sections 5 and 8. □

4. The Proof Method

In this section we explain the method that we used for proving **all** of the identities stated in this article. The method consists in a combination of various algorithmic steps which are briefly described below. Remarkably most of the proving steps can be represented in the form of *proof certificates* which are identities that can be verified by elementary calculations independently from the way the algorithm has obtained them. This allows to produce compact descriptions of the proofs of the identities under consideration which can be found in the remaining sections.

Except the task of combining recurrences described in Subsection 4.1, all other algorithmic steps were executed by the computer algebra package Sigma [Sch01, PS03, Sch04b] developed by the third author. It should be noted that in order to handle the conjectured TSPP multiple sum identities, the Sigma tool-box was extended significantly. A detailed description of this work can be found in [Sch04a].

4.1. Combining Recurrences.

Hypergeometric sequences are special instances of P-finite sequences where the latter are defined to be sequences that satisfy a linear recurrence with polynomial coefficients. Such recurrences are called P-finite recurrences. P-finite sequences enjoy various closure properties; see [Sta80].

For our purpose we need to combine P-finite recurrences additively and multiplicatively, namely:

Given P-finite sequences $\big(a_n\big)$ and $\big(b_n\big)$ satisfying the recurrences

$$p_\alpha(n)a_{n+\alpha} + \cdots + p_0(n)a_n = 0 \quad (n \geq 0)^4$$

and

$$q_\beta(n)b_{n+\beta} + \cdots + q_0(n)b_n = 0 \quad (n \geq 0)$$

respectively, **compute** a P-finite recurrence

$$r_\gamma(n)c_{n+\gamma} + \cdots + r_0(n)c_n = 0 \quad (n \geq 0) \tag{48}$$

which is satisfied by the sum sequence $\big(c_n\big)$ with $c_n := a_n + b_n$. — The analogous product problem is with $c_n := a_n\, b_n$.

Algorithms which compute the recurrence (48) for the sum (resp. product) sequence $\big(c_n\big)$ are described in [SZ94]. For our computations we have used Mallinger's Mathematica package GeneratingFunctions [Mal96].

4.2. The General Proof Strategy.

All the identities we need to prove in this article are of the form

$$c_n^{(1)} + \cdots + c_n^{(k)} = 0 \quad (n \geq 0) \tag{49}$$

where k is a fixed positive integer and where each of the $c_n^{(i)}$ is a P-finite sequence. To prove that (49) is valid indeed for all $n \geq 0$ we proceed as follows. First we compute P-finite recurrences for all of the $c_n^{(i)}$, unless such recurrences are already given. Then, as described in Section 4.1, a P-finite recurrence for the sum sequence $s_n := c_n^{(1)} + \cdots + c_n^{(k)}$ is computed. Finally, we show that $s_n = 0$ for all $n \geq 0$ by checking sufficiently many initial values. — Note that the leading polynomial coefficient of the recurrence for s_n must not have any nonnegative integer root.

In our context the $c_n^{(i)}$ are given as hypergeometric sequences, as single, double and triple sums over hypergeometric sequences, or as the product of such sequences. So in view of Section 4.1 and of our general strategy described above, there remains the task to derive P-finite recurrences for such sums.

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[4]For the sake of simplicity we have chosen $n \geq 0$; in concrete cases the initial values might be different from 0.

Concerning summation ranges the following remark is in place. In this section, for the sake of simplicity we restrict ourselves to (multiple) sums where all summations are taken over finite summand supports. With this restriction *homogeneous* sum recurrences are guaranteed.

4.3. **Single Sums.** Here the basic task is as follows.

Given a summand $F(r,s)$ which is hypergeometric[5] in r and s, **compute** a P-finite recurrence

$$p_\gamma(r)f(r+\gamma)+\cdots+p_0(r)f(r)=0 \quad (n\geq 0) \tag{50}$$

which is satisfied by the sum $f(r):=\sum_s F(r,s)$.

In case that $F(r,s)$ satisfies some mild side conditions this problem can be solved by applying Zeilberger's algorithm [PWZ96] which computes polynomials $p_i(r)$, free of s, and $G(r,s)$ such that

$$p_\gamma(r)F(r+\gamma,s)+p_{\gamma-1}(r)F(r+\gamma-1,s)+\cdots+p_0(r)F(r,s)=\Delta_s G(r,s). \tag{51}$$

Note that Δ_s denotes the difference operator defined as usual by $\Delta_s G(s)=G(s+1)-G(s)$.

One can show that $G(r,s)$ is a rational function multiple of $F(r,s)$. Hence recurrence (50) is obtained from (51) by summation over all s. — Consequently, all what is needed to prove the correctness of (50) is the knowledge of (51) which is called "certificate recurrence". Note that after dividing (51) by $F(r,s)$, its verification reduces to checking equality of rational functions, a simple check which is independent from the way the algorithm obtained (51).

Remark: It can be that for a fixed order γ there exists only the trivial solution, i.e., where all the $p_i(r)$ are 0. In this case one has to increase the order γ incrementally until a non-trivial solution is computed whose existence is guaranteed by the theory explained in [PWZ96].

4.3.1. *A slight but important variation.* Many TSPP identities involve summands in more than one independent variable. For instance, instead of the summand $F(r,s)$ take the summand $F(n,r,s)$, now hypergeometric in r,s and n. For the following it is important to note that completely analogously to (51) one can compute

$$p'_\gamma(n,r)F(n+1,r,s)+p'_{\gamma-1}(n,r)F(n,r+\gamma-1,s)+\cdots+p'_0(n,r)F(n,r,s)=\Delta_s G'(n,r,s) \tag{52}$$

if it exists. Also for such cases one can prove that $G'(m,r,s)$ is a rational function multiple of $F(n,r,s)$. — Recurrences like (52) are related to contiguous relations [Pau04]; see also [Sch04a]. For instance, summing (52) over all s (assuming finite support) yields

$$p'_\gamma(n,r)f(n+1,r)+p'_{\gamma-1}(n,r)f(n,r+\gamma-1)+\cdots+p'_0(n,r)f(n,r)=0 \quad (n\geq 0) \tag{53}$$

with $f(n,r)=\sum_s F(n,r,s)$ and where the $p'_i(n,r)$ are polynomials in n and r.

Remark: There remains the question whether relations like (53) or (52) do exist. However, in [Pau04] an existence theory is presented which closely relates to the situation of Zeilberger's algorithm; for multiple sums in [PS04] this question is analysed in further details.

4.4. **Double Sums.** Here the basic task is as follows.

Given a summand $F(n,r,s)$ which is hypergeometric in n,r and s, **compute** a P-finite recurrence

$$p_\gamma(n)S(n+\gamma)+\cdots+p_0(n)S(n)=0 \quad (n\geq 0) \tag{54}$$

which is satisfied by the sum $S(n):=\sum_r\sum_s F(n,r,s)$.

In principle, one could apply the WZ method which is based on ideas of Sister Celine Fasenmyer and which is described in [PWZ96] . However, it turns out that all available implementations of this approach or of variations of it (e.g., Wegschaider 's algorithm [Weg97]) meet serious problems of computational complexity when applied to the TSPP identities in question. As a consequence we will follow a different approach which can be viewed as a new, surprisingly simple variant of Chyzak's algorithm [Chy00]. The basic ideas of this method are as follows; a full account of the details and a comparison to [Chy00] is given in [Sch04a].

[5] $F(r)$ is hypergeometric in r iff $F(r+1)/F(r)=g(r)$ for some fixed rational function $g(r)$.

The overall goal of the method is to compute a certificate recurrence of type (51), i.e.,

$$p_\gamma(n)f(n+\gamma,r)+\cdots+p_0(n)f(n,r)=\Delta_r g(n,r) \tag{55}$$

where $f(n,r)$ is defined to be the inner sum, i.e.,

$$f(n,r):=\sum_s F(n,r,s),$$

and where $g(n,r)$ is suitably chosen. From (55) the desired recurrence (54) for $S(n)$ is obtained by summing over all r — as in Zeilberger's algorithm for single sums.

In order to find (55) we proceed as follows. First one computes recurrences of the following form,

$$f(n,r+\gamma)=\lambda_0(n,r)f(n,r)+\cdots+\lambda_{\gamma-1}(n,r)f(n,r+\gamma-1), \tag{56}$$

and

$$f(n+1,r)=\mu_0(n,r)f(n,r)+\cdots+\mu_{\gamma-1}(n,r)f(n,r+\gamma-1), \tag{57}$$

where the $\lambda_i(n,r)$ and $\mu_i(n,r)$ are rational functions in n and r. This can be accomplished by following the description to compute (50) and (53) via (51) and (52), respectively.

Second, for $g(n,r)$ one chooses an ansatz with undetermined coefficients of the following form,

$$g(n,r)=\phi_0(n,r)f(n,r)+\cdots+\phi_{\gamma-1}(n,r)f(n,r+\gamma-1). \tag{58}$$

In the third step, the unknown polynomials $p_i(n)$, free of r, and the unknown rational function coefficients $\phi_i(n,r)$ for $g(n,r)$ are computed such that the certificate recurrence (55) holds. In view of (56) and (57), the key observation is that any shift in n and r of $f(n,r)$ and also $g(n,r)$ can be represented as a linear combination of $f(n,r),\dots,f(n,r+\gamma-1)$ over rational functions in n and r. Then rewriting both sides of (55) in terms of these generators, allows — in all our applications — to compute the unknown data by comparing the coefficients of all the $f(n,r+i)$ involved.

The corresponding computational steps are carried out as follows. For the sake of simplicity we restrict to $\gamma=2$; the general case works completely analogously and is described in [Sch04a]. From the relations (56) and (57) one can find rational functions $\nu_i(n,r)$ such that

$$f(n+2,r)=\nu_0(n,r)f(n,r)+\nu_1(n,r)f(n,r+1). \tag{59}$$

This together with (57) implies for the hand side of (55) that

$$\begin{aligned}&p_2(n)f(n+2,r)+p_1(n)f(n+1,r)+p_0(n)f(n,r)\\&=\big(p_0(n)+p_1(n)\mu_0(n,r)+p_2(n)\nu_0(n,r)\big)f(n,r)+\big(p_1(n)\mu_1(n,r)+p_2(n)\nu_1(n,r)\big)f(n,r+1).\end{aligned} \tag{60}$$

To represent the right hand side of (55) in terms of the generators $f(n,r+i)$ one invokes (56) which gives that

$$\begin{aligned}\Delta_r g(n,r)=\big(-\phi_0(n,r)+\phi_1(n,r+1)\lambda_0(n,r)\big)f(n,r)\\+\big(\phi_0(n,r+1)-\phi_1(n,r)+\phi_1(n,r+1)\lambda_1(n,r)\big)f(n,r+1).\end{aligned} \tag{61}$$

Finally comparing the coefficients of $f(n,r)$ and $f(n,r+1)$ on the right hand sides of (60) and (61), respectively, after triangulation leads to the problem of solving the system

$$\begin{aligned}&\lambda_0(n,r+1)\phi_1(n,r+2)+\lambda_1(n,r)\phi_1(n,r+1)-\phi_1(n,r)\\&\qquad=p_0(n)+\big(\mu_0(n,r+1)+\mu_1(n,r)\big)p_1(n)+\big(\nu_0(n,r+1)+\nu_1(n,r)\big)p_2(n)\end{aligned} \tag{62}$$

and

$$\phi_0(n,r)=\phi_1(n,r+1)\lambda_0(n,r)-\big(p_0(n)+p_1(n)\mu_0(n,r)+p_2(n)\nu_0(n,r)\big). \tag{63}$$

Equation (62) is a parameterized difference equation which has to be solved for a rational function $\phi_1(n,r)$ and for the polynomials $p_i(n)$. This is done by the Sigma package by using a refinement of Abramov's algorithm [Abr89]. Finally $\phi_0(n,r)$ is computed from (63). It is important to note that not only for $\gamma=2$, but also for general γ the approach works entirely the same. In particular, as pointed out in [Sch04a] triangularization of the system arising from this coefficient comparison can be avoided completely since the uncoupled system can be represented by a generic formula.

SUMMARY: The key identity for deriving a P-finite recurrence for the double sum $S(n) = \sum_r \sum_s F(n, r, s) = \sum_r f(n, r)$ is the certificate identity (55). Knowing (56) and (57) together with the $\phi_i(n, r)$ in (58), the reader can check the correctness of (55) independently from the steps of the method. Note that the correctness of (56) and (57) can be verified by standard creative telescoping. As a consequence, to certify that double sums satisfy certain P-finite recurrences, in the remaining sections we restrict ourselves to provide the data contained in (54), (55), (56), and (57).

4.5. **Triple Sums.** Based on what we said about single and double sums we are in the position to solve the following problem.

Given a summand $F(m, n, r, s)$ which is hypergeometric in m, n, r and s, **compute** a P-finite recurrence

$$p_\gamma(m)S(m + \gamma) + \cdots + p_0(m)S(m) = 0 \quad (m \geq 0) \tag{64}$$

which is satisfied by the sum $S(m) := \sum_n \sum_r \sum_s F(m, n, r, s)$.

As with double sums the overall goal of the method is to compute a certificate recurrence of the form

$$p_\gamma(m)h(m + \gamma, n) + \cdots + p_0(m)h(m, n) = \Delta_n g(m, n) \tag{65}$$

where we define $h(m, n)$ as

$$h(m, n) := \sum_r \sum_s F(m, n, r, s), \tag{66}$$

and where $g(m, n)$ is suitably chosen. Then from (65) the desired recurrence (64) for $S(m)$ is obtained by summation over all n.

To find (65) we proceed analogously to the double sum case. Namely, we first derive recurrences of the form

$$h(m, n + \gamma) = \lambda_0(m, n)h(m, n) + \cdots + \lambda_{\gamma-1}(m, n)h(m, n + \gamma - 1), \tag{67}$$

and

$$h(m + 1, n) = \mu_0(m, n)h(m, n) + \cdots + \mu_{\gamma-1}(m, n)h(m, n + \gamma - 1), \tag{68}$$

and afterwards we apply the same method as in the double sum case in order to compute all the components for the certificate recurrence (65). In particular, due to our ansatz, $g(m, n)$ will be of the form

$$g(m, n) = \phi_0(m, n)h(m, n) + \cdots + \phi_{\gamma-1}(m, n)h(m, n + \gamma - 1), \tag{69}$$

where the $\phi_i(m, n)$, $\lambda_i(m, m)$ and $\mu_i(m, n)$ are rational functions in m and n.

Obviously, from (67), (68) and (69) the correctness of (65) can be verified independently from the steps of our algorithm.

In order to apply the above strategy there remains the task to compute the recurrences (67) and (68). In principle, we could apply our techniques from above. Namely, with our description from Subsection 4.4 we can obtain a recurrence of the type (67) for the double sum (66). Similarly we can derive a recurrence of the form (68) by a slight variation of the same strategy which is described in [Sch04a]. Roughly spoken, this way we reduce triple summation first to double and then to single summation by recursion. Summarizing, we have a general method in hand to derive (67) and (68). But, by observing that the summand in the given TSPP triple sums are all of the type

$$h(m, n) = \sum_r H(m, n, r) \sum_{s=0}^{r} F(m, n, s)$$

where $H(m, n, r)$ is hypergeometric in m, n and r and where $F(m, n, s)$ is free of r, we can follow a more direct approach. Namely, as Zeilberger's algorithm can compute the required recurrences (56) and (57) for the double sum case, Sigma can produce in an analogous fashion the recurrences (67) and (68) together with recurrence certificates for $h(m,n)$.

5. A first step to prove Identities (35) and (47)

First observe that

$$t_1(2m) = 2\prod_{i=1}^{m}\frac{3(6i-5)(6i-1)}{4(4i-3)(4i-1)} \qquad \text{and} \qquad t_1(2m-1) = -\frac{2}{3}\prod_{i=1}^{m}\frac{-3(3i-4)(3i-2)}{(4i-5)(4i-3)},$$

i.e., they satisfy the recurrences

$$-3(6m+1)(6m+5)t_1(2m) + 4(4m+1)(4m+3)t_1(2m+2) = 0 \tag{70}$$

and

$$-3(3m-1)(3m+1)t_1(2m-1) + (4m-1)(4m+1)t_1(2m+1) = 0 \tag{71}$$

for all $m \geq 1$.

Using Maple's gfun package [SZ94] or the Mathematica package GeneratingFunctions described in [Mal96], from (70) and (71) we can derive a recurrence which is satisfied by the right side of (35). More precisely, first we combine (70) and (71) into the recurrence

$$-9(3m-1)(3m+1)(6m+1)(6m+5)P(m) + 4(4m-1)(4m+1)^2(4m+3)P(1+m) = 0$$

which is satisfied by the sequence $P(m) := t_1(2m)t_1(2m-1)$ for all $m \geq 1$. Next, we combine this recurrence with (71) to obtain the recurrence

$$\begin{aligned}
&27(-1+3m)(1+3m)(2+3m)(4+3m)(1+6m)(5+6m)(91+132m+44m^2)S(m)\\
&-3(2+3m)(4+3m)(-1+4m)(1+4m)(1785+18544m+39544m^2+30272m^3+7568m^4)S(1+m)\\
&+4(-1+4m)(1+4m)(3+4m)(5+4m)^2(7+4m)(3+44m+44m^2)S(2+m) = 0
\end{aligned} \tag{72}$$

which is satisfied by $S(m) := P(m) - 3t_1(2m-1)$. Finally, in order to add the constant sequence 2, we combine (72) with the recurrence $C(m) - C(m+1) = 0$ and obtain

$$\begin{aligned}
&27(-1+3m)(1+3m)(2+3m)(4+3m)(1+6m)(5+6m)(35582085+208705770m+502426266m^2\\
&+659838718m^3+522259397m^4+256674880m^5+76794344m^6+12820192m^7+915728m^8)b(m)\\
&-3(2+3m)(4+3m)(-5091730875-64116259830m-268357073220m^2\\
&-344918244168m^3+764136680690m^4+3625956026718m^5+6449149601689m^6\\
&+6689203564428m^7+4423817173116m^8+1893138648192m^9\\
&+508247453296m^{10}+77855194560m^{11}+5190346304m^{12})b(m+1)\\
&+(3+4m)(5+4m)(-7125148800-88649425620m-376507014210m^2\\
&-591098983182m^3+431261276465m^4+3340499936442m^5\\
&+6290733211249m^6+6633648559692m^7+4412582090796m^8\\
&+1892139974208m^9+508247453296m^{10}+77855194560m^{11}+5190346304m^{12})b(m+2)\\
&-4(3+4m)(5+4m)(7+4m)(9+4m)^2(11+4m)(-61740-644340m-1950794m^2-910998m^3\\
&+6194397m^4+13852080m^5+12693384m^6+5494368m^7+915728m^8)b(m+3) = 0
\end{aligned} \tag{73}$$

that is satisfied by $b(m) := S(m) + 2 = t_1(2m)t_1(2m-1) - 3t_1(2m-1) + 2$ for $m \geq 1$.

In Section 8 we show that $B_0(2m+1,m)$ satisfies the same recurrence (73) for $m \geq 1$. Since the coefficient of $b(m+3)$ in (73) is nonzero for all $m \geq 1$ and since $b(m)$ and $B_0(2m+1,m)$ are equal for $m \in \{1,2,3\}$, this will prove identity (35).

In complete analogous fashion one derives the recurrence

$$\begin{aligned}
&27(2+3m)(4+3m)(1+6m)(5+6m)(7+6m)(11+6m)(389987325\\
&\quad+1563210054m+2677225618m^2+2567677266m^3+1512020037m^4\\
&\quad+560774016m^5+128075112m^6+16483104m^7+915728m^8)a(m)\\
&-3(7+6m)(11+6m)(188256814200+3218702456970m\\
&\quad+17894686721409m^2+51759068834382m^3+91965480308606m^4\\
&\quad+108563491759548m^5+88575755915749m^6+50780656348884m^7\\
&\quad+20427064666140m^8+5647619363232m^9+1022091737392m^{10}\\
&\quad+108997272384m^{11}+5190346304m^{12})a(1+m)\\
&+4(5+4m)(7+4m)(83466403020+2609224361478m+16370962889247m^2\\
&\quad+49594957358604m^3+90028764304859m^4+107424612589224m^5\\
&\quad+88133765355661m^6+50671172949012m^7+20411335550892m^8\\
&\quad+5646620689248m^9+1022091737392m^{10}+108997272384m^{11}+5190346304m^{12})a(2+m)\\
&-16(5+4m)(7+4m)(9+4m)(11+4m)^2(13+4m)(79380+7185780m\\
&\quad+39195762m^2+91462910m^3+116468957m^4+87187760m^5\\
&\quad+38333768m^6+9157280m^7+915728m^8)a(3+m)=0 \qquad (74)
\end{aligned}$$

which is satisfied by $a(m) := t_1(2m+1)t_1(2m) - 3t_1(2m) + 2$ for all $m \geq 1$. In Section 8 we will show that this recurrence is also satisfied by $A_0(2m+2, m)$. Since the coefficient of $a(m+3)$ is nonzero for $m \geq 1$ and since $a(m)$ and $A_0(2m+2, m)$ are equal for the first three initial values $m \in \{1, 2, 3\}$, identity (47) must hold for all $m \geq 2$.

6. A Proof of Identity (33)

In this section we will use the abbreviation $h_1(k, m) := f_1(2m-k-1, m)$. By Lemma 2 we may write

$$B_1(i,m) = \sum_{k=0}^{i-3} (-1)^k h_1(k,m). \qquad (75)$$

Moreover the sum in Proposition 3 can be rewritten as

$$-\frac{1}{2}t_1(2m-1) = \sum_{k=0}^{2m-1} (-1)^k h_1(k,m) = B_1(i,m) - (-1)^i\, h_1(i-1,m) + (-1)^i\, h_1(i-2,m) + B_2(i,m).$$

Consequently to show identity (33) is equivalent to prove

$$\boxed{6(-1)^i B_2(i,m) - B_0(i,m) + 2h_1(i-2,m) - 5h_1(i-1,m) + 3(-1)^i t_1(2m-1) = 0, \qquad (76)}$$

or, equivalently,

$$\boxed{6(-1)^i B_1(i,m) + B_0(i,m) + 4h_1(i-2,m) - h_1(i-1,m) = 0 \qquad (77)}$$

for all $2 \leq i \leq 2m$.

Now it suffices to verify one of those two identities. To this end we will follow the same proof strategy as sketched in Section 5: derive recurrences for all the ingredients in (77), namely $h_1(i-2, m)$, $h_1(i-1, m)$, $B_0(i, m)$, and $B_1(i, m)$, or for all objects in (76), namely $h_1(i-2, m)$, $h_1(i-1, m)$, $B_0(i, m)$, and $B_2(i, m)$, and combine them into one homogeneous recurrence that contains the left side of (77), respectively (76) as solution. For the sake of simplicity we refrain from computing a recurrence for $B_1(i, m)$ (see Remark 2) and therefore compute a recurrence for the left side in (76).

However, as shown below, for a full proof of (76) we also need to validate that certain instances of (77) hold.

Using Sigma [Sch01] we first compute a recurrence for $h_1(k,m)$ that holds for all $m, k \geq 0$, namely

$$\begin{aligned}
&-2(2+k)^2(1+k-2m)(k+2m)h_1(k,m)\\
&+(29k^3+5k^4+k(46+20m-40m^2)+k^2(58+6m-12m^2)+12(1+m-2m^2))h_1(1+k,m)\\
&-(26k^3+4k^4+k(55+14m-28m^2)+k^2(59+6m-12m^2)+6(3+m-2m^2))h_1(2+k,m)\\
&\qquad+(1+k)^2(3+k-2m)(2+k+2m)h_1(3+k,m)=0 \quad (78)
\end{aligned}$$

Note that (78) can be computed with any package like [PS95] that implements Zeilberger's algorithm [Zei90].

Remark 1. In order to derive this recurrence, we first tried to apply Zeilberger's algorithm on the input of $h_1(k,m) = \sum_{s=0}^{\lfloor \frac{2m-k-1}{2} \rfloor} q(s,k,m)$ with

$$q(s,k,m) = \frac{(-1)^{s+k-1}\binom{m-s-1}{2m-2s-k-1}(1-3m)_s(m)_s}{(3/2-2m)_s\, s!\, 4^s}\frac{3m-3s-1}{3m-1}. \quad (79)$$

More precisely, we computed constants $c_i(k,m) \in \mathbb{Q}[k,m]$, free of s, and $g(s,k,m) := r(s,k,m)q(s,k,m)$ given by a rational function $r(s,k,m) \in \mathbb{Q}(s,k,m)$ such that

$$g(s+1,k,m) - g(s,k,m) = c_0(k,m)q(s,k,m) + \cdots + c_3(k,m)q(s,k+3,m).$$

The usual strategy is now to sum this "creative telescoping equation" over s from 0 to $\lfloor \frac{2m-k-1}{2} \rfloor$, in order to obtain a recurrence for $h_1(k,m)$. But we failed, since the denominator of $q(s,k,m)$ contains the factors $(m-s-k-1)(m-s-l-2)$ which causes poles in the given summation range. By rewriting $q(s,k,m)$ to $q'(s,k,m)$ by replacing $\binom{m-s-1}{2m-2s-k-1}$ with $\frac{(m-s-k-1)(m-s-k-2)}{(2m-2s-k-1)(2m-2s-k-2)}\binom{m-s-1}{2m-2s-k-3}$ these poles can be eliminated in the result of Zeilberger's algorithm. But in this case the sum $h_1(k,m)$ can be only defined in the summation range $0 \leq s \leq \lfloor \frac{2m-k-1}{2} \rfloor - 1$. i.e., $h_1'(k,m) := \sum_{s=0}^{\lfloor \frac{2m-k-1}{2} \rfloor -1} q'(s,k,m)$. Applying this representation of the definite sum $h_1'(k,m)$ to Zeilberger's algorithm gives the creative telescoping equation

$$g(s+1,k,m) - g(s,k,m) = c_0(k,m)q'(s,k,m) + \cdots + c_3(k,m)q'(s,k+3,m). \quad (80)$$

with

$$\begin{aligned}
c_0(k,m) =& -2(2+k)^2(1+k-2m)(k+2m)(-1+3m),\\
c_1(k,m) =&(1-3m)(-29k^3-5k^4-12((1+m-2m^2)+2k^2(-29-3m+6m^2)\\
&+2k(-23-10m+20m^2)),\\
c_2(k,m) =&(-1+3m)(-26k^3-4k^4-6((3+m-2m^2)+k^2(-59-6m+12m^2)\\
&+k(-55-14m+28m^2)),\\
c_3(k,m) =&(1+k)^2(3+k-2m)(2+k+2m)((-1+3m),
\end{aligned}$$

and

$$\begin{aligned}
g(s,k,m) =& \frac{(-1)^{s+k-1}\binom{m-s-1}{2m-2s-k-3}(1-3m)_s(m)_s}{(3/2-2m)_s\, s!\, 4^s}\\
&\times \frac{-2(3+k)(-1+4m-2s)(m-s)s(3k+2k^2+4m+6km+3k^2m-4s-6ks-3k^2s)}{(2m-2s-k-1)(2m-2s-k-2)}.
\end{aligned}$$

Now we have to check that this summand telescoping equation (80) holds within the summation range. For this denote $p(s,k,m) = (-1)^k\binom{m-s-1}{2m-2s-k-3}$. In a first attempt we tried to express $p(s+1,k,m)$, that occurs in $g(s+1,k,m)$, and $p(s,k+i,m)$, that occurs in $q(s,k+i,m)$, in terms of $p(s,k,m)$ times a rational function in $\mathbb{Q}(s,k,m)$. But we failed to show (80) in this representation, since we entered again into pole problems within our summation range $0 \leq s \leq \lfloor \frac{2m-k-1}{2} \rfloor - 1$:

for instance we have $p(s,k+1,m)=\frac{2m-2s-k-1}{m-s-k-1}p(s,k,m)$. In order to avoid this, we represent all ingredients in (80) not in terms of $p(s,k,m)$ but in terms of $p(s+1,k+3,m)$, i.e.,

$$p(s+1,k,m)=p(s+1,k+3,m)\prod_{j=0}^{2}\frac{m-s-k-j-4}{2m-2s-k-j-5}, \tag{81}$$

$$p(s,k+i,m)=-p(s+1,k+3,m)\frac{m-s-1}{2m-2s-k-7}\prod_{j=0}^{3-i}\frac{m-s-k+j-6}{2m-2s-k+j-6}, \quad 0\leq i\leq 3. \tag{82}$$

With these representations in $p(s+1,k+3,m)$ we finally manage to verify that (80) holds for $0\leq s\leq\lfloor\frac{2m-k-1}{2}\rfloor-4$.

Hence, summing equation (80) over s from 0 to an arbitrary d with $0\leq d\leq\lfloor\frac{2m-k-1}{2}\rfloor-4$ gives the relation

$$g(d+1,k,m)-g(0,k,m)=c_0(k,m)\,y(k,m)+\cdots+c_3\,y(k+3,m) \tag{83}$$

for the sum $y(k,m)=\sum_{s=0}^{d}q(s,k,m)=\sum_{s=0}^{d}q'(s,k,m)$. Now we do a case distinction on k.

If k is even, we can choose $d=m-k/2-5$, and obtain

$$\begin{aligned}&c_0(k,m)\,h_1(k,m)+\cdots+c_3(k,m)\,h_1(k+3,m)\\&\qquad=g(m-\frac{k}{2}-4,k,m)-g(0,k,m)+\sum_{i=0}^{3}c_i(k,m)\sum_{j=1}^{4}q(m-\frac{k}{2}-j,k+i,m).\end{aligned}$$

By term rewriting one can now show that the right hand side is equal to 0. Hence for $m-k/2-5\geq 0$, k even, the relation (78) holds. Similarly, for odd k, one can choose $d=m-(k-1)/2-4$, and obtains

$$\begin{aligned}&c_0(k,m)\,h_1(k,m)+\cdots+c_3(k,m)\,h_1(k+3,m)\\&\qquad=g(m-\frac{k-1}{2}-3,k,m)-g(0,k,m)+\sum_{i=0}^{3}c_i(k,m)\sum_{j=0}^{3}q(m-\frac{k-1}{2}-j,k+i,m).\end{aligned}$$

Again one can prove that the right hand side vanishes to 0, and hence the relation (78) holds for $m-(k-1)/2-4\geq 0$, k odd. Summarizing, for all $m\geq 5$ and $0\leq k\leq 2m-10$ the recurrence (78) contains the solution $h_1(k,m)$. The special cases $0\leq m\leq 5$ and $k\geq 2m-9$ for $m\geq 5$ can be verified by simple evaluation. This shows that the recurrence holds for all $m,k\geq 0$. ◊

We want to emphasize that recurrence (78) enables us to describe the k-shifts $h_1(k+j,m)$, $j\geq 3$, by linear combinations in $h_1(k,m)$, $h_1(k+1,m)$ and $h_1(k+2,m)$. Given this information, in a next step we can obtain with Sigma a recurrence for B_0, namely,

$$\begin{aligned}&(-2-i-i^2)(-1+i-2m)(-2+i+2m)B_0(i,m)\\&\quad+(3+i)(-2+2i-i^2+i^3-2m+4m^2)B_0(i+1,m)\\&\quad+(-3+i)(2+2i+i^2+i^3+2m-4m^2)B_0(i+2,m)\\&\quad+(-2+i-i^2)(2+i-2m)(1+i+2m)B_0(i+3,m)=0\end{aligned}\tag{84}$$

which holds for $2\leq i\leq 2m$.

Remark 2. More precisely, Sigma is able to compute constants

$$\begin{aligned}c_0(i,m)&=(1-i)i(1+i)(2+i+i^2)(-1+i-2m)(-2+i+2m),\\c_1(i,m)&=(-1+i)i(1+i)(3+i)(-2+2i-i^2+i^3-2m+4m^2),\\c_2(i,m)&=(-3+i)(-1+i)i(1+i)(2+2i+i^2+i^3+2m-4m^2),\\c_3(i,m)&=(1-i)i(1+i)(2-i+i^2)(2+i-2m)(1+i+2m),\end{aligned}$$

and

$$g(k,i,m)=\frac{p_0(i,m)h_1(k,i,m)+p_1(i,m)h_1(k+1,i,m)+p_2(i,m)h_1(k+2,i,m)}{(1+k)^2}(k-1)\binom{i+k-3}{i-2} \tag{85}$$

where

$$\begin{aligned}
p_0(i,m)=&-(12-6i-12i^2+6i^3+26k-82ik+7i^2k+16i^3k-3i^4k-24k^2\\
&-106ik^2+73i^2k^2-4i^3k^2+i^4k^2-60k^3+16ik^3+59i^2k^3-16i^3k^3\\
&+9i^4k^3-12k^4+60ik^4-i^2k^4+4i^3k^4+5i^4k^4+14k^5+12ik^5-4i^2k^5+6i^3k^5+4k^6\\
&-2ik^6+2i^2k^6+12m-90im+54i^2m+10i^3m-2i^4m-46km-20ikm\\
&+76i^2km-14i^3km+8i^4km-24k^2m+64ik^2m-4i^2k^2m+2i^3k^2m+6i^4k^2m\\
&+14k^3m+12ik^3m-4i^2k^3m+6i^3k^3m+4k^4m-2ik^4m+2i^2k^4m-24m^2+180im^2\\
&-108i^2m^2-20i^3m^2+4i^4m^2+92km^2+40ikm^2-152i^2km^2+28i^3km^2-16i^4km^2\\
&+48k^2m^2-128ik^2m^2+8i^2k^2m^2-4i^3k^2m^2-12i^4k^2m^2-28k^3m^2-24ik^3m^2+8i^2k^3m^2\\
&-12i^3k^3m^2-8k^4m^2+4ik^4m^2-4i^2k^4m^2),\\
p_1(i,m)=&(-2+i+k)(-10+10i^2-32k+39ik+22i^2k-i^3k-11k^2+85ik^2+5i^2k^2\\
&+5i^3k^2+34k^3+53ik^3-7i^2k^3+10i^3k^3+29k^4+4ik^4+3i^2k^4+4i^3k^4+6k^5-3ik^5+3i^2k^5\\
&-10m+18im+10i^2m-2i^3m-2km+40ikm+4i^2km+2i^3km+20k^2m+16ik^2m\\
&-6i^2k^2m+6i^3k^2m+8k^3m-4ik^3m+4i^2k^3m+20m^2-36im^2-20i^2m^2\\
&+4i^3m^2+4km^2-80ikm^2-8i^2km^2-4i^3km^2-40k^2m^2-32ik^2m^2\\
&+12i^2k^2m^2-12i^3k^2m^2-16k^3m^2+8ik^3m^2-8i^2k^3m^2),\\
p_2(i,m)=&-(-2+i+k)(-1+i+k)(2+2i+5k+ik+2k^2\\
&-ik^2+i^2k^2)(2+k-2m)(1+k+2m).
\end{aligned}$$

This information is sufficient to produce (84). Namely, suppose that $m\geq 2$. Then for $q(k,i,m)=h_1(k,m)\binom{i+k-3}{i-2}$ we have that

$$g(k+1,i,m)-g(k,i,m)=c_0(i,m)\,q(k,i,m)+\cdots+c_3(i,m)\,q(k,i+3,m) \tag{86}$$

for $0\leq k<2m-3$ and $2\leq i\leq 2m$ which can be verified as follows. By applying the shift in k to the representation (85) of $g(k,i,m)$ the expression $g(k+1,i,m)$ depends on $\binom{i+k-2}{i-2}$, $h_1(k+1,m)$, $h_1(k+2,m)$, and $h_1(k+3,m)$. Then the term $k\binom{i+k-2}{i-2}$ can be rewritten with $(i+k-2)\binom{i+k-3}{i-2}$, and $h_1(k+3,m)$ can be expressed by $h_1(k,m)$, $h_1(k+1,m)$ and $h_1(k+2,m)$ by applying the relation given in (78) for any $0\leq k<2m-3$. Moreover, $q(k,i+j,m)=q_j\,q(k,i,m)$ with $q_1=\frac{i+k-2}{i-1}$, $q_2=\frac{(i+k-2)(i+k-1)}{(i-1)i}$, and $q_3=\frac{(i+k-2)(i+k-1)(i+k)}{(i-1)i(i+1)}$. In this representation the equation (86) can be verified by simple term rewriting for $i\geq 2$. Summing equation (86) over k from 0 to $2m-4$ gives

$$\begin{aligned}
&c_0(i,m)\,B_0(i,m)(k,m)+\cdots+c_3(i,m)\,B_0(i+3,m)\\
&\qquad=g(2m-3,i,m)-g(0,i,m)+\sum_{j=0}^{3}c_j(i,m)\sum_{r=2m-3}^{2m-1}\binom{i+r-3}{i-2}h_1(r,m).
\end{aligned}$$

Observe that under the assumption $2\leq i\leq 2m$ in $g(2m-3,i,m)$ the binomial $\binom{2m+i-7}{i-2}$ is well defined for $m>2$. In this case one can show that the right hand side vanishes to 0 which gives recurrence (84). Since $B_0(i,m)$ evaluates to a polynomial in $\mathbb{Q}[i]$ for any $m\in\mathbb{N}$, the special cases $m=1,2$ follow by plugging in those polynomials for the specific evaluations at $m=1,2$ in $\sum_{r=0}^{3}c_r(i,m)\,B_0(i+r,m)$ and showing that these expressions are the 0 polynomial. Summarizing, we have proven that $B_0(i,m)$ is a solution of recurrence (84) for all $2\leq i\leq 2m$. ◊

Next we compute a recurrence for $B_2(i,m)$, namely

$$\begin{aligned}2(2+i)(3+i)(1+i-2m)(i+2m)B_2(i,m)\\ +(3+i)(3i^2+i^3+8m(-1+2m)+i(2-2m+4m^2))B_2(i+1,m)\\ -2(1+i)(24+14i^2+2i^3+5m-10m^2+i(32+m-2m^2))B_2(i+2,m)\\ -2(2+i)(6+6i^2+i^3+i(11+2m-4m^2))B_2(i+3,m)\\ +2(1+i)(3+i)(8+6i+i^2+m-2m^2)B_2(i+4,m)\\ +(1+i)(2+i)(4+i-2m)(3+i+2m)B_2(i+5,m)=0 \quad (87)\end{aligned}$$

for all $2\leq i\leq 2m$.

Remark 3. In order to achieve this, we actually compute a creative telescoping equation for the sum

$$B_2^{(e)}(i,m,e):=\sum_{k=0}^{2m-4}q(k,i,m,e)$$

where

$$q(k,i,m,e)=(-1)^k\frac{(k-i+e)!}{(k-i)!}h_1(k,m).$$

Namely, with Sigma we can compute

$$\begin{aligned}c_0(i,m,e):=&2(e^2-e(7+3i)+3(6+5i+i^2))(1+i-2m)(i+2m),\\ c_1(i,m,e):=&4e^3(1+i)-e^2(26+37i+11i^2+2m-4m^2)+e(58+56i^2+9i^3+14m-28m^2\\ &+i(105+6m-12m^2))+3(3+i)(3i^2+i^3+8m(-1+2m)+i(2-2m+4m^2)),\\ c_2(i,m,e):=&-4e^3+2e^4-e^2(37+51i+16i^2+2m-4m^2)-6(1+i)(24+14i^2+2i^3+5m-10m^2\\ &+i(32+m-2m^2))+e(165+169i^2+30i^3+26m-52m^2+6i(50+m-2m^2)),\\ c_3(i,m,e):=&5e^4-e^3(35+17i)+e^2(82+82i+19i^2-4m+8m^2)\\ &-6(2+i)(6+6i^2+i^3+i(11+2m-4m^2))-2e(17+5i^2-5m+10m^2+i(19-6m+12m^2)),\\ c_4(i,m,e):=&4e^4-2e^3(19+9i)-e(7+3i)(33+37i+8i^2+2m-4m^2)+e^2(139+139i+32i^2+2m-4m^2)\\ &+6(3+4i+i^2)(8+6i+i^2+m-2m^2),\\ c_5(i,m,e):=&(e^2-e(4+3i)+3(2+3i+i^2))(-3+e-i-2m)(-4+e-i+2m),\end{aligned}$$

and

$$\begin{aligned}g(k,i,m,e)=-(p_0(k,i,m,e)h_1(k,i,m)+p_1(k,i,m,e)h_1(k{+}1,i,m)+p_2(k,i,m,e)h_1(k{+}2,i,m))\\ \times\frac{q(k,i,e)}{d(k,i,e)}(-1)^k\frac{(k-i+e)!}{(k-i)!}\end{aligned}$$

where

$$\begin{aligned}q(k,i,e)&:=(-1+e)e(i-k),\\ d(k,i,e)&:=(1+k)^2(-4+e-i+k)(-3+e-i+k)(-2+e-i+k)(-1+e-i+k)(e-i+k),\end{aligned}$$

and

$$
\begin{aligned}
p_0(k,i,m,e) = &-((e^4(9k^3+5k^4+2m(-1+2m)+k(-3+8m-16m^2)+k^2(1+6m-12m^2))-e^3(3-40k+20k^2\\
&+126k^3+49k^4-14k^5-26m+122km+66k^2m-18k^3m+52m^2-244km^2-132k^2m^2+36k^3m^2+i(3-10k+10k^2\\
&+52k^3+29k^4-8m+48km+36k^2m+16m^2-96km^2-72k^2m^2))+e^2(33-202k+146k^2+688k^3+165k^4\\
&-132k^5+10k^6-108m+720km+258k^2m-176k^3m+14k^4m+216m^2-1440km^2-516k^2m^2+352k^3m^2-28k^4m^2\\
&+i^2(12-2k+45k^2+126k^3+67k^4+2m+124km+86k^2m-4m^2-248km^2-172k^2m^2)+i(45-87k+160k^2+579k^3\\
&+225k^4-62k^5-42m+590km+310k^2m-82k^3m+84m^2-1180km^2-620k^2m^2+164k^3m^2))-e(120-447k+459k^2\\
&+1681k^3+153k^4-442k^5+60k^6-150m+1894km+362k^2m-606k^3m+84k^4m+300m^2-3788km^2-724k^2m^2+1212k^3m^2\\
&-168k^4m^2+i^3(15+14k+69k^2+142k^3+72k^4+28m+152km+96k^2m-56m^2-304km^2-192k^2m^2)+i^2(102+13k+405k^2\\
&+933k^3+337k^4-102k^5+110m+1018km+482k^2m-138k^3m-220m^2-2036km^2-964k^2m^2+276k^3m^2)+i(207-238k\\
&+765k^2+2116k^3+454k^4-422k^5+30k^6+40m+2358km+754k^2m-574k^3m+42k^4m-80m^2-4716km^2-1508k^2m^2\\
&+1148k^3m^2-84k^4m^2))+2(72-180k+262k^2+753k^3-65k^4-249k^5+55k^6+917km+7k^2m-353k^3m+77k^4m-1834km^2\\
&-14k^2m^2+706k^3m^2-154k^4m^2+i^4(3+4k+16k^2+30k^3+15k^4+11m+36km+21k^2m-22m^2-72km^2-42k^2m^2)\\
&+i^3(30+28k+141k^2+260k^3+87k^4-30k^5+83m+310km+129k^2m-42k^3m-166m^2-620km^2-258k^2m^2\\
&+84k^3m^2)+i(150-100k+571k^2+1256k^3+17k^4-358k^5+60k^6+169m+205m+1017km+258k^2m-249k^3m+21k^4m\\
&-410m^2-2034km^2-516k^2m^2+498k^3m^2-42k^4m^2)))),
\end{aligned}
$$

$$
\begin{aligned}
p_1(k,i,m,e) = &(-4+e-i+k)(-60-88k-257k^2-291k^3-7k^4+55k^5+18m-148km-174k^2m+88k^3m-36m^2\\
&+296km^2+348k^2m^2-176k^3m^2-i^3(10+32k+55k^2+48k^3+15k^4+10m+30km+24k^2m-20m^2-60km^2\\
&-48k^2m^2)-3i^2(20+56k+97k^2+83k^3+17k^4-5k^5+14m+52km+42k^2m-8k^3m-28m^2-104km^2-84k^2m^2\\
&16k^3m^2)-i(110+256k+497k^2+450k^3+39k^4-60k^5+38m+270km+240k^2m-96k^3m-76m^2-540km^2\\
&-480k^2m^2+192k^3m^2)+e^3(10k^3+4k^4+2m(-1+2m)+k^2(5+6m-12m^2)+k(-1+2m-4m^2))\\
&-e^2(5-2k+50k^2+88k^3+26k^4-5k^5-16m+24km+54k^2m-8k^3m+32m^2-48km^2\\
&-108k^2m^2+16k^3m^2+i(5+9k+32k^2+45k^3+17k^4-4m+14km+26k^2m+8m^2-28km^2-52k^2m^2))\\
&+e(35+33k+197k^2+285k^3+56k^4-30k^5-38m+104km+174k^2m-48k^3m+76m^2-208km^2\\
&-348k^2m^2+96k^3m^2+i^2(15+41k+79k^2+80k^3+27k^4+6m+38km+42k^2m-12m^2-76km^2\\
&-84k^2m^2)+i(50+108k+262k^2+298k^3+79k^4-15k^5-4m+128km+166k^2m-24k^3m+8m^2-256km^2\\
&-332k^2m^2+48k^3m^2))),\\
p_2(k,i,m,e) = &-((-4+e-i+k)(-3+e-i+k)(4+3k+11k^2+e^2k^2+i^2(2+3k+3k^2)+2i(3+4k+6k^2)\\
&-e(1+6k^2+i(1+k+3k^2)))(2+k-2m)(1+k+2m)).
\end{aligned}
$$

Note that for all $e \notin \mathbb{Z}$ and $k, i, m \in \mathbb{Z}$ we have that $d(k,i,e) \neq 0$. Now suppose that $2 \leq i \leq 2m-4$, $m \geq 3$. Then by using the relation (78) one can verify that the creative telescoping equation

$$g(k+1,i,m,e) - g(k,i,m,e) = c_0(i,m,e)\, q(k,i,m,e) + \cdots + c_5(i,m,e)\, q(k,i+5,m,e) \tag{88}$$

holds for $0 \leq k \leq 2m-4$ and $e \notin \mathbb{Z}$ as we have sketched it for the sum case $B_0(i,m)$ in Remark 2. Note that the factor $i-k$ in $q(k,i,e)$ becomes essential in order to avoid poles in this verification. Then summing this equation over k from 0 to $2m-4$ results the inhomogeneous recurrence

$$c_0(i,m,e)\, B_2^{(e)}(i,m,e) + \cdots + c_0(i+5,m,e)\, B_2^{(e)}(i+5,m,e) = g(2m-3,i,m,e) - g(0,i,m,e) \tag{89}$$

for all $e \notin \mathbb{Z}$ and $2 \leq i \leq 2m-4$.

But this means that also

$$B_2'(i,m) := \lim_{e \to 0} B_2^{(e)}(i,m) = \sum_{k=i}^{2m-4} (-1)^k\, h_1(k,m)$$

is a solution of the recurrence (89) in the limit $e \to 0$. Observe that $g(k,i,m,0) = 0$ which comes from the fact that $q(k,i,m,0) = 0$. Therefore $B_2'(i,m)$ is a solution of the recurrence

$$c_0(i,m,0)\, B_2'(i,m) + \cdots + c_5(i,m,0)\, B_2'(i+5,m) = 0 \tag{90}$$

for all $2 \leq i \leq 2m-4$ and $m \geq 3$. Note that this recurrence is nothing but recurrence (87) given from above. In the final step we observe that $\sum_{j=0}^{5} c_j(i,m,0) = 0$ which shows that any constant is a solution of (90) or (87). Therefore $B_2(i,m) = B_2'(i,m) + \sum_{j=2m-3}^{2m-1} (-1)^j h_1(j,m)$ is a solution of (87) for all $2 \leq i \leq 2m$ and $m \geq 3$. Similarly, for the specific values $m = 1, 2$ with $2 \leq r \leq 2m$ the constant sum $B_2(r, m)$ is a solution of (87). This finally shows that $B_2(i, m)$ is a solution of (87) for all $2 \leq i \leq 2m$.

As a side remark we would like to mention that we also could have derived a recurrence for $B_1(i, m)$ that holds for all $2 \leq i \leq m$. In a naive attempt we might set up the definite summation problem with $B_1^{(e)}(i, m, e) := \sum_{k=0}^{2m-4} q'(k, i, m, e)$ where $q'(k, i, m, e) = (-1)^k \frac{(i-k-3+e)!}{(i-k-3)!} h_1(k, m)$ and derive a creative telescoping solution with Sigma. But now note that $\frac{(i-k-3+e)!}{(i-k-3)!}$ shifted in i equals $\frac{(i-k-3+e)!}{(i-k-3)!} \frac{i-k-2+e}{i-k-2}$. Hence, during the verification for the corresponding creative telescoping equation similarly to (88), we have a pole problem in the summation range $2 \leq i \leq 2m - 4$. This could be avoided by choosing a different representation of $q'(k, i, m, e)$, namely in $q'(k, i-3, m, e)$, similarly to Remark 1. ♢

Let us denote by $L(i, m)$ the left hand side of (76), i.e.,

$$L(i, m) = 6(-1)^i B_2(i, m) - B_0(i, m) + 2h_1(i-2, m) - 5h_1(i-1, m) + 3(-1)^i t_1(2m-1). \quad (91)$$

Then we have all the ingredients in hand in order to compute a recurrence that contains $L(i, m)$ as solution for any $2 \leq i \leq 2m$. Verifying that sufficiently many initial values of the recurrence for $L(i, m)$ vanish, will prove that $L(i, m)$ indeed equals 0.

To this end, we first derive, for instance with the packages [SZ94] or [Mal96], a recurrence in i that contains the solutions

$$c_1 (-1)^i + c_2 (-1)^i B_2(i, m) + c_3 h_1(i-1, m) + c_4 h_2(i-2, m) + c_5 B_0(i, m) \quad (92)$$

for any constants $c_i \in \mathbb{Q}$ and any $2 \leq i \leq 2m$.

As already pointed out in Remark 3 the recurrence (87) contains besides $B_2(i, m)$ also the constant solution 1. Therefore changing the signs of the coefficients of $B_2(i, m)$, $B_2(i+2, m)$ and $B_2(i+4, m)$ in (87) gives a recurrence that contains the solutions $c_1 (-1)^i + c_2 (-1)^i B_2(i, m)$ for $2 \leq i \leq 2m$. Moreover one obtains immediately recurrences for $h_1(i-1, m)$ and $h_1(i-2, m)$, if one substitutes k by $i-1$ or $i-2$ in the recurrence (78); those recurrences are both valid in the range $2 \leq i \leq 2m$. Then combining these two recurrences with the recurrence (84) for $B_0(i, m)$ and the recurrence for $c_1 (-1)^i + c_2 (-1)^i B_2(i, m)$ by using procedures from the packages [SZ94] or [Mal96] enables us to compute a single homogeneous recurrence of order 9,

$$c_0(i, m) S(i, m) + c_1(i, m) S(i+1, m) + \cdots + c_9(i, m) S(i+9, m) = 0, \quad (93)$$

which contains the solution for (92) and therefore also (91) for all $2 \leq i \leq 2m$. The explicit expressions for the $c_j(i, m)$ can be easily recomputed with the Maple package [SZ94] gfun or the Mathematica package [Mal96] GeneratingFunctions.

Moreover the following remark shows that in the range $2 \leq i \leq 2m - 9$ the leading coefficient $c_9(i, m)$ in (93) does not vanish.

Remark 4. The leading coefficient $c_9(i, m)$ in (93) is

$$(i-1)i(6+i-2m)(7+i-2m)(8+i-2m)(5+i+2m)(6+i+2m)(7+i+2m)\, p(i, m)$$

with the polynomial $p(i, m) \in \mathbb{Z}[i, m]$ defined as

$$
\begin{aligned}
p(i,m) := {} & 286934400\, i + 1022116320\, i^2 + 1518851088\, i^3 + 1277708580\, i^4 + 764090658\, i^5 + 449853750\, i^6 + 295140024\, i^7 + 169048296\, i^8 + \\
& 70449102\, i^9 + 20137410\, i^{10} + 3830652\, i^{11} + 462672\, i^{12} + 32076\, i^{13} + 972\, i^{14} + 1012435200\, m + 5482935360\, i\, m + \\
& 14278564656\, i^2\, m + 23213538792\, i^3\, m + 26151610626\, i^4\, m + 21597662355\, i^5\, m + 13483008435\, i^6\, m + 6458953002\, i^7\, m + \\
& 2388711936\, i^8\, m + 682573911\, i^9\, m + 150177099\, i^{10}\, m + 25219524\, i^{11}\, m + 3183792\, i^{12}\, m + 295824\, i^{13}\, m + 19752\, i^{14}\, m + 912\, i^{15}\, m + \\
& 24\, i^{16}\, m + 2024870400\, m^2 + 6988032000\, i\, m^2 + 10391171184\, i^2\, m^2 + 7815524436\, i^3\, m^2 + 1403744502\, i^4\, m^2 - 3433378715\, i^5\, m^2 - \\
& 4306205557\, i^6\, m^2 - 2795766398\, i^7\, m^2 - 1193680594\, i^8\, m^2 - 353952095\, i^9\, m^2 - 72888049\, i^{10}\, m^2 - 9860680\, i^{11}\, m^2 - \\
& 699794\, i^{12}\, m^2 + 13148\, i^{13}\, m^2 + 7588\, i^{14}\, m^2 + 608\, i^{15}\, m^2 + 16\, i^{16}\, m^2 - 13619181888\, m^3 - 62360574624\, i\, m^3 - 138500395704\, i^2\, m^3 - \\
& 196534634088\, i^3\, m^3 - 197750758622\, i^4\, m^3 - 148547525347\, i^5\, m^3 - 85793698617\, i^6\, m^3 - 38793616174\, i^7\, m^3 - \\
& 13884128774\, i^8\, m^3 - 3953893039\, i^9\, m^3 - 896125845\, i^{10}\, m^3 - 160785656\, i^{11}\, m^3 - 22549686\, i^{12}\, m^3 - 2412656\, i^{13}\, m^3 - \\
& 188176\, i^{14}\, m^3 - 9728\, i^{15}\, m^3 - 256\, i^{16}\, m^3 + 1472884992\, m^4 + 17277933456\, i\, m^4 + 55255553160\, i^2\, m^4 + 97387495728\, i^3\, m^4 + \\
& 114340534314\, i^4\, m^4 + 96961318441\, i^5\, m^4 + 61869893729\, i^6\, m^4 + 30390885826\, i^7\, m^4 + 11641806632\, i^8\, m^4 + 3500762485\, i^9\, m^4 + \\
& 827525561\, i^{10}\, m^4 + 153150896\, i^{11}\, m^4 + 21942460\, i^{12}\, m^4 + 2380016\, i^{13}\, m^4 + 187216\, i^{14}\, m^4 + 9728\, i^{15}\, m^4 + 256\, i^{16}\, m^4 + \\
& 25007101632\, m^5 + 96066457488\, i\, m^5 + 181925812632\, i^2\, m^5 + 221098537536\, i^3\, m^5 + 189184506516\, i^4\, m^5 + 118702812708\, i^5\, m^5 + \\
& 55719392004\, i^6\, m^5 + 19763963784\, i^7\, m^5 + 5314961340\, i^8\, m^5 + 1080128100\, i^9\, m^5 + 164117244\, i^{10}\, m^5 + 18299232\, i^{11}\, m^5 + \\
& 1456536\, i^{12}\, m^5 + 78336\, i^{13}\, m^5 + 2304\, i^{14}\, m^5 - 3283424064\, m^6 - 24487119216\, i\, m^6 - 62740182552\, i^2\, m^6 - 91955201268\, i^3\, m^6 - \\
& 89825565356\, i^4\, m^6 - 62243885388\, i^5\, m^6 - 31512673972\, i^6\, m^6 - 11837036352\, i^7\, m^6 - 3321422232\, i^8\, m^6 - 695467968\, i^9\, m^6 - \\
& 107666704\, i^{10}\, m^6 - 12118848\, i^{11}\, m^6 - 968336\, i^{12}\, m^6 - 52224\, i^{13}\, m^6 - 1536\, i^{14}\, m^6 - 21238328448\, m^7 - 64255523184\, i\, m^7 - \\
& 96366768720\, i^2\, m^7 - 92096792976\, i^3\, m^7 - 60749842800\, i^4\, m^7 - 28468056528\, i^5\, m^7 - 9555442224\, i^6\, m^7 - 2283228864\, i^7\, m^7 - \\
& 379758816\, i^8\, m^7 - 42201312\, i^9\, m^7 - 2991072\, i^{10}\, m^7 - 138240\, i^{11}\, m^7 - 4608\, i^{12}\, m^7 + 6290303040\, m^8 + 23115529440\, i\, m^8 + \\
& 38148908736\, i^2\, m^8 + 38686906320\, i^3\, m^8 + 26702560752\, i^4\, m^8 + 13015763424\, i^5\, m^8 + 4522579392\, i^6\, m^8 + 1111374432\, i^7\, m^8 + \\
& 188339328\, i^8\, m^8 + 21100656\, i^9\, m^8 + 1495536\, i^{10}\, m^8 + 69120\, i^{11}\, m^8 + 2304\, i^{12}\, m^8 + 5912676864\, m^9 + 12381602112\, i\, m^9 + \\
& 13532698560\, i^2\, m^9 + 9775140480\, i^3\, m^9 + 4854527040\, i^4\, m^9 + 1615086720\, i^5\, m^9 + 339571200\, i^6\, m^9 + 40320000\, i^7\, m^9 + \\
& 2053440\, i^8\, m^9 - 2620173312\, m^{10} - 5651547264\, i\, m^{10} - 5713697280\, i^2\, m^{10} - 3829696704\, i^3\, m^{10} - 1857903168\, i^4\, m^{10} - \\
& 627501888\, i^5\, m^{10} - 134592960\, i^6\, m^{10} - 16128000\, i^7\, m^{10} - 821376\, i^8\, m^{10} + 209811456\, m^{11} + 681806592\, i\, m^{11} + \\
& 312221952\, i^2\, m^{11} - 87664896\, i^3\, m^{11} - 91535616\, i^4\, m^{11} - 20217600\, i^5\, m^{11} - 1347840\, i^6\, m^{11} + 46356480\, m^{12} - 92874240\, i\, m^{12} - \\
& 77865984\, i^2\, m^{12} + 29221632\, i^3\, m^{12} + 30511872\, i^4\, m^{12} + 6739200\, i^5\, m^{12} + 449280\, i^6\, m^{12} - 111992832\, m^{13} - \\
& 124056576\, i\, m^{13} - 24192000\, i^2\, m^{13} + 38633472\, m^{14} + 35444736\, i\, m^{14} + 6912000\, i^2\, m^{14} - 5308416\, m^{15} + 1327104\, m^{16}.
\end{aligned}
$$

By inspection the first 8 factors in $c_9(i,m)$ do not vanish in the range $2 \le i \le 2m-9$. Moreover by the cylindric algebraic decomposition method [CH91] one can decide that there does not exist any solution to the problem

$$\exists i \in \mathbb{R} \exists m \in \mathbb{R} \big[p(i,m) = 0 \wedge 3 \le i \le 2m \big]$$

where $\mathbb{R}$ stands for the real numbers. More precisely, in our instance the algorithm returns the equivalent formula *FALSE* which means that there does not exist any solution $i, m \in \mathbb{R}$ with $3 \le i \le 2m$ and $p(i,m) = 0$. In particular it follows that for any integer i, m with $3 \le i \le 2m$ there does not exist any diophantine solution $p(i,m) = 0$. We want to mention that we verified this result with the cylindric algebraic decomposition implementations of Mathematica 5.0 and with the package QEPCAD B version 1.30 of Hoon Hong. Note that for $2 \le i < 3$ there are solutions $m \in \mathbb{R}$ with $p(i,m)$. As a side remark note that the implementations mentioned above failed to compute a cylindric algebraic decomposition for the larger range $2 \le i \le 2m$ in a reasonable amount of time. But by checking that $p(2,m)$ does not have any integer root (by factoring the polynomial $p(2,m) \in \mathbb{Z}[m]$ over $\mathbb{Z}$), we complete our verification that $p(i,m) \neq 0$ for any integer with $2 \le i \le 2m$. $\diamondsuit$

In order to prove Identity (76) for $2 \le i \le 2m$ it suffices to show that the first nine initial values of $L(i,m)$ are 0, i.e., we will verify that $L(r,m) = 0$ for all $2 \le r \le 10$ and all $2m \ge r$. Then together with the fact that $L(i,m)$ is a solution of (93) for all $2 \le i \le 2m$ and $c_9(i,m) \neq 0$ for all $2 \le i \le 2m-9$, it follows that $L(i,m) = 0$ for all $2 \le i \le 2m$.

To accomplish this task we will use the equivalent relation (77). Then showing that $L(r,m) = 0$ for $2 \le r \le 10$ and all m with $2m \ge r$ is equivalent to proving

$$6(-1)^r B_1(r,m) + B_0(r,m) + 4h_1(r-2,m) - h_1(r-1,m) = 0 \tag{94}$$

for the same range of values.

For $r = 2$ we can prove this last identity for instance as follows. With Sigma we compute the expression

$$
\begin{aligned}
g(k,m) := {} & \frac{1}{2\,(1+k)\,m\,(-1+2\,m)} \Big(-2\,\big(3\,k^2 + k^3 + 3\,(1-2\,m)\,m + k\,\big(2 + m - 2\,m^2\big)\big)\,h_1(k,m) \\
& + \big(9\,k^2 + 3\,k^3 + 2\,(1-2\,m)\,m + k\,\big(6 + 4\,m - 8\,m^2\big)\big)\,h_1(k+1,m)
\end{aligned}
$$

$$-k\left(2+3\ k+k^2+2\ m-4\ m^2\right)\ h_1(k+2,m)\Big) \quad (95)$$

such that

$$g(k+1,m)-g(k,m)=h_1(k,m) \quad (96)$$

holds for all $0\le k\le 2m-4$. We want to emphasize that this equation can be verified by using the relation (78). Now summing this equation from 0 to $2m-4$ gives $\sum_{k=0}^{2m-4}h_1(k,m)=g(2m-3,m)-g(0,m)$. Adding $h_1(2m-3,m)+h_1(2m-2,m)+h_1(2m-1,m)$ to both sides together with Lemma 2 and

$$h_1(2m-3,m)=\frac{-6+21m-18m^2+4m^3}{2(-3+4m)},\quad h_1(2m-2,m)=1-m,\quad h_1(2m-1,m)=1$$

finally gives

$$B_0(2,m)=h_1(1,m),\quad m\ge 1. \quad (97)$$

Hence for $r=2$ we have that

$$\begin{aligned}6(-1)^iB_1(r,m)+B_0(r,m)+4h_1(r-2,m)-h_1(r-1,m)\\ =h_1(1,m)+4h_1(0,m)-h_1(1,m)=4h_1(0,m)=0,\quad \forall m\ge 1,\end{aligned}$$

which verifies the identity (94) for the case $r=2$.

Similar to the derivation of (97) one can obtain the following representations[6] of $B_0(r,m)$ that hold for all m with $2m\ge r$, $2\le r\le 10$.

$$\begin{aligned}
B_0(2,m)=&h_1(1,m),\\
B_0(3,m)=&-4h_1(1,m)+h_1(2,m),\\
B_0(4,m)=&\frac{-12h_1(1,m)+(3+m-2m^2)h_1(2,m)}{(1+m)(-3+2m)},\\
B_0(5,m)=&\frac{3(-24-3m+7m^2-4m^3+4m^4)h_1(1,m)+(18+3m-7m^2+4m^3-4m^4)h_1(2,m)}{(-2+m)(1+m)(-3+2m)(3+2m)},\\
B_0(6,m)=&\frac{(120+19m-31m^2-28m^3+28m^4)h_1(1,m)+(-30-13m+25m^2+4m^3-4m^4)h_1(2,m)}{(1+m)(2+m)(-5+2m)(-3+2m)},\\
B_0(7,m)=&(-10800-3060m+6339m^2-838m^3+647m^4+464m^5-328m^6+32m^7-16m^8)h_1(1,m)\\
&+2(1350+495m-954m^2-134m^3+85m^4+112m^5-56m^6-32m^7+16m^8)h_1(2,m)\Big/\\
&((-3+m)(-2+m)(1+m)(2+m)(-5+2m)(-3+2m)(3+2m)(5+2m)),\\
B_0(8,m)=&(-226800-98820m+172035m^2+100077m^3-88313m^4-28501m^5+19850m^6-1288m^7+224m^8\\
&+560m^9-224m^{10})h_1(1,m)+2(28350+19305m-34983m^2-14460m^3+14195m^4+775m^5-974m^6\\
&+760m^7-320m^8-80m^9+32m^{10})h_1(2,m)\Big/((-3+m)(-2+m)(1+m)(2+m)(3+m)(-7+2m)\\
&(-5+2m)(-3+2m)(3+2m)(5+2m)),\\
B_0(9,m)=&-10(-211680-8442m+15159m^2+5996m^3-1538m^4-10353m^5+5754m^6+1944m^7-912m^8\\
&-80m^9+32m^{10})h_1(1,m)+(-529200-76860m+125976m^2+107021m^3-87398m^4\\
&-46245m^5+28002m^6+4824m^7-2352m^8-80m^9+32m^{10})h_1(2,m))\Big/\\
&((-4+m)(-3+m)(-2+m)(2+m)(3+m)(-7+2m)(-5+2m)(3+2m)(5+2m)(7+2m)),\\
B_0(10,m)=&2(-114307200-56269080m+98214930m^2+56493117m^3-52362987m^4-10634788m^5+9427703m^6\\
&-3727349m^7+1168354m^8+909516m^9-328536m^{10}-38768m^{11}+13408m^{12}-448m^{13}\\
&+128m^{14})h_1(1,m)-3(3+m-2m^2)^2(-2116800-114240m+247244m^2-74384m^3+70991m^4\\
&+8327m^5-6158m^6+1016m^7-448m^8-80m^9+32m^{10})h_1(2,m)\Big/((-4+m)(-3+m)(-2+m)\\
&(1+m)(2+m)(3+m)(4+m)(-9+2m)(-7+2m)(-5+2m)(-3+2m)(3+2m)(5+2m)(7+2m)).
\end{aligned}$$

[6] Note that identity $B_0(2,m)=h_1(1,m)$ for $m\ge 1$ proves Lemma 5.

Remark 5. As for the case $r=2$, for $B_0(r,m)=\sum_{k=0}^{2m}\binom{r+k-3}{r-2}h_1(k,m)$ with $r\in\{3,\dots,10\}$ we can compute with Sigma expressions $g_r(k,m)$, listed in Appendix A, such that

$$g_r(k+1,m)-g_r(k,m)=\binom{r+k-3}{r-2}h_1(k,m)$$

holds for all $0\le k\le 2m-4$. Similarly to the case $r=2$, the corresponding equations can be verified by using the relation (78). Finally with telescoping, as for the case $r=2$, one derives the above closed form evaluations of the $B_0(r,m)$. ♢

Hence we see that for $r\in\{2,\dots,10\}$ each $B_0(r,m)$ can be expressed in terms of $h_1(1,m)$ and $h_1(2,m)$. Moreover, by (75) each corresponding $B_1(r,m)$ can be written as a linear combination of the $h_1(1,m),\dots,h_1(r-3,m)$. In particular by applying relation (78) each of the $h_1(j,m)$ for $3\le j\le r-3$ can be reduced to $h_1(1,m)$ and $h_1(2,m)$ where $m>r/2$. In other words, the left side of identity (94) can be expressed in terms of $h_1(1,m)$ and $h_1(2,m)$ that occur only linearly. Since this expression collapses to 0, identity (94) holds for $m>r/2$. The special case $2m=r$ of identity (94) can be shown separately by simple evaluation.

Applying this proof strategy for all $r\in\{2,\dots,10\}$ shows that $L(r,m)=0$ for all r,m with $2m\ge r$ and $2\le r\le 10$ which finally proves that $L(i,m)=0$ for all $2\le i\le 2m$.

7. A proof of identity (32)

In this section we will use the abbreviation $h_2(k,m):=f_2(2m-k,m)$. By redefining $A_1(i,m)$ from above with

$$A_1(i,m)=\sum_{k=0}^{i-3}(-1)^k h_2(k,m) \tag{98}$$

and Lemma 2 it is immediate that showing (32) is equivalent to proving

$$-\frac{3}{2}(-1)^i t_1(2m)=h_2(i-2,m)+2h_2(i-1,m)+3(-1)^i(A_1(i,m)-A_2(i,m))+A_0(i,m) \tag{99}$$

for $3\le i\le 2m+1$. Moreover the sum in Proposition 3 can be rewritten as

$$\frac{1}{2}t_1(2m)=\sum_{k=0}^{2m}(-1)^k h_2(k,m)=A_1(i,m)+A_2(i,m)+(-1)^i h_2(i-2,m)-(-1)^i h_2(i-1,m)$$

for $i\ge 3$. Consequently to show identity (99) is equivalent to prove[7]

$$6(-1)^i A_2(i,m)-A_0(i,m)+2h_2(i-2,m)-5h_2(i-1,m)-3(-1)^i t_1(2m-1)=0 \tag{100}$$

or, equivalently,

$$6(-1)^i A_1(i,m)+A_0(i,m)+4h_2(i-2,m)-h_2(i-1,m)=0 \tag{101}$$

for all $3\le i\le 2m+1$. To verify these relations we proceed analogously to the proof of Identity (33).

Using Sigma we first compute a recurrence of $h_2(k,m)$ for all $m,k\ge 0$.

$$\begin{aligned}&2(2+k)^2(k-2m)(1+k+2m)h_2(k,m)\\&-(29k^3+5k^4+k(46-20m-40m^2)-12(-1+m+2m^2)-2k^2(-29+3m+6m^2))h_2(1+k,m)\\&+(26k^3+4k^4+k(55-14m-28m^2)+k^2(59-6m-12m^2)-6(-3+m+2m^2))h_2(2+k,m)\\&-(1+k)^2(2+k-2m)(3+k+2m)h_2(3+k,m)=0\end{aligned} \tag{102}$$

[7] Note that equations (101) and (100) are analogous to (77) and (76), except a sign change of $3(-1)^i t_1(2m-1)$ in (76) and (100).

Remark 6. Denote by $q(s,k,m)$ the summand of $h_2(k,m)$, i.e., $h_2(k,m)=\sum_{s=0}^{\lfloor\frac{2m-k-1}{2}\rfloor} q(s,k,m)$. First note that $\binom{m-s-1}{2m-2s-k}=\frac{s-m+k}{m-s}\binom{m-s}{2m-2s-k}$ for all $0\leq s<m$. Therefore with

$$h_2'(k,m):=\sum_{s=0}^{\lfloor\frac{2m-k}{2}\rfloor-1}\frac{k}{m-s}\binom{m-s}{2m-2s-k}\frac{(-1)^{s+k}}{2m\,4^s}\sum_{r=0}^{s}\frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r} \tag{103}$$

we have that $h_2(k,m)=h_2'(k,m)+q(\lfloor\frac{2m-k}{2}\rfloor,k,m)$ for $0\leq k\leq 2m$. Now similarly as in Remark 1 we transform the sum $h_2'(s,k,m)=\sum_{s=0}^{\lfloor\frac{2m-k}{2}\rfloor-1}q'(s,k,m)$ to $h_2''(s,k,m)=\sum_{s=0}^{\lfloor\frac{2m-k}{2}\rfloor-1}q''(s,k,m)$ by replacing $\binom{m-s}{2m-2s-k}$ with $\frac{(m-s-k-1)(m-s-k-2)}{(2m-2s-k)(2m-2s-k-1)}\binom{m-s}{2m-2s-k-2}$ in $q'(s,k,m)$.

Afterwards, we compute with Sigma the constants $c_i(k,m)$ and $g(s,k,m)$ for the creative telescoping equation

$$g(s+1,k,m)-g(s,k,m)=c_0(k,m)q''(s,k,m)+\cdots+c_3q''(s,k+3,m). \tag{104}$$

More precisely, those constants turn out to be

$$c_0(k,m)=2m(2+k)^2(k-2m)(1+k+2m),$$
$$c_1(k,m)=-m(29k^3+5k^4+k(46-20m-40m^2)-12(-1+m+2m^2)-2k^2(-29+3m+6m^2)),$$
$$c_2(k,m)=m(26k^3+4k^4+k(55-14m-28m^2)+k^2(59-6m-12m^2)-6(-3+m+2m^2)),$$
$$c_3(k,m)=-m(1+k)^2(2+k-2m)(3+k+2m)h_2(3+k)=0,$$

and

$$g(s,k,m)=\binom{m-s}{2m-2s-k-2}\frac{-2(-1)^{s+k}}{4^s(2m-2s-k-2)(2m-2s-k-1)}\Big[p_1(s,k,m)\,\frac{(m)_s(-3m-1)_s}{s!(\frac{1}{2}-2m)_s}+p_2(s,k,m)\,m\sum_{r=0}^{s}\frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r}\Big]$$

where

$$p_1(s,k,m):=(k(-1+2m-2s)+2(-1+m-s)+k^2(m-s))\times(1+3m-s)(m+s)(k-2m+2s)(1+k-2m+2s)$$

and

$$\begin{aligned}p_2(s,k,m):=&\,k^4(m+3m^2+s+2ms-s^2)+k^3(-1-12m^3+10s+24ms-9s^2+m^2(5+12s))\\&+4(6m^4-3(-1+s)s+2m^2s(2+3s)-m^3(7+12s)+m(1+11s+3s^2))\\&+2k(-1+12m^4+16s-15s^2-4m^3(5+6s)+2m^2(5+7s+6s^2)+m(3+43s+6s^2))\\&+k^2(-3+12m^4+29s-26s^2-2m^3(13+12s)+2m^2(7+10s+6s^2)+m(-1+68s+6s^2)).\end{aligned}$$

In order to show the correctness of the creative telescoping equation (104) within the summation range we proceed as in Remark (1). If we denote $p(s,k,m)=(-1)^k\binom{m-s-1}{2m-2s-k-2}$, we represent all ingredients in (104) in terms of $p(s+1,k+3,m)$, i.e.,

$$p(s+1,k,m)=p(s+1,k+3,m)\prod_{j=0}^{2}\frac{m-s-k-j-4}{2m-2s-k-j-4}, \tag{105}$$

$$p(s,k+i,m)=-p(s+1,k+3,m)\frac{m-s}{2m-2s-k-6}\prod_{j=0}^{3-i}\frac{m-s-k+j-6}{2m-2s-k+j-5},\quad 0\leq i\leq 3. \tag{106}$$

With these representations in $p(s+1,k+3,m)$ we finally manage to verify that (80) holds for $0\leq s\leq\lfloor\frac{2m-k}{2}\rfloor-4$. Hence, summing equation (104) over s from 0 to an arbitrary d with $0\leq$

$d \le \lfloor \frac{2m-k}{2} \rfloor - 4$ gives the relation (83) for the sum $b(k,m) = \sum_{s=0}^{d} q''(s,k,m) = \sum_{s=0}^{d} q'(s,k,m)$. Now we do a case distinction on k. If k is even, we can choose $d = m - k/2 - 4$, and obtain

$$\begin{aligned} &c_0(k,m)\, h_2(k,m) + \cdots + c_3(k,m)\, h_2(k+3,m) \\ &\qquad = g(m - \frac{k}{2} - 3, k, m) - g(0,k,m) + \sum_{i=0}^{3} c_i(k,m) \sum_{j=0}^{3} h_2(m - \frac{k}{2} - j, k+i, m). \end{aligned}$$

By term rewriting one can now show that the right hand side is equal to 0. Hence for $m - k/2 - 4 \ge 0$, k even, the relation (102) holds. Similarly, for odd k, one can choose $d = m - (k-1)/2 - 4$, and obtains

$$\begin{aligned} &c_0(k,m)\, h_2(k,m) + \cdots + c_3(k,m)\, h_2(k+3,m) \\ &\qquad = g(m - \frac{k-1}{2} - 3, k, m) - g(0,k,m) + \sum_{i=0}^{3} c_i(k,m) \sum_{j=0}^{3} h_2(m - \frac{k-1}{2} - j, k+i, m). \end{aligned}$$

Again one can prove that the right hand side vanishes to 0, and hence the relation (102) holds for $m - (k-1)/2 - 4 \ge 0$, k odd.

Summarizing, for all $m \ge 5$ and $0 \le k \le 2m-9$ the recurrence (102) contains the solution $h_2(k,m)$. The special cases $0 \le m \le 4$ or $k \ge 2m-8$ for $m \ge 4$ can be easily verified by evaluation. This shows that the recurrence holds for all $m, k \ge 0$. ◊

In the next step we compute with Sigma a recurrence for A_0, namely,

$$\begin{aligned} &(-2 - i - i^2)(2 - 3i + i^2 - 2m - 4m^2) A_0(i,m) \\ &\quad + (3+i)(-2 + 2i - i^2 + i^3 + 2m + 4m^2) A_0(i+1,m) \\ &\quad + (-3+i)(2 + 2i + i^2 + i^3 - 2m - 4m^2) A_0(i+2,m) \\ &\quad - (2 - i + i^2)(2 + 3i + i^2 - 2m - 4m^2) A_0(i+3,m) = 0 \end{aligned} \tag{107}$$

which holds for $3 \le i \le 2m+1$.

Remark 7. More precisely, Sigma is able to compute constants for the creative telescoping summand recurrence (109) below, namely,

$$\begin{aligned} c_0(i,m) &:= (1-i)i(1+i)(2+i+i^2)(-2+i-2m)(-1+i+2m), \\ c_1(i,m) &:= (-1+i)i(1+i)(3+i)(-2+2i-i^2+i^3+2m+4m^2), \\ c_2(i,m) &:= (-3+i)(-1+i)i(1+i)(2+2i+i^2+i^3-2m-4m^2), \\ c_3(i,m) &:= (1-i)i(1+i)(2-i+i^2)(1+i-2m)(2+i+2m), \end{aligned}$$

and

$$g(k,i,m) = -\Big[p_0(i,m)h_2(i,m) + p_1(i,m)h_2(i+1,m) + p_2(i,m)h_2(i+2,m)\Big] \frac{k-1}{(k+1)^2} \binom{i+k-3}{i-2} \tag{108}$$

where

$$\begin{aligned} p_0(i,m) := {} & 2(-6 - k + 2k^2)(-1 + 4k^3 + k^4 + m + 2m^2 - k^2(-2 + m + 2m^2) - 2k(1 + 2m + 4m^2)) \\ & + 2i^3(3 + 2k^4 + 3k^5 - 5m - 10m^2 - k^2(2 + m + 2m^2) - k^3(8 + 3m + 6m^2) + k(8 + 7m + 14m^2)) \\ & + i^4(9k^3 + 5k^4 + 2m(1+2m) + k^2(1 - 6m - 12m^2) - k(3 + 8m + 16m^2)) \\ & + i^2(-4k^5 + 2k^6 + k(7 - 76m - 152m^2) - k^4(1 + 2m + 4m^2) + k^3(59 + 4m + 8m^2) \\ & + k^2(73 + 4m + 8m^2) - 6(2 + 9m + 18m^2)) - 2i(3 - 6k^5 + k^6 - 45m - 90m^2 + k(41 - 10m - 20m^2) \\ & - k^4(30 + m + 2m^2) + 2k^3(-4 + 3m + 6m^2) + k^2(53 + 32m + 64m^2)), \\ p_1(i,m) := {} & (2 - i - k)(29k^4 + 6k^5 + 2k(-16 + m + 2m^2) + 10(-1 + m + 2m^2) - 2k^3(-17 + 4m + 8m^2) \\ & - k^2(11 + 20m + 40m^2) + i^3(10k^3 + 4k^4 + 2m(1+2m) + k^2(5 - 6m - 12m^2) \end{aligned}$$

$$- k(1+2m+4m^2)) + i(4k^4 - 3k^5 - 18m(1+2m) + k(39 - 40m - 80m^2)$$
$$+ k^2(85 - 16m - 32m^2) + k^3(53 + 4m + 8m^2)) + i^2(3k^4 + 3k^5 - 10(-1 + m + 2m^2)$$
$$- 2k(-11 + 2m + 4m^2) - k^3(7 + 4m + 8m^2) + k^2(5 + 6m + 12m^2))),$$
$$(-2 + i + k)(-1 + i + k)(2 + 5k + 2k^2 + i^2k^2 + i(2 + k - k^2))(1 + k - 2m)(2 + k + 2m).$$

Then for the summand $q(k,i,m) := h_2(k,m)\binom{i+k-3}{i-2}$ of $A_0(i,m)$ we have that

$$g(k+1,i,m) - g(k,i,m) = c_0(i,m)\,q(k,i,m) + \cdots + c_3(i,m)\,q(k,i+3,m) \tag{109}$$

for $m \geq 2$, $0 \leq k \leq 2m-3$ and $3 \leq i \leq 2m+1$. The verification of this creative telescoping equation and correctness of the recurrence (107) for $3 \leq i \leq 2m+1$ is now completely analogous to Remark 2. $\Diamond$

Next we compute a recurrence for $A_2(i,m)$, namely

$$\begin{aligned}
&2(6+5i+i^2)(i+i^2-2m(1+2m))A_2(i,m)\\
&\quad + (3+i)(3i^2+i^3+8m(1+2m)+2i(1+m+2m^2))A_2(1+i,m)\\
&\quad - 2(1+i)(24+14i^2+2i^3-5m-10m^2-i(-32+m+2m^2))A_2(2+i,m)\\
&\quad - 2(2+i)(6+6i^2+i^3+i(11-2m-4m^2))A_2(3+i,m)\\
&\quad + 2(3+4i+i^2)(8+6i+i^2-m-2m^2)A_2(4+i)\\
&\quad + (2+3i+i^2)(7i+i^2-2(-6+m+2m^2))A_2(5+i,m) = 0
\end{aligned} \tag{110}$$

for all $3 \leq i \leq 2m+1$.

Remark 8. In order to achieve this, we actually derive the creative telescoping equation (111) below for the sum

$$A_2^{(e)}(i,m,e) := \sum_{k=0}^{2m-3} q(k,i,k,e)$$

where

$$q(k,i,m,e) = (-1)^k \frac{(k-i+e)!}{(k-i)!} h_2(k,m).$$

More precisely, with Sigma we can compute

$$\begin{aligned}
c_0(i,m,e) :=& 2(e^2 - e(7+3i) + 3(6+5i+i^2))(i-2m)(1+i+2m),\\
c_1(i,m,e) :=& 4e^3(1+i) + e^2(-26-37i-11i^2+2m+4m^2) + 3(3+i)(3i^2+i^3+8m(1+2m)\\
&+ 2i(1+m+2m^2)) + e(58+56i^2+9i^3-14m-28m^2-3i(-35+2m+4m^2))\\
&- 4e^2m^2 - 36im^2 - 12eim^2 - 12i^2m^2,\\
c_2(i,m,e) :=& -4e^3 + 2e^4 + e^2(-37-51i-16i^2+2m+4m^2) + e(165+169i^2+30i^3\\
&- 26m - 52m^2 - 6i(-50+m+2m^2)) - 6(1+i)(24+14i^2+2i^3-5m-10m^2\\
&- i(-32+m+2m^2)),\\
c_3(i,m,e) :=& 5e^4 - e^3(35+17i) + e^2(82+82i+19i^2+4m+8m^2) - 6(2+i)(6+6i^2+i^3\\
&+ i(11-2m-4m^2)) - 2e(17+5i^2+5m+10m^2+i(19+6m+12m^2)),\\
c_4(i,m,e) :=& 4e^4 - 2e^3(19+9i) - e(7+3i)(33+37i+8i^2-2m-4m^2)\\
&+ e^2(139+139i+32i^2-2m-4m^2) + 6(3+4i+i^2)(8+6i+i^2-m-2m^2),\\
c_5(i,m,e) :=& (e^2 - e(4+3i) + 3(2+3i+i^2))(-4+e-i-2m)(-3+e-i+2m),
\end{aligned}$$

and

$$g(k,i,m) = -(p_0(k,i,m,e)h_2(k,i,m) + p_1(k,i,m,e)h_2(k+1,i,m) + p_2(k,i,m,e)h_2(k+2,i,m)) \times \frac{q(k,i,e)}{d(k,i,e)}(-1)^k \frac{(k-i+e)!}{(k-i)!}$$

where

$$q(k,i,e) := (-1+e)e(i-k),$$
$$d(k,i,e) := (1+k)^2(-4+e-i+k)(-3+e-i+k)(-2+e-i+k)(-1+e-i+k)(e-i+k),$$

and

$$\begin{aligned}
p_0 :=& -(e^4(9k^3+5k^4+2m(1+2m)+k^2(1-6m-12m^2)-k(3+8m+16m^2)))+e^3(3-40k+20k^2+126k^3+49k^4\\
&-14k^5+26m-122km-66k^2m+18k^3m+52m^2-244km^2-132k^2m^2+36k^3m^2+i(3-10k+10k^2+52k^3\\
&+29k^4+8m-48km-36k^2m+16m^2-96km^2-72k^2m^2))-e^2(33-202k+146k^2+688k^3+165k^4\\
&-132k^5+10k^6+108m-720km-258k^2m+176k^3m-14k^4m+216m^2-1440km^2-516k^2m^2\\
&+352k^3m^2-28k^4m^2+i^2(12-2k+45k^2+126k^3+67k^4-2m-124km-86k^2m-4m^2\\
&-248km^2-172k^2m^2)+i(45-87k+160k^2+579k^3+225k^4-62k^5+42m-590km-310k^2m+82k^3m+84m^2\\
&-1180km^2-620k^2m^2+164k^3m^2))-2(72-180k+262k^2+753k^3-65k^4-249k^5+55k^6-917km\\
&-7k^2m+353k^3m-77k^4m-1834km^2-14k^2m^2+706k^3m^2-154k^4m^2\\
&+i^4(3+4k+16k^2+30k^3+15k^4-11m-36km-21k^2m-22m^2-72km^2-42k^2m^2)\\
&+i^3(30+28k+141k^2+260k^3+87k^4-30k^5-83m-310km-129k^2m+42k^3m-166m^2\\
&-620km^2-258k^2m^2+84k^3m^2)+i(150-100k+571k^2+1256k^3+17k^4\\
&-358k^5+60k^6-169m-1534km-159k^2m+506k^3m-84k^4m-338m^2-3068km^2-318k^2m^2\\
&+1012k^3m^2-168k^4m^2)+i^2(105+32k+434k^2+843k^3+144k^4-177k^5+15k^6-205m\\
&-1017km-258k^2m+249k^3m-21k^4m-410m^2-2034km^2-516k^2m^2\\
&+498k^3m^2-42k^4m^2))-e(-120+447k-459k^2-1681k^3-153k^4\\
&+442k^5-60k^6-150m+1894km+362k^2m-606k^3m+84k^4m-300m^2+3788km^2\\
&+724k^2m^2-1212k^3m^2+168k^4m^2+i^3(-15-14k-69k^2-142k^3-72k^4+28m+152km+96k^2m\\
&+56m^2+304km^2+192k^2m^2)+i^2(-102-13k-405k^2-933k^3-337k^4+102k^5+110m+1018km\\
&+482k^2m-138k^3m+220m^2+2036km^2+964k^2m^2-276k^3m^2)+i(-207+238k-765k^2\\
&-2116k^3-454k^4+422k^5-30k^6+40m+2358km+754k^2m-574k^3m+42k^4m+80m^2+4716km^2\\
&+1508k^2m^2-1148k^3m^2+84k^4m^2)),\\
p_1 :=&(-4+e-i+k)(-60-88k-257k^2-291k^3-7k^4+55k^5-18m+148km+174k^2m-88k^3m-36m^2\\
&+296km^2+348k^2m^2-176k^3m^2+i^3(-10-32k-55k^2-48k^3-15k^4+10m\\
&+30km+24k^2m+20m^2+60km^2+48k^2m^2)+3i^2(-20-56k-97k^2-83k^3-17k^4\\
&+5k^5+14m+52km+42k^2m-8k^3m+28m^2+104km^2+84k^2m^2-16k^3m^2)\\
&+e^3(10k^3+4k^4+2m(1+2m)+k^2(5-6m-12m^2)-k(1+2m+4m^2))+i(-39k^4+60k^5\\
&-6k^3(75+16m+32m^2)+2(-55+19m+38m^2)+2k(-128+135m+270m^2)+k^2(-497+240m+480m^2))\\
&-e^2(5-2k+50k^2+88k^3+26k^4-5k^5+16m-24km-54k^2m+8k^3m+32m^2\\
&-48km^2-108k^2m^2+16k^3m^2+i(5+9k+32k^2+45k^3+17k^4+4m-14km-26k^2m+8m^2\\
&-28km^2-52k^2m^2))+e(35+33k+197k^2+285k^3+56k^4-30k^5+38m-104km-174k^2m+48k^3m\\
&+76m^2-208km^2-348k^2m^2+96k^3m^2+i^2(15+41k+79k^2+80k^3+27k^4\\
&-6m-38km-42k^2m-12m^2-76km^2-84k^2m^2)+i(50+108k+262k^2+298k^3\\
&+79k^4-15k^5+4m-128km-166k^2m+24k^3m+8m^2-256km^2-332k^2m^2+48k^3m^2))),\\
p_2 :=&(4-e+i-k)(-3+e-i+k)(4+3k+11k^2+e^2k^2+i^2(2+3k+3k^2)+2i(3+4k+6k^2)-e(1+6k^2\\
&+i(1+k+3k^2)))(1+k-2m)(2+k+2m).
\end{aligned}$$

The verification of recurrence (110) is analogous to Remark (8). For completeness, we summarize the steps. Note that for all $e \notin \mathbb{Z}$ and $k, i, m \in \mathbb{Z}$ we have that $d(k,i,e) \neq 0$. Now suppose that

$3 \leq i \leq 2m-3$, $m \geq 2$. Then by using the relation (78) one can verify that the creative telescoping equation

$$g(k+1,i,m,e) - g(k,i,m,e) = c_0(i,m,e)\,q(k,i,m,e) + \cdots + c_5(i,m,e)\,q(k,i+5,m,e) \quad (111)$$

holds for $0 \leq k \leq 2m-3$ and $e \notin \mathbb{Z}$ as we have sketched it for the sum case $B_0(i,m)$ in Remark 2. Then summing this equation over k from 0 to $2m-3$ results the inhomogeneous recurrence

$$c_0(i,m,e)\,A_2^{(e)}(i,m,e) + \cdots + c_5(i,m,e)\,A_2^{(e)}(i+5,m,e) = g(2m-2,i,m,e) - g(0,i,m,e) \quad (112)$$

for all $e \notin \mathbb{Z}$ and $3 \leq i \leq 2m-3$.

But this means that also

$$A_2'(i,m) := \lim_{e\to 0} A_2^{(e)}(i,m) = \sum_{k=i}^{2m-3} (-1)^k\, h_2(k,m)$$

is a solution of the recurrence (112) in the limit $e \to 0$. Observe that $g(k,i,m,0) = 0$ which comes from the fact that $q(k,i,m,0) = 0$. Therefore $A_2'(i,m)$ is a solution of the recurrence

$$c_0(i,m,0)\,A_2'(i,m) + \cdots + c_0(i+5,m,0)\,A_2'(i+5,m) = 0 \quad (113)$$

for all $3 \leq i \leq 2m-3$ and $m \geq 2$. Note that this recurrence is nothing but recurrence (110) from above. In the next step observe that $\sum_{j=0}^{5} c_j(i,m,0) = 0$ which shows that any constant is a solution of (113) or (110). Therefore $A_2(i,m) = A_2'(i,m) + \sum_{j=2m-2}^{2m}(-1)^j h_2(j,m)$ is a solution of (110) for all $3 \leq i \leq 2m+1$ and $m \geq 2$.

$\diamondsuit$

Also for the final part we follow the strategy of the proof for identity (76) and compute a recurrence that contains the left hand side

$$L(i,m) := 6(-1)^i A_2(i,m) - A_0(i,m) + 2h_2(i-2,m) - 5h_2(i-1,m) - 3(-1)^i t_1(2m-1) \quad (114)$$

of (100) as solution for any $3 \leq i \leq 2m+1$. More precisely, we derive a recurrence in i that is satisfied by all solutions of the form

$$c_1\,(-1)^i + c_2\,(-1)^i\,A_2(i,m) + c_3\,h_2(i-1) + c_4\,h_2(i-2) + c_5\,A_0(i,m) \quad (115)$$

where the constants $c_i \in \mathbb{Q}$ and $3 \leq i \leq 2m+1$.

By Remark 8 the recurrence (110) contains besides $A_2(i,m)$ also the constant solution 1. Therefore changing the signs of the coefficients of $A_2(i,m)$, $A_2(i+2,m)$ and $A_2(i+4,m)$ in (110) gives a recurrence that contains the solutions $c_1\,(-1)^i + c_2\,(-1)^i\,A_2(i,m)$ for $3 \leq i \leq 2m+1$. Moreover one obtains immediately recurrences for $h_2(i-1)$ and $h_2(i-2)$, if one substitutes k by $i-1$ or $i-2$ in the recurrence (102); those recurrences are both valid in the range $3 \leq i \leq 2m$. Then adding these two recurrences with the recurrence (107) for $A_0(i,m)$ and the recurrence for $c_1\,(-1)^i + c_2\,(-1)^i\,A_2(i,m)$ for instance with the packages [SZ94] or [Mal96] gives a homogeneous recurrence of order 9,

$$c_0(i,m)\,S(i,m) + c_1(i,m)\,S(i+1,m) + \cdots + c_9(i,m)\,S(i+9,m) = 0, \quad (116)$$

which contains the solution for (115) and therefore also (114) for all $3 \leq i \leq 2m+1$. The explicit expressions for the $c_j(i,m)$ can be easily recomputed with the Maple package [SZ94] gfun or the Mathematica package [Mal96] GeneratingFunctions.

The following remark shows that in the range $3 \leq i \leq 2m-8$ the leading coefficient $c_9(i,m)$ in (116) does not vanish.

Remark 9. The leading coefficient $c_9(i,m)$ in (116) is

$$(-1+i)i(5+i-2m)(6+i-2m)(7+i-2m)(6+i+2m)(7+i+2m)(8+i+2m)\,p(i,m)$$

with the polynomial

$$
\begin{aligned}
p(i,m) = {} & 286934400\, i + 1022116320\, i^2 + 1518851088\, i^3 + 1277708580\, i^4 + 764090658\, i^5 + 449853750\, i^6 + 295140024\, i^7 + 169048296\, i^8 + \\
& 70449102\, i^9 + 20137410\, i^{10} + 3830652\, i^{11} + 462672\, i^{12} + 32076\, i^{13} + 972\, i^{14} - 1012435200\, m - 5482935360\, i\, m - \\
& 14278564656\, i^2 m - 23213538792\, i^3 m - 26151610626\, i^4 m - 21597662355\, i^5 m - 13483008435\, i^6 m - 6458953002\, i^7 m - \\
& 2388711936\, i^8 m - 682573911\, i^9 m - 150177099\, i^{10} m - 25219524\, i^{11} m - 3183792\, i^{12} m - 295824\, i^{13} m - 19752\, i^{14} m - 912\, i^{15} m - \\
& 24\, i^{16} m + 2024870400\, m^2 + 6988032000\, i\, m^2 + 10391171184\, i^2 m^2 + 7815524436\, i^3 m^2 + 1403744502\, i^4 m^2 - 3433378715\, i^5 m^2 - \\
& 4306205557\, i^6 m^2 - 2795766398\, i^7 m^2 - 1193680594\, i^8 m^2 - 353952095\, i^9 m^2 - 72888049\, i^{10} m^2 - 9860680\, i^{11} m^2 - \\
& 699794\, i^{12} m^2 + 13148\, i^{13} m^2 + 7588\, i^{14} m^2 + 608\, i^{15} m^2 + 16\, i^{16} m^2 + 13619181888\, m^3 + 62360574624\, i\, m^3 + \\
& 138500395704\, i^2 m^3 + 196534634088\, i^3 m^3 + 197750758622\, i^4 m^3 + 148547525347\, i^5 m^3 + 85793698617\, i^6 m^3 + \\
& 38793616174\, i^7 m^3 + 13884128774\, i^8 m^3 + 3953893039\, i^9 m^3 + 896125845\, i^{10} m^3 + 160785656\, i^{11} m^3 + 22549686\, i^{12} m^3 + \\
& 2412656\, i^{13} m^3 + 188176\, i^{14} m^3 + 9728\, i^{15} m^3 + 256\, i^{16} m^3 + 1472884992\, m^4 + 17277933456\, i\, m^4 + 55255553160\, i^2 m^4 + \\
& 97387495728\, i^3 m^4 + 114340534314\, i^4 m^4 + 96961318441\, i^5 m^4 + 61869893729\, i^6 m^4 + 30390885826\, i^7 m^4 + 11641806632\, i^8 m^4 + \\
& 3500762485\, i^9 m^4 + 827525561\, i^{10} m^4 + 153150896\, i^{11} m^4 + 21942460\, i^{12} m^4 + 2380016\, i^{13} m^4 + 187216\, i^{14} m^4 + 9728\, i^{15} m^4 + \\
& 256\, i^{16} m^4 - 25007101632\, m^5 - 96066457488\, i\, m^5 - 181925812632\, i^2 m^5 - 221098537536\, i^3 m^5 - 189184506516\, i^4 m^5 - \\
& 118702812708\, i^5 m^5 - 55719392004\, i^6 m^5 - 19763963784\, i^7 m^5 - 5314961340\, i^8 m^5 - 1080128100\, i^9 m^5 - 164117244\, i^{10} m^5 - \\
& 18299232\, i^{11} m^5 - 1456536\, i^{12} m^5 - 78336\, i^{13} m^5 - 2304\, i^{14} m^5 - 3283424064\, m^6 - 24487119216\, i\, m^6 - 62740182552\, i^2 m^6 - \\
& 91955201268\, i^3 m^6 - 89825565356\, i^4 m^6 - 62243885388\, i^5 m^6 - 31512673972\, i^6 m^6 - 11837036352\, i^7 m^6 - 3321422232\, i^8 m^6 - \\
& 695467968\, i^9 m^6 - 107666704\, i^{10} m^6 - 12118848\, i^{11} m^6 - 968336\, i^{12} m^6 - 52224\, i^{13} m^6 - 1536\, i^{14} m^6 + 21238328448\, m^7 + \\
& 64255523184\, i\, m^7 + 96366768720\, i^2 m^7 + 92096792976\, i^3 m^7 + 60749842800\, i^4 m^7 + 28468056528\, i^5 m^7 + 9555442224\, i^6 m^7 + \\
& 2283228864\, i^7 m^7 + 379758816\, i^8 m^7 + 42201312\, i^9 m^7 + 2991072\, i^{10} m^7 + 138240\, i^{11} m^7 + 4608\, i^{12} m^7 + 6290303040\, m^8 + \\
& 23115529440\, i\, m^8 + 38148908736\, i^2 m^8 + 38686906320\, i^3 m^8 + 26702560752\, i^4 m^8 + 13015763424\, i^5 m^8 + 4522579392\, i^6 m^8 + \\
& 1111374432\, i^7 m^8 + 188339328\, i^8 m^8 + 21100656\, i^9 m^8 + 1495536\, i^{10} m^8 + 69120\, i^{11} m^8 + 2304\, i^{12} m^8 - 5912676864\, m^9 - \\
& 12381602112\, i\, m^9 - 13532698560\, i^2 m^9 - 9775140480\, i^3 m^9 - 4854527040\, i^4 m^9 - 1615086720\, i^5 m^9 - 339571200\, i^6 m^9 - \\
& 40320000\, i^7 m^9 - 2053440\, i^8 m^9 - 2620173312\, m^{10} - 5651547264\, i\, m^{10} - 5713697280\, i^2 m^{10} - 3829696704\, i^3 m^{10} - \\
& 1857903168\, i^4 m^{10} - 627501888\, i^5 m^{10} - 134592960\, i^6 m^{10} - 16128000\, i^7 m^{10} - 821376\, i^8 m^{10} - 209811456\, m^{11} - \\
& 681806592\, i\, m^{11} - 312221952\, i^2 m^{11} + 87664896\, i^3 m^{11} + 91535616\, i^4 m^{11} + 20217600\, i^5 m^{11} + 1347840\, i^6 m^{11} + 46356480\, m^{12} - \\
& 92874240\, i\, m^{12} - 77865984\, i^2 m^{12} + 29221632\, i^3 m^{12} + 30511872\, i^4 m^{12} + 6739200\, i^5 m^{12} + 449280\, i^6 m^{12} + 111992832\, m^{13} + \\
& 124056576\, i\, m^{13} + 24192000\, i^2 m^{13} + 38633472\, m^{14} + 35444736\, i\, m^{14} + 6912000\, i^2 m^{14} + 5308416\, m^{15} + 1327104\, m^{16}.
\end{aligned}
$$

By inspection the first 8 linear factors in $c_9(i,m)$ do not vanish in the range $3 \le i \le 2m-8$. Moreover by the cylindric algebraic decomposition method [CH91] (QEPCAD B version 1.30 of Hoon Hong) there does not exist any solution $i \in \mathbb{R}$ and $m \in \mathbb{R}$ such that $p(i,m) = 0$ and $3 \le i \le 2m$. Hence there also does not exist any positive integer i, m with $3 \le i \le 2m-8$ and $c_9(i,m) \neq 0$. ♢

Now we show that the first nine initial values of $L(i,m)$ are 0, i.e., we will verify that $L(r,m) = 0$ for all $3 \le r \le 11$ and all $2m+1 \ge r$. Then together with the fact that $L(i,m)$ is a solution of (116) for all $3 \le i \le 2m+1$ and $c_9(i,m) \neq 0$ for all $3 \le i \le 2m-8$, it follows that $L(i,m) = 0$ for all $3 \le i \le 2m+1$. The crucial step is that $L(r,m) = 0$ for $3 \le r \le 11$ and all m with $2m+1 \ge r$ if and only if

$$6(-1)^r A_1(r,m) + A_0(r,m) + 4h_2(r-2,m) - h_2(r-1,m) = 0. \tag{117}$$

Similarly as in the case of identity (94), we derive representations[8] for $A_0(r,m)$ that hold for all m with $2m+1 \ge r$, namely

$$A_0(2,m) = -3h_2(0,m) + h_2(1,m), \quad m \ge 1, \tag{118}$$

and

$$
\begin{aligned}
A_0(3,m) &= 6\, h_2(0,m) - 4\, h_2(1,m) + h_2(2,m), \\
A_0(4,m) &= \frac{2\,(9 + m + 2m^2)\, h_2(0,m) - 12\, h_2(1,m) - (-3 + m + 2m^2)\, h_2(2,m)}{(-1+m)(3+2m)}, \\
A_0(5,m) &= \frac{-12\,(-9 + 2m + 4m^2)\, h_2(0,m) + 3\,(-24 + 3m + 7m^2 + 4m^3 + 4m^4)\, h_2(1,m) - (-18 + 3m + 7m^2 + 4m^3 + 4m^4)\, h_2(2,m)}{(-1+m)\,(2+m)\,(-3+2m)\,(3+2m)}, \\
A_0(6,m) &= \big(-12\,(15 - m - m^2 + 4m^3 + 4m^4)\, h_2(0,m) + (120 - 19m - 31m^2 + 28m^3 + 28m^4)\, h_2(1,m) + \\
&\qquad (-30 + 13m + 25m^2 - 4m^3 - 4m^4)\, h_2(2,m)\big)/(-2+m)\,(-1+m)\,(3+2m)\,(5+2m), \\
A_0(7,m) &= \big(-4\,(-4050 + 1440m + 2865m^2 - 82m^3 - 191m^4 - 256m^5 - 152m^6 + 32m^7 + 16m^8)\, h_2(0,m) + \\
&\qquad (-10800 + 3060m + 6339m^2 + 838m^3 + 647m^4 - 464m^5 - 328m^6 - 32m^7 - 16m^8)\, h_2(1,m) + \\
&\qquad 2\,(1350 - 495m - 954m^2 + 134m^3 + 85m^4 - 112m^5 - 56m^6 + 32m^7 + 16m^8)\, h_2(2,m)\big)/ \\
&\quad ((-2+m)\,(-1+m)\,(2+m)\,(3+m)\,(-5+2m)\,(-3+2m)\,(3+2m)\,(5+2m)), \\
A_0(8,m) &= \big(4\,(85050 - 31050m - 51525m^2 + 40671m^3 + 32579m^4 - 19123m^5 - 11750m^6 + 1736m^7 + 928m^8 + 80m^9 + 32m^{10})\, h_2(0,m) + \\
&\qquad (-226800 + 98820m + 172035m^2 - 100077m^3 - 88313m^4 + 28501m^5 + 19850m^6 + 1288m^7 + 224m^8 - 560m^9 - 224m^{10})\, h_2(1,m) + \\
&\qquad 2\,(28350 - 19305m - 34983m^2 + 14460m^3 + 14195m^4 - 775m^5 - 974m^6 - 760m^7 - 320m^8 + 80m^9 + 32m^{10})\, h_2(2,m)\big)/ \\
&\quad ((-3+m)\,(-2+m)\,(-1+m)\,(2+m)\,(3+m)\,(-5+2m)\,(-3+2m)\,(3+2m)\,(5+2m)\,(7+2m)),
\end{aligned}
$$

[8] Note that identity (118) implies Lemma 4.

$$\begin{aligned}
A_0(9,m) = \big(&18\,(-176400 + 16800\,m + 34090\,m^2 + 2353\,m^3 + 4274\,m^4 + 4365\,m^5 + 2098\,m^6 - 1368\,m^7 - 624\,m^8 + 80\,m^9 + 32\,m^{10})\,h_2(0,m) - \\
&10\,(-211680 + 8442\,m + 15159\,m^2 - 5996\,m^3 - 1538\,m^4 + 10353\,m^5 + 5754\,m^6 - 1944\,m^7 - 912\,m^8 + 80\,m^9 + 32\,m^{10})\,h_2(1,m) + \\
&(-529200 + 76860\,m + 125976\,m^2 - 107021\,m^3 - 87398\,m^4 + 46245\,m^5 + 28002\,m^6 - 4824\,m^7 - 2352\,m^8 + 80\,m^9 + 32\,m^{10})\,h_2(2,m)\big)/ \\
&((-3+m)\,(-2+m)\,(2+m)\,(3+m)\,(4+m)\,(-7+2\,m)\,(-5+2\,m)\,(-3+2\,m)\,(5+2\,m)\,(7+2\,m)),
\end{aligned}$$

$$\begin{aligned}
A_0(10,m) = \big(&6\,(57153600 - 24872400\,m - 41202000\,m^2 + 33029298\,m^3 + 27322823\,m^4 - 13673507\,m^5 - 9012177\,m^6 + \\
&296549\,m^7 + 441274\,m^8 + 375684\,m^9 + 120424\,m^{10} - 32272\,m^{11} - 10272\,m^{12} + 448\,m^{13} + 128\,m^{14})\,h_2(0,m) + \\
&2\,(-114307200 + 56269080\,m + 98214930\,m^2 - 56493117\,m^3 - 52362987\,m^4 + 10634788\,m^5 + 9427703\,m^6 + 3727349\,m^7 + \\
&1168354\,m^8 - 909516\,m^9 - 328536\,m^{10} + 38768\,m^{11} + 13408\,m^{12} + 448\,m^{13} + 128\,m^{14})\,h_2(1,m) - 3\,(-3 + m + 2\,m^2)^2 \\
&(-2116800 + 114240\,m + 247244\,m^2 + 74384\,m^3 + 70991\,m^4 - 8327\,m^5 - 6158\,m^6 - 1016\,m^7 - 448\,m^8 + 80\,m^9 + 32\,m^{10})\,h_2(2,m)\big)\Big/ \\
&((-4+m)\,(-3+m)\,(-2+m)\,(-1+m)\,(2+m)\,(3+m)\,(4+m)\,(-7+2\,m)\,(-5+2\,m)\,(-3+2\,m)\,(3+2\,m)\,(5+2\,m)\,(7+2\,m)\,(9+2\,m)),
\end{aligned}$$

$$\begin{aligned}
A_0(11,m) = \big(&-4\,(-3857868000 + 1828915200\,m + 3455037180\,m^2 - \\
&769680063\,m^3 - 573285609\,m^4 + 410292357\,m^5 + 75600887\,m^6 - 317122186\,m^7 - 103854870\,m^8 + 70617976\,m^9 + \\
&23616736\,m^{10} - 5008224\,m^{11} - 1596608\,m^{12} + 68992\,m^{13} + 22272\,m^{14} + 2048\,m^{15} + 512\,m^{16})\,h_2(0,m) + \\
&(-10287648000 + 4421239200\,m + 8271754020\,m^2 - 2100636018\,m^3 - \\
&1223321355\,m^4 + 1917190986\,m^5 + 662194436\,m^6 - 1008261760\,m^7 - 385832709\,m^8 + 155348368\,m^9 + \\
&57307408\,m^{10} - 5439936\,m^{11} - 2089952\,m^{12} - 245504\,m^{13} - 56064\,m^{14} + 11264\,m^{15} + 2816\,m^{16})\,h_2(1,m) - \\
&3\,(-857304000 + 470156400\,m + 809515080\,m^2 - \\
&491016438\,m^3 - 333442341\,m^4 + 359794142\,m^5 + 179100812\,m^6 - 102079240\,m^7 - 45758047\,m^8 + \\
&7260176\,m^9 + 3326576\,m^{10} + 426304\,m^{11} + 84832\,m^{12} - 51968\,m^{13} - 13568\,m^{14} + 1024\,m^{15} + 256\,m^{16})\,h_2(2,m)\big)/ \\
&((-4+m)\,(-3+m)\,(-2+m)\,(-1+m) \\
&(2+m)\,(3+m)\,(4+m)\,(5+m)\,(-9+2\,m)\,(-7+2\,m)\,(-5+2\,m)\,(-3+2\,m)\,(3+2\,m)\,(5+2\,m)\,(7+2\,m)\,(9+2\,m)).
\end{aligned}$$

Details about the way how these representations have been derived and proved can be found in

Remark 10. For $A_0(r,m)=\sum_{k=0}^{2m}\binom{r+k-3}{r-2}h_2(k,m)$ with $r\in\{3,\dots,11\}$ we can compute with Sigma expressions $g_r(k,m)$ such that

$$g_r(k+1,m)-g_r(k,m)=\binom{r+k-3}{r-2}h_2(k,m)$$

holds for all $0\leq k\leq 2m-3$. The g_r are listed explicitly in Appendix B. Note that the telescoping equations for each case $2\leq r\leq 11$ can be verified by applying the relation (102). Moreover, the identities for $A_0(r,m)$ can be derived by telescoping and using additionally the representations

$$h_2(2m-2,m)=\frac{1+12\,m-24\,m^2+8\,m^3}{-4+16\,m},\qquad h_2(2m-1,m)=\frac{1-2\,m}{2},\qquad h_2(2m,m)=1$$

that hold for $m\geq 1$. $\diamondsuit$

Hence we see that for $r\in\{2,\dots,12\}$ each $A_0(r,m)$ can be expressed in terms of $h_2(0,m)$, $h_2(1,m)$ and $h_2(2,m)$. Moreover, by (98) each corresponding $A_1(r,m)$ can be written as a linear combination in terms of $h_2(0,m),\dots,h_2(r-3,m)$.

In particular by applying relation (102) each of the $h_2(j,m)$ for $3\leq j\leq r-3$ can be reduced to $h_2(0,m)$, $h_2(1,m)$ and $h_2(2,m)$ where $m>(r-1)/2$. In other words, the left side of identity (94) can be expressed in terms of $h_2(0,m)$, $h_2(1,m)$ and $h_2(2,m)$ that occur only linearly. Since this expression collapses to 0, identity (94) holds for $m>(r-1)/2$. The special case $2m+1=r$ of identity (94) can be shown separately by simple evaluation.

Applying this proof strategy for all $r\in\{3,\dots,11\}$ shows that $L(r,m)=0$ for all m,r with $2m+1\geq r$ and $3\leq r\leq 11$ which finally proves that $L(i,m)=0$ for all $3\leq i\leq 2m+1$.

8. Completing missing proof steps

What remains to show are Propositions 3 and 5 and that

$$\begin{aligned}
b(m) &:= B_0(2m+1,m)=\sum_{k=0}^{2m-1}\binom{2m+k-2}{2m-1}h_1(k,m),\\
a(m) &:= A_0(2m+2,m)=\sum_{k=0}^{2m}\binom{2m+k-2}{2m-1}h_2(k,m) \qquad (119)
\end{aligned}$$

are solutions of (73) and (74) respectively for all $m \geq 1$. In order to accomplish this, we first compute

$$\begin{aligned}
&2(k+2m)(-6k-13k^2-3k^3+11k^4+9k^5+2k^6+12m-6km-72k^2m-72k^3m-2k^4m\\
&\quad+24k^5m+8k^6m+66m^2+138km^2-20k^2m^2-192k^3m^2-160k^4m^2-48k^5m^2+42m^3+342km^3\\
&\quad+306k^2m^3+186k^3m^3+96k^4m^3-240m^4+108km^4+72k^2m^4-108k^3m^4-312m^5-144km^5+72k^2m^5)\,h_1(k,m)\\
&(1+4m)(18k^2+39k^3+9k^4-33k^5-27k^6-6k^7-6km+46k^2m+112k^3m+100k^4m\\
&\quad+56k^5m+16k^6m-60m^2-162km^2+14k^2m^2+252k^3m^2+172k^4m^2+24k^5m^2-180m^3-528km^3\\
&\quad-504k^2m^3-192k^3m^3-24k^4m^3-240km^4-432k^2m^4-144k^3m^4+240m^5+432km^5+144k^2m^5)\,h_1(k+1,m)\\
&(2+k-2m)(1+4m)(-3k^2-5k^3+k^4+5k^5+2k^6-14k^2m-18k^3m-4k^4m+12m^2\\
&\quad+24km^2-42k^2m^2-54k^3m^2-12k^4m^2+48m^3+120km^3-36k^3m^3+48m^4+144km^4+72k^2m^4)\,h_1(k+2,m)\\
&\quad+4(1+k)^2(-1+k-2m)(k-2m)m(1+2m)(-1+4m)(1+4m)\,h_1(k,m+1)=0, \qquad (120)
\end{aligned}$$

which holds for all $m, k \geq 0$, with Sigma [Sch00, Sch01] or an extended version of [PS95].

Remark 11. As in Remark 1 we consider the sum $h_1'(k,m) := \sum_{s=0}^{\lfloor \frac{2m-k-1}{2} \rfloor - 1} q'(s,k,m)$. Recall that $q'(s,k,m) = q(s,k,m)$ for $0 \leq s \leq \lfloor \frac{2m-k-1}{2} \rfloor - 1$ where the summand $q(s,k,m)$ of $h_1(k,m) = \sum_{s=0}^{\lfloor \frac{2m-k-1}{2} \rfloor} q(s,k,m)$ is given in (79).

In the sequel denote

$$p(s,k,m) = (-1)^k \binom{m-s-1}{2m-2s-k-3} \qquad \text{and} \qquad u(s,m) = \frac{(-1)^s (1-3m)_s (m)_s}{(3/2-2m)_s \, s! \, 4^s},$$

i.e.,

$$q'(s,k,m) = -p(s,k,m)\,u(s,m) \frac{3m-3s-1}{3m-1} \frac{(m-s-k-1)(m-s-k-2)}{(2m-2s-k-1)(2m-2s-k-2)}.$$

Now applying this representation of the definite sum $h_1'(k,m)$ to Sigma gives the creative telescoping equation

$$\begin{aligned}
&g(s+1,k,m) - g(s,k,m)\\
&\qquad = c_0(k,m)q'(s,k,m) + \cdots + c_2(k,m)q'(s,k+2,m) + \kappa(k,m)\,q'(s,k,m+1). \qquad (121)
\end{aligned}$$

with

$$\begin{aligned}
c_0(k,m) &= 2(k+2m)(-1+3m)(-6k-13k^2-3k^3+11k^4+9k^5+2k^6+12m-6km-72k^2m-72k^3m-2k^4m\\
&\quad+24k^5m+8k^6m+66m^2+138km^2-20k^2m^2-192k^3m^2-160k^4m^2-48k^5m^2+42m^3+342km^3\\
&\quad+306k^2m^3+186k^3m^3+96k^4m^3-240m^4+108km^4+72k^2m^4-108k^3m^4-312m^5-144km^5+72k^2m^5),\\
c_1(k,m) &= (-1+3m)(1+4m)(18k^2+39k^3+9k^4-33k^5-27k^6-6k^7-6km+46k^2m+112k^3m+100k^4m\\
&\quad+56k^5m+16k^6m-60m^2-162km^2+14k^2m^2+252k^3m^2+172k^4m^2+24k^5m^2-180m^3-528km^3\\
&\quad-504k^2m^3-192k^3m^3-24k^4m^3-240km^4-432k^2m^4-144k^3m^4+240m^5+432km^5+144k^2m^5),\\
c_2(k,m) &= (2+k-2m)(-1+3m)(1+4m)(-3k^2-5k^3+k^4+5k^5+2k^6-14k^2m-18k^3m-4k^4m+12m^2\\
&\quad+24km^2-42k^2m^2-54k^3m^2-12k^4m^2+48m^3+120km^3-36k^3m^3+48m^4+144km^4+72k^2m^4),\\
\kappa(k,m) &= 4(1+k)^2(-1+k-2m)(k-2m)m(1+2m)(-1+3m)(-1+4m)(1+4m)
\end{aligned}$$

and

$$g(s,k,m) = -p(s,k,m)\,u(k,m)\,\frac{a(s,k,m)}{b(s,k,m)}$$

where

$$\begin{aligned}
b(s,k,m) &= (3m-s)(3m-s+1)(3m-s+2)\\
&\qquad (2m-2s-k+1)(2m-2s-k)(2m-2s-k-1)(2m-2s-k-2)
\end{aligned}$$

and

$$\begin{aligned}
a(s,k,m) = {} & 2\,(-1+4m-2s)\,(m-s)\,s\,(2+k-m+s)\,\big(12k^2+20k^3-16k^4-40k^5-4k^6+20k^7+8k^8-48km+38k^2m+368k^3m+\\
& 120k^4m-452k^5m-238k^6m+132k^7m+80k^8m+48m^2-396km^2-786k^2m^2+1937k^3m^2+2200k^4m^2-1476k^5m^2-\\
& 1720k^6m^2+103k^7m^2+282k^8m^2+504m^3-762km^3-6194k^2m^3+4457k^3m^3+9396k^4m^3-848k^5m^3-\\
& 4204k^6m^3-843k^7m^3+414k^8m^3+2280m^4+1080km^4-22806k^2m^4+10392k^3m^4+22080k^4m^4+1308k^5m^4-\\
& 3462k^6m^4-2016k^7m^4+216k^8m^4+7764m^5+4926km^5-72276k^2m^5+22866k^3m^5+49980k^4m^5-492k^5m^5+\\
& 864k^6m^5-1296k^7m^5+26400m^6+19260km^6-183792k^2m^6-16608k^3m^6+84012k^4m^6-1152k^5m^6+1944k^6m^6+\\
& 68940m^7+89892km^7-247536k^2m^7-132444k^3m^7+59004k^4m^7+432k^5m^7+106608m^8+192264km^8-104424k^2m^8-\\
& 137592k^3m^8+8424k^4m^8+84672m^9+171936km^9+43200k^2m^9-33696k^3m^9+26784m^{10}+53568km^{10}+26784k^2m^{10}+\\
& 24ks+10k^2s-148k^3s-120k^4s+140k^5s+134k^6s-16k^7s-24k^8s-48ms+288kms+886k^2ms-1176k^3ms-\\
& 2324k^4ms+508k^5ms+1414k^6ms+68k^7ms-192k^8ms-624m^2s+468km^2s+8320k^2m^2s-1403k^3m^2s-12894k^4m^2s-\\
& 2314k^5m^2s+5000k^6m^2s+1233k^7m^2s-474k^8m^2s-3456m^3s-4494km^3s+35338k^2m^3s+5388k^3m^3s-32542k^4m^3s-\\
& 12234k^5m^3s+5508k^6m^3s+3540k^7m^3s-360k^8m^3s-14076m^4s-21870km^4s+113340k^2m^4s+11634k^3m^4s-\\
& 64344k^4m^4s-14436k^5m^4s-2952k^6m^4s+2880k^7m^4s-56196m^5s-68496km^5s+313104k^2m^5s+68664k^3m^5s-\\
& 123732k^4m^5s-240k^5m^5s-6048k^6m^5s-171144m^6s-238824km^6s+508944k^2m^6s+280704k^3m^6s-126096k^4m^6s+\\
& 3312k^5m^6s-308280m^7s-535920km^7s+279240k^2m^7s+362448k^3m^7s-31248k^4m^7s-284976m^8s-561312km^8s-\\
& 127728k^2m^8s+117504k^3m^8s-103680m^9s-207360km^9s-103680k^2m^9s-60ks^2-232k^2s^2+79k^3s^2+468k^4s^2+\\
& 40k^5s^2-258k^6s^2-59k^7s^2+22k^8s^2+120ms^2-114kms^2-3266k^2ms^2-965k^3ms^2+4696k^4ms^2+1832k^5ms^2-\\
& 1788k^6ms^2-645k^7ms^2+130k^8ms^2+1176m^2s^2+2952km^2s^2-17924k^2m^2s^2-11450k^3m^2s^2+16234k^4m^2s^2+\\
& 11382k^5m^2s^2-2690k^6m^2s^2-2200k^7m^2s^2+168k^8m^2s^2+6756m^3s^2+17646km^3s^2-63760k^2m^3s^2-37310k^3m^3s^2+\\
& 29100k^4m^3s^2+21416k^5m^3s^2+3160k^6m^3s^2-2256k^7m^3s^2+37284m^4s^2+59616km^4s^2-196536k^2m^4s^2-\\
& 86712k^3m^4s^2+55860k^4m^4s^2+7776k^5m^4s^2+6792k^6m^4s^2+147432m^5s^2+211320km^5s^2-393120k^2m^5s^2-\\
& 237096k^3m^5s^2+84600k^4m^5s^2-6768k^5m^5s^2+327072m^6s^2+544704km^6s^2-284064k^2m^6s^2-356256k^3m^6s^2+\\
& 38880k^4m^6s^2+362304m^7s^2+688320km^7s^2+134208k^2m^7s^2-157248k^3m^7s^2+155520m^8s^2+311040km^8s^2+\\
& 155520k^2m^8s^2+348k^2s^3+271k^3s^3-454k^4s^3-350k^5s^3+112k^6s^3+79k^7s^3-6k^8s^3-534kms^3+3486k^2ms^3+\\
& 3936k^3ms^3-2786k^4ms^3-3286k^5ms^3+284k^6ms^3+508k^7ms^3-24k^8ms^3-324m^2s^3-4410km^2s^3+15680k^2m^2s^3+\\
& 18170k^3m^2s^3-4716k^4m^2s^3-9632k^5m^2s^3-1520k^6m^2s^3+768k^7m^2s^3-7212m^3s^3-17760km^3s^3+54968k^2m^3s^3+\\
& 43056k^3m^3s^3-6188k^4m^3s^3-7200k^5m^3s^3-3456k^6m^3s^3-50160m^4s^3-73776km^4s^3+140256k^2m^4s^3+\\
& 97248k^3m^4s^3-18672k^4m^4s^3+3600k^5m^4s^3-152784m^5s^3-241488km^5s^3+140928k^2m^5s^3+164304k^3m^5s^3-\\
& 19440k^4m^5s^3-213600m^6s^3-388800km^6s^3-57312k^2m^6s^3+100224k^3m^6s^3-112320m^7s^3-224640km^7s^3-\\
& 112320k^2m^7s^3+60ks^4-158k^2s^4-306k^3s^4+118k^4s^4+270k^5s^4+40k^6s^4-24k^7s^4-120ms^4+468kms^4-\\
& 1356k^2ms^4-2580k^3ms^4+344k^4ms^4+1668k^5ms^4+376k^6ms^4-96k^7ms^4-324m^2s^4+1428km^2s^4-6520k^2m^2s^4-\\
& 8376k^3m^2s^4-800k^4m^2s^4+2208k^5m^2s^4+864k^6m^2s^4+4812m^3s^4+7524km^3s^4-23088k^2m^3s^4-18684k^3m^3s^4-\\
& 180k^4m^3s^4-576k^5m^3s^4+28656m^4s^4+41784km^4s^4-35448k^2m^4s^4-36072k^3m^4s^4+3384k^4m^4s^4+56832m^5s^4+\\
& 98208km^5s^4+6912k^2m^5s^4-30240k^3m^5s^4+38880m^6s^4+77760km^6s^4+38880k^2m^6s^4-24ks^5+20k^2s^5+84k^3s^5+\\
& 4k^4s^5-60k^5s^5-24k^6s^5+48ms^5-96kms^5+152k^2ms^5+456k^3ms^5+64k^4ms^5-240k^5ms^5-96k^6ms^5+120m^2s^5+\\
& 24km^2s^5+1104k^2m^2s^5+1056k^3m^2s^5+192k^4m^2s^5-1272m^3s^5-1344km^3s^5+3768k^2m^3s^5+3168k^3m^3s^5-\\
& 5232m^4s^5-8352km^4s^5+720k^2m^4s^5+3456k^3m^4s^5-5184m^5s^5-10368km^5s^5-5184k^2m^5s^5\big).
\end{aligned}$$

In the next step we have to check that this summand telescoping equation (121) holds within the summation range. In order to avoid poles in the evaluation, similarly to Remark 1, we represent all ingredients in (121) not in terms of $p(s,k,m)$ but in terms of $p(s+1,k+2,m)$, i.e.,

$$p(s+1,k,m) = p(s+1,k+2,m)\frac{(m-s-k-4)(m-s-k-5)}{(2m-2s-k-5)(2m-2s-k-6)}$$

and

$$p(s,k+i,m) = p(s+1,k+2,m)\,p_i(s,k,m), \qquad 0 \le i \le 2,$$

where

$$p_i(s,k,m) = -\frac{m-s-1}{2m-2s-k-6}\prod_{j=0}^{2-i}\frac{m-s-k+j-5}{2m-2s-k+j-5}.$$

Moreover, the shifted version of $p(s,k,m)$ in m can be expressed by

$$p(s,k,m+1) = p(s+1,k+2,m)\,p_0(s,k,m)\,\frac{-(m-s)(m-s-k-2)}{(2m-2s-k-2)(2m-2s-k-1)}.$$

Note that the representation of $u(s,k,m)$ will not be changed. Hence, with

$$u(s,m+1) = u(s,m)\,\frac{3(3m+1)(3m+2)(4m-2s-1)(4m-2s+1)(m+s)}{(4m-1)(4m+1)(3m-s)(3m-s+1)(3m-s+2)}$$

we finally manage to verify that (121) holds for $0 \le s \le \lfloor \frac{2m-k-1}{2} \rfloor - 3$ in terms of $u(s,k,m)$ and $p(s+1,k+2,m)$.

Therefore, summing equation (80) over s from 0 to an arbitrary d with $0 \leq d \leq \lfloor \frac{2m-k-1}{2} \rfloor - 3$ gives the relation

$$g(d+1,k,m) - g(0,k,m) = c_0(k,m)\,y(k,m) + \cdots + c_2(k,m)\,y(k+2,m) + \kappa(k,m)\,y(k,m+1) \quad (122)$$

for the sum $y(k,m) = \sum_{s=0}^{d} q(s,k,m) = \sum_{s=0}^{d} q'(s,k,m)$. Now we do a case distinction on k.

If k is even, we can choose $d = m - k/2 - 4$, and obtain

$$\begin{aligned} &c_0(k,m)\,h_1(k,m) + \cdots + c_2(k,m)\,h_1(k+2,m) + \kappa(k,m+1)\,h_1(k,m+1) \\ &\quad = g(m - \frac{k}{2} - 3, k, m) - g(0,k,m) + \sum_{i=0}^{2} c_i(k,m) \sum_{j=1}^{3} q(m - \frac{k}{2} - j, k+i, m) \\ &\qquad\qquad + \kappa(k,m) \sum_{j=1}^{4} q(m + 1 - \frac{k}{2} - j, k, m+1). \end{aligned}$$

By term rewriting one can now show that the right hand side is equal to 0. Hence for $m - k/2 - 4 \geq 0$, k even, the relation (120) holds. Similarly, for odd k, one can choose $d = m - (k-1)/2 - 3$, and obtains

$$\begin{aligned} &c_0(k,m)\,h_1(k,m) + \cdots + c_2(k,m)\,h_1(k+2,m) \\ &\quad = g(m - \frac{k-1}{2} - 2, k, m) - g(0,k,m) + \sum_{i=0}^{2} c_i(k,m) \sum_{j=0}^{2} q(m - \frac{k-1}{2} - j, k+i, m) \\ &\qquad\qquad + \kappa(k,m) \sum_{j=0}^{3} q(m + 1 - \frac{k-1}{2} - j, k, m+1). \end{aligned}$$

Again one can prove that the right hand side vanishes to 0, and hence the relation (120) holds for $m - (k-1)/2 - 3 \geq 0$, k odd. Summarizing, for all $m \geq 4$ and $0 \leq k \leq 2m - 8$ the recurrence (120) contains the solution $h_1(k,m)$. The special cases $0 \leq m \leq 4$ and $k \geq 2m - 7$ for $m \geq 4$ can be verified by simple evaluation. This shows that the recurrence holds for all $m, k \geq 0$. $\diamondsuit$

Now let us turn to the identities (12), (17) and the sum expression $b(m)$.

Given the recurrences (78) and (120) we are able to compute with Sigma the recurrence

$$3(3m-1)(3m+1)S(m) - (4m-1)(4m+1)S(m+1) = 0 \quad (123)$$

that is satisfied by the left sides of (12) and (17) for $m \geq 1$. The verification is explained in

Remark 12. Denote $f(k,m) = (-1)^k h_1(k,m)$. With Sigma we obtain polynomials

$$c_0(m) = 12m(1+2m)(-1+3m)(1+3m), \qquad c_1(m) = -4m(1+2m)(-1+4m)(1+4m)$$

and the expression

$$\begin{aligned} g(k,m) = &\frac{1}{(1+k)^2\,(-1+k-2m)\,(k-2m)} \\ &\big(-2(-1)^k k(1+4m)\big(-2k - 5k^2 - 2k^3 + 4k^4 + 4k^5 + k^6 + 8m + 6km - 11k^2m - 19k^3m - 15k^4m - 5k^5m + 8m^2 + 32km^2 + 6k^2m^2 - \\ &\quad 2k^3m^2 + 4k^4m^2 - 64m^3 - 36km^3 + 52k^2m^3 + 36k^3m^3 - 192km^4 - 144k^2m^4 + 192m^5 + 144km^5\big)\,h_1(k,m) + \\ &(-1)^k k(1+4m)\big(-6k - 15k^2 - 6k^3 + 12k^4 + 12k^5 + 3k^6 + 16m + 2km - 37k^2m - 45k^3m - 33k^4m - 11k^5m + 40m^2 + 16km^2 - \\ &\quad 54k^2m^2 - 46k^3m^2 - 4k^4m^2 - 8m^3 + 12km^3 + 92k^2m^3 + 60k^3m^3 - 96km^4 - 144k^2m^4 + 96m^5 + 144km^5\big)\,h_1(1+k,m) - \\ &(-1)^k k(2+k-2m)(1+k+2m)(1+4m)\big(-k - k^2 + k^3 + k^4 + 2m - 3km - 2k^2m - 3k^3m + 6m^2 - 12km^2 + 12m^3\big)\,h_1(2+k,m)\big) \end{aligned}$$

such that

$$c_0(m)f(k,m) + c_1(m)f(k,m+1) = g(k+1,m) - g(k,m) \quad (124)$$

holds for all $0 \leq k \leq 2m - 4$. The verification can be achieved as follows. By applying the shift in k to $g(k,m)$ the expression $g(k+1,m)$ depends on $(-1)^{k+1}$, $h_1(k+1,m)$, $h_1(k+2,m)$, and $h_1(k+3,m)$. Then the term $(-1)^{k+1}$ can be rewritten with $-(-1)^k$, and $h_1(k+3,m)$ can be expressed by $h_1(k,m)$, $h_1(k+1,m)$ and $h_1(k+2,m)$ by applying the relation given in (78) for any $0 \leq k < 2m - 3$. Moreover, the expression $f(m+1,k)$ depends on $(-1)^k$ and $h_1(k,m+1)$, $h_1(k+1,m+1)$ and $h_1(k+2,m+1)$. Now the crucial step is to apply rule (120) and represent

$h_1(k, m+1)$ in terms of $h_1(k, m)$ and $h_1(k+1, m)$. This rewriting holds for all $0 \leq k < 2m$. In this representation the equation (124) can be verified by simple term rewriting for $0 \leq k \leq 2m-4$. Summing equation (124) over k from 0 to $2m-4$ gives

$$\begin{aligned} &c_0(m)S(m) + c_1(m)S(m+1) \\ &\qquad = g(2m-3, i, m) - g(0, i, m) + c_0(m) \sum_{r=2m-3}^{2m-1} f(r, m) + c_1(m) \sum_{r=2m-3}^{2m+1} f(r, m+1) \end{aligned} \tag{125}$$

where $S(m)$ denotes the left hand side of (12). In this case one can show that the right hand side of (125) vanishes to 0 which gives recurrence (123). The special cases $m = 1, 2$ follow by simple evaluation.

Completely analogously, by choosing $f(k, m) = 2^k h_1(k, m)$,

$$c_0(m) = -12m(1+2m)(-1+3m)(1+3m), \qquad c_1(m) = 4m(1+2m)(-1+4m)(1+4m)$$

and

$$\begin{aligned} g(k,m) = &\frac{1}{(1+k)^2\,(-1+k-2m)\,(k-2m)}\Big(-2^{1+k}\,(1+4m)\,(2k^2-k^3-7k^4-k^5+5k^6+2k^7-2km- \\ &\quad 18k^2m+8k^3m+16k^4m-24k^5m-16k^6m+12m^2+10km^2-38k^2m^2-60k^3m^2+68k^4m^2+56k^5m^2+ \\ &\quad 36m^3-8km^3-16k^3m^3-24k^4m^3+48km^4-240k^2m^4-144k^3m^4-48m^5+240km^5+144k^2m^5)\,h_1(k,m)+ \\ &2^k\,(1+4m)\,(6k^2-3k^3-21k^4-3k^5+15k^6+6k^7-34km-62k^2m+40k^3m+ \\ &\quad 48k^4m-60k^5m-40k^6m+60m^2+2km^2-262k^2m^2-120k^3m^2+160k^4m^2+112k^5m^2+180m^3+ \\ &\quad 248km^3-120k^2m^3-152k^3m^3+240km^4-48k^2m^4-432k^3m^4-240m^5+48km^5+432k^2m^5)\,h_1(1+k,m)- \\ &2^k\,(2+k-2m)\,(1+k+2m)\,(1+4m) \\ &\quad (k^2-2k^3-k^4+2k^5-8km+14k^3m-12k^4m+12m^2-12km^2-42k^2m^2+36k^3m^2+24m^3+24km^3-36k^2m^3)\,h_1(2+k,m)\Big) \end{aligned}$$

one can verify that the left hand side of (17) is a solution of recurrence (123). ♢

Since $t_1(2m-1)$ is also a solution of (71), identities (12) and (17) follow by checking their equalities at $m = 1$.

Similarly, we are able to derive the recurrence (73) for $b(m)$. The correctness of this result can be found in

Remark 13. Denote $f(k,m) = \binom{2m+k-2}{2m-1} h_1(k,m)$. With Sigma we obtain polynomials

$$\begin{aligned} c_0(m) = &-3456m^2(1+m)^2(2+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(-1+3m)(1+3m)(2+3m)(4+3m) \\ &(-1+4m)(1+4m)(1+6m)(5+6m)(35582085+208705770m+502426266m^2+659838718m^3+522259397m^4 \\ &+256674880m^5+76794344m^6+12820192m^7+915728m^8), \end{aligned}$$

$$\begin{aligned} c_1(m) = &\,384m^2(1+m)^2(2+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(2+3m)(4+3m)(-1+4m)(1+4m) \\ &(-5091730875-64116259830m-268357073220m^2-344918244168m^3+764136680690m^4+3625956026718m^5 \\ &+6449149601689m^6+6689203564428m^7+4423817173116m^8+1893138648192m^9+508247453296m^{10} \\ &+77855194560m^{11}+5190346304m^{12}), \end{aligned}$$

$$\begin{aligned} c_2(m) = &-128m^2(1+m)^2(2+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(-1+4m)(1+4m)(3+4m)(5+4m) \\ &(-7125148800-88649425620m-376507014210m^2-591098983182m^3+431261276465m^4+3340499936442m^5 \\ &+6290733211249m^6+6633648559692m^7+4412582090796m^8+1892139974208m^9+508247453296m^{10} \\ &+77855194560m^{11}+5190346304m^{12}), \end{aligned}$$

$$\begin{aligned} c_3(m) = &\,512m^2(1+m)^2(2+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(-1+4m)(1+4m)(3+4m)(5+4m)(7+4m) \\ &(9+4m)^2(11+4m)(-61740-644340m-1950794m^2-910998m^3+6194397m^4+13852080m^5 \\ &+12693384m^6+5494368m^7+915728m^8), \end{aligned}$$

and the expression

$$g(k,m) = \frac{-(k-1)\big[p_0(k,m)h(k,m) + p_1(k,m)h(k+1,m) + p_2(k,m)h(k+2,m)\big]}{(k+1)^2 \prod_{i=0}^{5}(2m-k+i) \prod_{i=2}^{4}(2m+k+i)} \times \binom{2m+k-2}{2m-1}, \tag{126}$$

where the polynomials $p_i(k,m)$ in k and m are given in Appendix D, such that

$$c_0(m)f(k,m)+\cdots+c_3(m)f(k,m+3)=g(k+1,m)-g(k,m) \qquad (127)$$

holds for all $0\le k\le 2m-4$. The verification can be achieved as follows. By applying the shift in k to $g(k,m)$ the expression $g(k+1,m)$ depends on $\binom{2m+k-1}{2m-1}$, $h_1(k+1,m)$, $h_1(k+2,m)$, and $h_1(k+3,m)$. Then the term $\binom{2m+k-1}{2m-1}$ can be rewritten with $\frac{2m+k-1}{k}\binom{2m+k-1}{2m-1}$, and $h_1(k+3,m)$ can be expressed by $h_1(k,m)$, $h_1(k+1,m)$ and $h_1(k+2,m)$ by applying the relation given in (78) for any $0\le k<2m-3$. Hence $g(k+1,m)$ can be expressed in terms of $h_1(k,m)$, $h_1(k+1,m)$, $h_1(k+2,m)$, and $\binom{2m+k-1}{2m-1}$ for all $0\le k\le 2m-4$; for the correctness at $k=0$ the factor $k-1$ in the numerator of (126) needed. Moreover, the expressions $f(m+i,k)$ depend on $\binom{2m+2i+k-1}{2m+2i-1}$ and $h_1(m+i,k)$, $h_1(m+i,k+1)$ and $h_1(m+i,k+2)$. $\binom{2m+2i+k-1}{2m+2i-1}$ can be represented in the form $\frac{q_1(k,m)}{q_2(k,m)}\binom{2m+k-1}{2m-1}$ with $q_i(k,m)\in\mathbb{Z}[k,m]$ where $q_2(k,m)\neq 0$ for $0\le k\le 2m-4$. Now the crucial step is to apply rule (120) and represent $h_1(k,m+1)$ in terms of $h_1(k,m)$ and $h_1(k+1,m)$. Similarly, the shifted version of (120) in m enables one to reduce $h_1(k,m+2)$ first to expressions in $h_1(k,m+1)$, $h_1(k+1,m+1)$, and $h_1(k+2,m+1)$ for $0\le k<2m-1$. Second, by applying the rules (120) and (78) those expressions can be again represented in terms of $h_1(k,m),\dots,h_1(k+2,m)$ for $0\le k\le 2m-4$. Following this rewriting process recursively, all expressions in $f(k+j,m+i)$, $0\le i\le 2$ amd $0\le j\le 3$ can be represented in terms of $h_1(k,m)$, $h_1(k+1,m)$ and $h_1(k+2,m)$ for $0\le k\le 2m-4$. In this representation the equation (126) can be verified by simple term rewriting for $0\le k\le 2m-4$. Summing equation (127) over k from 0 to $2m-4$ gives

$$\begin{aligned}&c_0(m)b(m)+\cdots+c_3(m)b(m+3)\\&\qquad=g(2m-3,i,m)-g(0,i,m)+\sum_{j=0}^{3}c_j(m)\sum_{r=2m-3}^{2(m+j)-1}f(m+j,r).\end{aligned}$$

In this case one can show that the right hand side vanishes to 0 which gives recurrence (74) for $b(m)$. The special cases $m=1,2$ follow by simple evaluation. ◊

Together with Section 5 this result shows identity (35).

Finally, we show the remaining identities in Propositions 3 and 5 and we prove that $a(m)$ is a solution of (74) for all $m\ge 1$. In order to achieve this, we compute the following recurrence of $h_2(k,m)$ for all $m,k\ge 0$ with Sigma.

$$\begin{aligned}&(1+k+2m)(6+141k-18k^2-141k^3-36k^4+18k^5+12k^6-216m+795km\\&\quad+392k^2m-357k^3m-180k^4m-48k^5m+16k^6m-1242m^2+1266km^2+1274k^2m^2\\&\quad-150k^3m^2-32k^4m^2-96k^5m^2-2436m^3+396km^3+1260k^2m^3-60k^3m^3\\&\quad+192k^4m^3-2040m^4-504km^4+504k^2m^4-216k^3m^4-624m^5-288km^5+144k^2m^5)\,h_2(k,m)\\&\quad+(3+4m)(-30-111k-41k^2+125k^3+99k^4+k^5-19k^6-6k^7-120m\\&\quad-549km-489k^2m+148k^3m+254k^4m+80k^5m+16k^6m-30m^2-774km^2\\&\quad-1210k^2m^2-252k^3m^2+136k^4m^2+24k^5m^2+420m^3+72km^3-1008k^2m^3-480k^3m^3\\&\quad-24k^4m^3+600m^4+840km^4-72k^2m^4-144k^3m^4+240m^5+432km^5+144k^2m^5)\,h_2(k+1,m)\\&\quad-(-1-k+2m)(3+4m)(12+30k-16k^2-32k^3-4k^4+5k^5+2k^6+72m\\&\quad+186km-20k^2m-99k^3m-16k^4m+156m^2+420km^2+66k^2m^2-108k^3m^2\\&\quad-12k^4m^2+144m^3+408km^3+144k^2m^3-36k^3m^3+48m^4+144km^4+72k^2m^4)\,h_2(k+2,m)\\&\quad+4(1+k)^2(-2+k-2m)(-1+k-2m)(1+m)(1+2m)(1+4m)(3+4m)\,h_2(k,m+1)=0 \qquad (128)\end{aligned}$$

Remark 14. As in Remark 6 we consider the sum

$$h_2''(s,k,m)=\sum_{s=0}^{\lfloor\frac{2m-k}{2}\rfloor-1}q''(s,k,m)$$

Recall that $q''(s,k,m) = q(s,k,m)$ for $0 \le s \le \lfloor\frac{2m-k}{2}\rfloor - 1$ where $q(s,k,m)$ is the summand representation of $f_2(k, 2m-k) = h_2(k,m)$. In the sequel denote

$$p(s,k,m) = (-1)^k \binom{m-s}{2m-2s-k-2},$$

and

$$u_1(r,m) = \frac{(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r}, \qquad u_2(s,k,m) = \sum_{r=0}^{s}(m-r)u_1(r,m),$$

i.e.,

$$q''(s,k,m) = \frac{k}{2m(m-s)}\frac{(m-s-k-1)(m-s-k-2)}{(2m-2s-k)(2m-2s-k-1)}\frac{(-1)^s}{4^s}\,p(s,k,m)\,u_2(s,k,m).$$

Now applying this representation of the definite sum $h_2''(k,m)$ to Sigma gives the creative telescoping equation

$$g(s+1,k,m)-g(s,k,m) = c_0(k,m)q''(s,k,m)+\cdots+c_2(k,m)q''(s,k+2,m)+\kappa(k,m)\,q''(s,k,m+1). \tag{129}$$

with

$$\begin{aligned}
c_0(k,m) &= m(1+k+2m)(6+141k-18k^2-141k^3-36k^4+18k^5+12k^6-216m+795km+392k^2m-357k^3m\\
&\quad -180k^4m-48k^5m+16k^6m-1242m^2+1266km^2+1274k^2m^2-150k^3m^2-32k^4m^2-96k^5m^2-2436m^3\\
&\quad +396km^3+1260k^2m^3-60k^3m^3+192k^4m^3-2040m^4-504km^4+504k^2m^4-216k^3m^4-624m^5\\
&\quad -288km^5+144k^2m^5),\\
c_1(k,m) &= m(3+4m)(-30-111k-41k^2+125k^3+99k^4+k^5-19k^6-6k^7-120m-549km-489k^2m+148k^3m\\
&\quad +254k^4m+80k^5m+16k^6m-30m^2-774km^2-1210k^2m^2-252k^3m^2+136k^4m^2+24k^5m^2+420m^3+72km^3\\
&\quad -1008k^2m^3-480k^3m^3-24k^4m^3+600m^4+840km^4-72k^2m^4-144k^3m^4+240m^5+432km^5+144k^2m^5),\\
c_2(k,m) &= -(m(-1-k+2m)(3+4m)(12+30k-16k^2-32k^3-4k^4+5k^5+2k^6+72m+186km-20k^2m-99k^3m\\
&\quad -16k^4m+156m^2+420km^2+66k^2m^2-108k^3m^2-12k^4m^2+144m^3+408km^3\\
&\quad +144k^2m^3-36k^3m^3+48m^4+144km^4+72k^2m^4)),\\
\kappa(k,m) &= 4(1+k)^2(-2+k-2m)(-1+k-2m)m(1+m)(1+2m)(1+4m)(3+4m),
\end{aligned}$$

and

$$g(s,k,m) = p(s,k,m)\frac{(-1)^s}{4^s}\frac{u_1(s,m)\,a_1(s,k,m)+u_2(s,k,m)\,a_2(s,k,m)}{b(s,k,m)}$$

where r is substituted with s in $u(r,m)$,

$$\begin{aligned}
b(s,k,m) = (3m-s+2)(3m-s+3)&\\
&(2m-2s-k+2)(2m-2s-k+1)(2m-2s-k)(2m-2s-k-1),
\end{aligned}$$

and

$$
\begin{aligned}
a_1(s,k,m) = {} & (-2-k+m-s)(m+s)\big(-216 - 432 k + 1278 k^2 + 864 k^3 - 1476 k^4 - 468 k^5 + 432 k^6 + 36 k^7 - 18 k^8 - 2988 m - 8100 k m + \\
& 15333 k^2 m + 17766 k^3 m - 17406 k^4 m - 8136 k^5 m + 5076 k^6 m + 234 k^7 m - 159 k^8 m + 36 k^9 m - 17100 m^2 - 64818 k m^2 + 64500 k^2 m^2 + \\
& 146391 k^3 m^2 - 76314 k^4 m^2 - 59949 k^5 m^2 + 25098 k^6 m^2 + 1170 k^7 m^2 - 792 k^8 m^2 + 138 k^9 m^2 - 49752 m^3 - 283356 k m^3 + \\
& 54240 k^2 m^3 + 631371 k^3 m^3 - 123441 k^4 m^3 - 237219 k^5 m^3 + 65184 k^6 m^3 + 4266 k^7 m^3 - 1851 k^8 m^3 + 174 k^9 m^3 - 61776 m^4 - \\
& 724608 k m^4 - 477300 k^2 m^4 + 1561185 k^3 m^4 + 129030 k^4 m^4 - 550083 k^5 m^4 + 94842 k^6 m^4 + 8310 k^7 m^4 - 1920 k^8 m^4 + 72 k^9 m^4 + \\
& 57348 m^5 - 1037808 k m^5 - 2013969 k^2 m^5 + 2211387 k^3 m^5 + 887745 k^4 m^5 - 771255 k^5 m^5 + 76608 k^6 m^5 + 7680 k^7 m^5 - 720 k^8 m^5 + \\
& 358452 m^6 - 531342 k m^6 - 3824454 k^2 m^6 + 1512048 k^3 m^6 + 1620966 k^4 m^6 - 644538 k^5 m^6 + 31248 k^6 m^6 + 2664 k^7 m^6 + \\
& 654336 m^7 + 780048 k m^7 - 4066884 k^2 m^7 - 139788 k^3 m^7 + 1511748 k^4 m^7 - 296100 k^5 m^7 + 4752 k^6 m^7 + 669312 m^8 + \\
& 1715184 k m^8 - 2344680 k^2 m^8 - 1036296 k^3 m^8 + 734184 k^4 m^8 - 57672 k^5 m^8 + 408960 m^9 + 1443456 k m^9 - 507168 k^2 m^9 - \\
& 694656 k^3 m^9 + 147744 k^4 m^9 + 139968 m^{10} + 603936 k m^{10} + 124416 k^2 m^{10} - 158112 k^3 m^{10} + 20736 m^{11} + 103680 k m^{11} + \\
& 62208 k^2 m^{11} + 828 s + 3564 k s - 4209 k^2 s - 8064 k^3 s + 6642 k^4 s + 3636 k^5 s - 2340 k^6 s - 36 k^7 s + 87 k^8 s - 36 k^9 s + 8412 m s + \\
& 53580 k m s - 20785 k^2 m s - 125706 k^3 m s + 52019 k^4 m s + 52960 k^5 m s - 22914 k^6 m s - 1138 k^7 m s + 1028 k^8 m s - 168 k^9 m s + \\
& 26160 m^2 s + 324470 k m^2 s + 132903 k^2 m^2 s - 771404 k^3 m^2 s + 81512 k^4 m^2 s + 310086 k^5 m^2 s - 85148 k^6 m^2 s - 7990 k^7 m^2 s + \\
& 3289 k^8 m^2 s - 250 k^9 m^2 s - 26496 m^3 s + 988470 k m^3 s + 1441421 k^2 m^3 s - 2410212 k^3 m^3 s - 439263 k^4 m^3 s + 940994 k^5 m^3 s - \\
& 153414 k^6 m^3 s - 20540 k^7 m^3 s + 4060 k^8 m^3 s - 120 k^9 m^3 s - 420588 m^4 s + 1382778 k m^4 s + 5393078 k^2 m^4 s - 3966280 k^3 m^4 s - \\
& 2232326 k^4 m^4 s + 1613374 k^5 m^4 s - 138960 k^6 m^4 s - 22248 k^7 m^4 s + 1728 k^8 m^4 s - 1372572 m^5 s - 303546 k m^5 s + \\
& 10829192 k^2 m^5 s - 2658282 k^3 m^5 s - 4422872 k^4 m^5 s + 1579620 k^5 m^5 s - 56736 k^6 m^5 s - 8640 k^7 m^5 s - 2431584 m^6 s - \\
& 4434180 k m^6 s + 12497240 k^2 m^6 s + 1549124 k^3 m^6 s - 4598480 k^4 m^6 s + 825160 k^5 m^6 s - 6576 k^6 m^6 s - 2627808 m^7 s - \\
& 7556208 k m^7 s + 7743264 k^2 m^7 s + 4133472 k^3 m^7 s - 2484864 k^4 m^7 s + 178704 k^5 m^7 s - 1734144 m^8 s - 6400256 k m^8 s + \\
& 1625216 k^2 m^8 s + 2824576 k^3 m^8 s - 552384 k^4 m^8 s - 644544 m^9 s - 2824320 k m^9 s - 659904 k^2 m^9 s + 688704 k^3 m^9 s - \\
& 103680 m^{10} s - 518400 k m^{10} s - 311040 k^2 m^{10} s - 360 s^2 - 8802 k s^2 - 3105 k^2 s^2 + 24399 k^3 s^2 - 6195 k^4 s^2 - 10671 k^5 s^2 + \\
& 4608 k^6 s^2 + 360 k^7 s^2 - 324 k^8 s^2 + 30 k^9 s^2 + 5496 m s^2 - 94460 k m s^2 - 130204 k^2 m s^2 + 283891 k^3 m s^2 + 9557 k^4 m s^2 - \\
& 124737 k^5 m s^2 + 30964 k^6 m s^2 + 5060 k^7 m s^2 - 1713 k^8 m s^2 + 82 k^9 m s^2 + 98832 m^2 s^2 - 342106 k m^2 s^2 - 1187447 k^2 m^2 s^2 + \\
& 1246970 k^3 m^2 s^2 + 429561 k^4 m^2 s^2 - 550642 k^5 m^2 s^2 + 72370 k^6 m^2 s^2 + 17762 k^7 m^2 s^2 - 2800 k^8 m^2 s^2 + 56 k^9 m^2 s^2 + \\
& 580296 m^3 s^2 - 167196 k m^3 s^2 - 4962936 k^2 m^3 s^2 + 2454572 k^3 m^3 s^2 + 2092520 k^4 m^3 s^2 - 1209448 k^5 m^3 s^2 + 67904 k^6 m^3 s^2 + \\
& 23480 k^7 m^3 s^2 - 1456 k^8 m^3 s^2 + 1819992 m^4 s^2 + 2383628 k m^4 s^2 - 11351580 k^2 m^4 s^2 + 1292292 k^3 m^4 s^2 + 4653772 k^4 m^4 s^2 - \\
& 1418296 k^5 m^4 s^2 + 16512 k^6 m^4 s^2 + 10584 k^7 m^4 s^2 + 3450000 m^5 s^2 + 8175768 k m^5 s^2 - 14731512 k^2 m^5 s^2 - 3272216 k^3 m^5 s^2 + \\
& 5478328 k^4 m^5 s^2 - 851488 k^5 m^5 s^2 - 5584 k^6 m^5 s^2 + 4089936 m^6 s^2 + 13010288 k m^6 s^2 - 9950560 k^2 m^6 s^2 - 6520608 k^3 m^6 s^2 + \\
& 3325712 k^4 m^6 s^2 - 206176 k^5 m^6 s^2 + 2974272 m^7 s^2 + 11472896 k m^7 s^2 - 1897088 k^2 m^7 s^2 - 4647040 k^3 m^7 s^2 + \\
& 822912 k^4 m^7 s^2 + 1215936 m^8 s^2 + 5415360 k m^8 s^2 + 1413696 k^2 m^8 s^2 - 1224192 k^3 m^8 s^2 + 214272 m^9 s^2 + 1071360 k m^9 s^2 + \\
& 642816 k^2 m^9 s^2 - 2448 s^3 + 5154 k s^3 + 23109 k^2 s^3 - 31650 k^3 s^3 - 6972 k^4 s^3 + 15942 k^5 s^3 - 2712 k^6 s^3 - 1008 k^7 s^3 + \\
& 243 k^8 s^3 - 6 k^9 s^3 - 42432 m s^3 + 3778 k m s^3 + 347291 k^2 m s^3 - 251692 k^3 m s^3 - 151185 k^4 m s^3 + 130030 k^5 m s^3 - 8322 k^6 m s^3 - \\
& 6156 k^7 m s^3 + 708 k^8 m s^3 - 8 k^9 m s^3 - 291768 m^2 s^3 - 346540 k m^2 s^3 + 1961590 k^2 m^2 s^3 - 610590 k^3 m^2 s^3 - 886438 k^4 m^2 s^3 + \\
& 396250 k^5 m^2 s^3 + 1064 k^6 m^2 s^3 - 11024 k^7 m^2 s^3 + 512 k^8 m^2 s^3 - 1076040 m^3 s^3 - 2279388 k m^3 s^3 + 5565216 k^2 m^3 s^3 + \\
& 83480 k^3 m^3 s^3 - 2358368 k^4 m^3 s^3 + 572884 k^5 m^3 s^3 + 20672 k^6 m^3 s^3 - 6144 k^7 m^3 s^3 - 2367504 m^4 s^3 - 6764632 k m^4 s^3 + \\
& 8484056 k^2 m^4 s^3 + 2863472 k^3 m^4 s^3 - 3235208 k^4 m^4 s^3 + 396840 k^5 m^4 s^3 + 14512 k^6 m^4 s^3 - 3212496 m^5 s^3 - \\
& 11179144 k m^5 s^3 + 6351984 k^2 m^5 s^3 + 5222808 k^3 m^5 s^3 - 2241392 k^4 m^5 s^3 + 106160 k^5 m^5 s^3 - 2640960 m^6 s^3 - \\
& 10639024 k m^6 s^3 + 932896 k^2 m^6 s^3 + 3976208 k^3 m^6 s^3 - 623616 k^4 m^6 s^3 - 1207488 m^7 s^3 - 5469120 k m^7 s^3 - \\
& 1569600 k^2 m^7 s^3 + 1144320 k^3 m^7 s^3 - 235776 m^8 s^3 - 1178880 k m^8 s^3 - 707328 k^2 m^8 s^3 + 4608 s^4 + 6498 k s^4 - 32829 k^2 s^4 + \\
& 15045 k^3 s^4 + 16785 k^4 s^4 - 9963 k^5 s^4 - 516 k^6 s^4 + 720 k^7 s^4 - 48 k^8 s^4 + 57012 m s^4 + 122932 k m s^4 - 338727 k^2 m s^4 + \\
& 42905 k^3 m s^4 + 165223 k^4 m s^4 - 53185 k^5 m s^4 - 6352 k^6 m s^4 + 2256 k^7 m s^4 - 64 k^8 m s^4 + 293940 m^2 s^4 + 815490 k m^2 s^4 - \\
& 1340342 k^2 m^2 s^4 - 218164 k^3 m^2 s^4 + 588342 k^4 m^2 s^4 - 99182 k^5 m^2 s^4 - 14608 k^6 m^2 s^4 + 1728 k^7 m^2 s^4 + 820896 m^3 s^4 + \\
& 2712600 k m^3 s^4 - 2542124 k^2 m^3 s^4 - 1224428 k^3 m^3 s^4 + 988652 k^4 m^3 s^4 - 76060 k^5 m^3 s^4 - 9408 k^6 m^3 s^4 + 1343712 m^4 s^4 + \\
& 5071136 k m^4 s^4 - 2157368 k^2 m^4 s^4 - 2273256 k^3 m^4 s^4 + 801144 k^4 m^4 s^4 - 19672 k^5 m^4 s^4 + 1291008 m^5 s^4 + 5426400 k m^5 s^4 - \\
& 129888 k^2 m^5 s^4 - 1897248 k^3 m^5 s^4 + 253728 k^4 m^5 s^4 + 674880 m^6 s^4 + 3109728 k m^6 s^4 + 963072 k^2 m^6 s^4 - 604896 k^3 m^6 s^4 + \\
& 148224 m^7 s^4 + 741120 k m^7 s^4 + 444672 k^2 m^7 s^4 - 3420 s^5 - 10230 k s^5 + 20196 k^2 s^5 + 1734 k^3 s^5 - 10692 k^4 s^5 + 1896 k^5 s^5 + \\
& 888 k^6 s^5 - 144 k^7 s^5 - 33756 m s^5 - 114602 k m s^5 + 147528 k^2 m s^5 + 51026 k^3 m s^5 - 66360 k^4 m s^5 + 4232 k^5 m s^5 + 3104 k^6 m s^5 - \\
& 192 k^7 m s^5 - 136704 m^2 s^5 - 512892 k m^2 s^5 + 387648 k^2 m^2 s^5 + 257732 k^3 m^2 s^5 - 149272 k^4 m^2 s^5 + 304 k^5 m^2 s^5 + \\
& 2560 k^6 m^2 s^5 - 290496 m^3 s^5 - 1184432 k m^3 s^5 + 391120 k^2 m^3 s^5 + 532560 k^3 m^3 s^5 - 146000 k^4 m^3 s^5 - \\
& 2624 k^5 m^3 s^5 - 341376 m^4 s^5 - 1497152 k m^4 s^5 - 34432 k^2 m^4 s^5 + 502720 k^3 m^4 s^5 - 52800 k^4 m^4 s^5 - 210240 m^5 s^5 - \\
& 985920 k m^5 s^5 - 322752 k^2 m^5 s^5 + 180480 k^3 m^5 s^5 - 52992 m^6 s^5 - 264960 k m^6 s^5 - 158976 k^2 m^6 s^5 + \\
& 1152 s^6 + 4800 k s^6 - 5496 k^2 s^6 - 3264 k^3 s^6 + 2472 k^4 s^6 + 288 k^5 s^6 - 192 k^6 s^6 + 9168 m s^6 + 39424 k m s^6 - \\
& 26264 k^2 m s^6 - 23792 k^3 m s^6 + 9368 k^4 m s^6 + 1440 k^5 m s^6 - 256 k^6 m s^6 + 28752 m^2 s^6 + 127744 k m^2 s^6 - \\
& 34944 k^2 m^2 s^6 - 61296 k^3 m^2 s^6 + 11408 k^4 m^2 s^6 + 1408 k^5 m^2 s^6 + 44352 m^3 s^6 + 204064 k m^3 s^6 + 11264 k^2 m^3 s^6 - \\
& 68192 k^3 m^3 s^6 + 4416 k^4 m^3 s^6 + 33600 m^4 s^6 + 160704 k m^4 s^6 + 54720 k^2 m^4 s^6 - 28032 k^3 m^4 s^6 + 9984 m^5 s^6 + \\
& 49920 k m^5 s^6 + 29952 k^2 m^5 s^6 - 144 s^7 - 744 k s^7 + 528 k^2 s^7 + 648 k^3 s^7 - 144 k^4 s^7 - 96 k^5 s^7 - 912 m s^7 - \\
& 4568 k m s^7 + 1184 k^2 m s^7 + 2616 k^3 m s^7 - 192 k^4 m s^7 - 128 k^5 m s^7 - 2112 m^2 s^7 - 10384 k m^2 s^7 - 800 k^2 m^2 s^7 + \\
& 3632 k^3 m^2 s^7 - 2112 m^3 s^7 - 10368 k m^3 s^7 - 3648 k^2 m^3 s^7 + 1728 k^3 m^3 s^7 - 768 m^4 s^7 - 3840 k m^4 s^7 - 2304 k^2 m^4 s^7\big),
\end{aligned}
$$

$$\begin{aligned}
a_2(s,k,m) = {} & -m\,(-2-k+m-s)\,(2+3m-s)\,(3+3m-s)\,\big(-36-72k+213k^2+144k^3-246k^4-78k^5+72k^6+6k^7-3k^8-\\
& 408m-1170km+2023k^2m+2601k^3m-2286k^4m-1161k^5m+666k^6m+24k^7m-19k^8m+6k^9m-1776m^2-7770km^2+\\
& 5373k^2m^2+17680k^3m^2-6635k^4m^2-6972k^5m^2+2410k^6m^2+126k^7m^2-80k^8m^2+8k^9m^2-3240m^3-26046km^3-7427k^2m^3+\\
& 57127k^3m^3-557k^4m^3-20365k^5m^3+3840k^6m^3+360k^7m^3-80k^8m^3+468m^4-43998km^4-69042k^2m^4+90860k^3m^4+\\
& 32850k^4m^4-30310k^5m^4+2592k^6m^4+296k^7m^4+13248m^5-23904km^5-151916k^2m^5+55724k^3m^5+66668k^4m^5-\\
& 22220k^5m^5+528k^6m^5+25920m^6+37200km^6-154056k^2m^6-23592k^3m^6+54216k^4m^6-6408k^5m^6+24384m^7+\\
& 72864km^7-64800k^2m^7-47904k^3m^7+16416k^4m^7+11712m^8+47904km^8+2304k^2m^8-17568k^3m^8+2304m^9+11520km^9+\\
& 6912k^2m^9+216s+516ks-582k^2s-897k^3s+489k^4s+423k^5s-150k^6s-66k^7s+9k^8s+6k^9s+1944ms+5952kms-\\
& 1554k^2ms-10101k^3ms+1242k^4ms+3869k^5ms-660k^6ms-256k^7ms+12k^8ms+8k^9ms+6240m^2s+26400km^2s+\\
& 16028k^2m^2s-42078k^3m^2s-8466k^4m^2s+14552k^5m^2s-1136k^6m^2s-464k^7m^2s+5736m^3s+51396km^3s+\\
& 101424k^2m^3s-76068k^3m^3s-48944k^4m^3s+26552k^5m^3s-448k^6m^3s-320k^7m^3s-14280m^4s+18096km^4s+\\
& 238792k^2m^4s-33712k^3m^4s-97720k^4m^4s+22816k^5m^4s+336k^6m^4s-46320m^5s-101040km^5s+266448k^2m^5s+\\
& 77328k^3m^5s-87264k^4m^5s+7248k^5m^5s-54816m^6s-185088km^6s+117216k^2m^6s+109728k^3m^6s-29376k^4m^6s-\\
& 30912m^7s-130944km^7s-15552k^2m^7s+42048k^3m^7s-6912m^8s-34560km^8s-20736k^2m^8s-396s^2-1032ks^2+\\
& 453k^2s^2+1701k^3s^2+87k^4s^2-687k^5s^2-114k^6s^2+72k^7s^2+24k^8s^2-2760ms^2-8130kms^2-2797k^2ms^2+\\
& 10899k^3ms^2+4087k^4ms^2-3087k^5ms^2-644k^6ms^2+72k^7ms^2+32k^8ms^2-5964m^2s^2-21186km^2s^2-33402k^2m^2s^2+\\
& 22208k^3m^2s^2+21206k^4m^2s^2-5442k^5m^2s^2-1088k^6m^2s^2-32k^7m^2s^2+912m^3s^2-5832km^3s^2-103748k^2m^3s^2-\\
& 1228k^3m^3s^2+44108k^4m^3s^2-3956k^5m^3s^2-576k^6m^3s^2+23520m^4s^2+74016km^4s^2-136008k^2m^4s^2-63048k^3m^4s^2+\\
& 41112k^4m^4s^2-744k^5m^4s^2+38784m^5s^2+149376km^5s^2-58848k^2m^5s^2-80064k^3m^5s^2+14112k^4m^5s^2+\\
& 26688m^6s^2+118176km^6s^2+24192k^2m^6s^2-31392k^3m^6s^2+6912m^7s^2+34560km^7s^2+20736k^2m^7s^2+\\
& 216s^3+684ks^3+180k^2s^3-804k^3s^3-540k^4s^3+60k^5s^3+108k^6s^3+24k^7s^3+1152ms^3+3828kms^3+\\
& 3876k^2ms^3-1936k^3ms^3-2808k^4ms^3-208k^5ms^3+144k^6ms^3+32k^7ms^3+1368m^2s^3+4392km^2s^3+\\
& 15576k^2m^2s^3+3312k^3m^2s^3-5376k^4m^2s^3-672k^5m^2s^3-2736m^3s^3-12288km^3s^3+22368k^2m^3s^3+\\
& 15744k^3m^3s^3-4320k^4m^3s^3-384k^5m^3s^3-8352m^4s^3-37152km^4s^3+6432k^2m^4s^3+18240k^3m^4s^3-\\
& 1152k^4m^4s^3-7488m^5s^3-35136km^5s^3-10944k^2m^5s^3+6912k^3m^5s^3-2304m^6s^3-11520km^6s^3-6912k^2m^6s^3\big).
\end{aligned}$$

In the next step we have to check that this summand telescoping equation (129) holds within the summation range. In order to avoid poles in the evaluation, similarly to Remark 6, we represent all ingredients in (129) not in terms of $p(s,k,m)$ but in terms of $p(s+1,k+2,m)$, i.e.,

$$p(s+1,k,m) = p(s+1,k+2,m)\frac{(m-s-k-4)(m-s-k-5)}{(2m-2s-k-4)(2m-2s-k-5)} \tag{130}$$

and

$$p(s,k+i,m) = p(s+1,k+2,m)\,p_i(s,k,m), \qquad 0 \le i \le 2, \tag{131}$$

where

$$p_i(s,k,m) = -\frac{m-s}{2m-2s-k-5}\prod_{j=0}^{2-i}\frac{m-s-k+j-5}{2m-2s-k+j-4}.$$

Moreover, the shifted version of $p(s,k,m)$ in m can be expressed by

$$p(s,k,m+1) = p(s,k,m)\,p_0(s,k,m)\,\frac{-(m-s+1)(m-s-k-2)}{(2m-2s-k-1)(2m-2s-k)}.$$

Note that we keep the representations in $u_1(s,m)$ and $u_2(s,m)$. In addition we need the nontrivial relation

$$\begin{aligned}
u_2(s,m+1) = \Big[&-3m(1+6m)(5+6m)(2+3m-s)(3+3m-s)\,u_2(s,m)\\
&+6(m+s)(18+169m+461m^2+496m^3+186m^4-97s-471ms-715m^2s\\
&-348m^3s+98s^2+278ms^2+198m^2s^2-27s^3-36ms^3)\,u_1(s,k,m)\Big]\Big/\\
&\big(m(1+4m)(3+4m)(2+3m-s)(3+3m-s)\big);
\end{aligned} \tag{132}$$

this identity can be verified by checking the telescoping equation

$$g'(s,k,m) - g'(s-1,k,m) = u_1(s,m+1)$$

where $g'(s,k,m)$ denotes the right side of (132).

With these relations we finally manage to verify that (129) holds for $0 \le s \le \lfloor \frac{2m-k}{2} \rfloor - 4$ in terms of $p(s+1,k+2,m)$, $u_1(s,m)$ and $u_2(s,m)$.

Hence, summing equation (129) over s from 0 to an arbitrary d with $0 \leq d \leq \lfloor \frac{2m-k}{2} \rfloor - 4$ gives the relation (122) for the sum $y(k,m) = \sum_{s=0}^{d} q(s,k,m) = \sum_{s=0}^{d} q''(s,k,m)$. Now we do a case distinction on k.

If k is even, we can choose $d = m - k/2 - 4$, and obtain

$$\begin{aligned} c_0(k,m)\,h_2(k,m) + \cdots + c_2(k,m)\,h_2(k+2,m) + \kappa(k,m+1)\,h_2(k,m+1) \\ = g(m - \frac{k}{2} - 3, k, m) - g(0,k,m) + \sum_{i=0}^{2} c_i(k,m) \sum_{j=0}^{3} q(m - \frac{k}{2} - j, k+i, m) \\ + \kappa(k,m) \sum_{j=0}^{4} q(m + 1 - \frac{k}{2} - j, k, m+1). \end{aligned}$$

By term rewriting one can now show that the right hand side is equal to 0. Hence for $m - k/2 - 4 \geq 0$, k even, the relation (78) holds. Similarly, for odd k, one can choose $d = m - (k-1)/2 - 4$, and obtains

$$\begin{aligned} c_0(k,m)\,h_2(k,m) + \cdots + c_2(k,m)\,h_2(k+2,m) \\ = g(m - \frac{k-1}{2} - 3, k, m) - g(0,k,m) + \sum_{i=0}^{2} c_i(k,m) \sum_{j=0}^{3} q(m - \frac{k-1}{2} - j, k+i, m) \\ + \kappa(k,m) \sum_{j=0}^{4} q(m + 1 - \frac{k-1}{2} - j, k, m+1). \end{aligned}$$

Again one can prove that the right hand side vanishes to 0, and hence the relation (128) holds for $m - (k-1)/2 - 4 \geq 0$, k odd. Summarizing, for all $m \geq 5$ and $0 \leq k \leq 2m - 9$ the recurrence (128) contains the solution $h_2(k,m)$. The special cases $0 \leq m \leq 5$ and $k \geq 2m - 8$ for $m \geq 4$ can be verified by simple evaluation. This shows that the recurrence holds for all $m, k \geq 0$. ♢

With this recurrence relation one can prove similarly as above that $T_1(m) = \sum_{k=0}^{2m} (-1)^k h_2(k,m)$ fulfills

$$3(6m+1)(6m+5)T_1(m) - 4(4m+1)(4m+3)T_1(m+1) = 0,$$

$T_2(m) = \sum_{k=0}^{2m} 2^k h_2(k,m)$ fulfills

$$\begin{aligned} 3(6m+1)(6m+5)T_2(m) - 4(4m+1)(4m+3)T_2(m+1) \\ = 3(2m-1)(4m+3)(2h_2(0,m) - 5h_2(1,m) + 2h_2(2,m)), \end{aligned}$$

and $a(m)$ fulfills (74) for all $m \geq 1$.

Remark 15. The verification of those recurrences is completely analogously to Remarks 12 and 13. For completeness we present the proof certificates only.

Denote $f(k,m) = (-1)^k h_2(k,m)$. Sigma provides the recurrence certificate (124) with

$$c_0(m) = 6(1+m)(1+2m)(1+6m)(5+6m), \quad c_1(m) = -8(1+m)(1+2m)(1+4m)(3+4m)$$

and

$$\begin{aligned} g(k,m) = {} & \frac{1}{(1+k)^2\,(-2+k-2m)\,(-1+k-2m)} \Big(-2\,(-1)^k\,k\,(3+4m) \\ & \big(8 - 6k - 23k^2 - 15k^3 - 5k^4 + 3k^5 + 2k^6 + 56m - 80km - 76k^2m + 12k^3m - 22k^4m - 10k^5m + 304m^2 - 260km^2 - 264k^2m^2 + {} \\ & \quad 104k^3m^2 + 8k^4m^2 + 832m^3 - 120km^3 - 472k^2m^3 + 72k^3m^3 + 960m^4 + 336km^4 - 288k^2m^4 + 384m^5 + 288km^5\big)\,h_2(k,m) + {} \\ & (-1)^k\,k\,(3+4m)\,\big(40 - 2k - 89k^2 - 65k^3 - 11k^4 + 13k^5 + 6k^6 + {} \\ & \quad 160m + 48km - 188k^2m - 92k^3m - 74k^4m - 22k^5m + 296m^2 + 140km^2 - 264k^2m^2 + 88k^3m^2 - 8k^4m^2 + {} \\ & \quad 464m^3 + 360km^3 - 392k^2m^3 + 120k^3m^3 + 480m^4 + 528km^4 - 288k^2m^4 + 192m^5 + 288km^5\big)\,h_2(1+k,m) - {} \\ & (-1)^k\,k\,(1+k-2m)\,(2+k+2m)\,(3+4m)\,\big(8 - 11k - 4k^2 - k^3 + 2k^4 + 34m - 30km - 4k^2m - 6k^3m + 48m^2 - 24km^2 + 24m^3\big) \\ & h_2(2+k,m)\Big). \end{aligned}$$

Similarly, denote $f(k,m) = 2^k h_2(k,m)$. Sigma provides the recurrence certificate (124) with

$$c_0(m) = -6(1+m)(1+2m)(1+6m)(5+6m), \quad c_1(m) = 8(1+m)(1+2m)(1+4m)(3+4m)$$

and

$$
\begin{aligned}
\mathtt{g(k,m)} = {} & \frac{1}{(1+k)^2\,(-2+k-2m)\,(-1+k-2m)} \\
& \Big(-2^{1+k}(3+4m)\big(6+11k-27k^2-23k^3+15k^4+k^5-3k^6+2k^7+24m+101km-131k^2m-136k^3m+66k^4m+ \\
& \quad 32k^5m-16k^6m+6m^2+370km^2-218k^2m^2-300k^3m^2+32k^4m^2+56k^5m^2-84m^3+688km^3-120k^2m^3- \\
& \quad 304k^3m^3-24k^4m^3-120m^4+648km^4+120k^2m^4-144k^3m^4-48m^5+240km^5+144k^2m^5\big)\,h_2(k,m)+ \\
& 2^k(3+4m)\big(30+31k-95k^2-59k^3+43k^4-5k^5-5k^6+6k^7+120m+289km-303k^2m-410k^3m+208k^4m+ \\
& \quad 52k^5m-40k^6m+30m^2+794km^2+26k^2m^2-996k^3m^2+160k^4m^2+112k^5m^2-420m^3+848km^3+ \\
& \quad 864k^2m^3-1016k^3m^3-600m^4+360km^4+1032k^2m^4-432k^3m^4-240m^5+48km^5+432k^2m^5\big)\,h_2(1+k,m)- \\
& 2^k(1+k-2m)(2+k+2m)(3+4m)\big(6-4k-14k^2+14k^3-7k^4+2k^5+ \\
& \quad 30m-2km-69k^2m+50k^3m-12k^4m+48m^2+24km^2-96k^2m^2+36k^3m^2+24m^3+24km^3-36k^2m^3\big) \\
& h_2(2+k,m)\Big).
\end{aligned}
$$

Finally, define $f(k,m) = \binom{2m+k-2}{2m-1} h_2(k,m)$, then Sigma produces the certificate recurrence (127) given by

$$
\begin{aligned}
c_0(m) &= -864(1+m)^2(2+m)^2(3+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(2+3m)(4+3m)(1+4m)(3+4m) \\
&\quad (1+6m)(5+6m)(7+6m)(11+6m)(389987325+1563210054m+2677225618m^2+2567677266m^3 \\
&\quad +1512020037m^4+560774016m^5+128075112m^6+16483104m^7+915728m^8), \\
c_1(m) &= 96(1+m)^2(2+m)^2(3+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(1+4m)(3+4m)(7+6m)(11+6m) \\
&\quad (188256814200+3218702456970m+17894686721409m^2+51759068834382m^3+91965480308606m^4 \\
&\quad +108563491759548m^5+88575755915749m^6+50780656348884m^7+20427064666140m^8 \\
&\quad +5647619363232m^9+1022091737392m^{10}+108997272384m^{11}+5190346304m^{12}), \\
c_2(m) &= -128(1+m)^2(2+m)^2(3+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(1+4m)(3+4m)(5+4m)(7+4m) \\
&\quad (83466403020+2609224361478m+16370962889247m^2+49594957358604m^3+90028764304859m^4 \\
&\quad +107424612589224m^5+88133765355661m^6+50671172949012m^7+20411335550892m^8 \\
&\quad +5646620689248m^9+1022091737392m^{10}+108997272384m^{11}+5190346304m^{12}), \\
c_3(m) &= 512(1+m)^2(2+m)^2(3+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(1+4m)(3+4m)(5+4m)(7+4m) \\
&\quad (9+4m)(11+4m)^2(13+4m)(79380+7185780m+39195762m^2+91462910m^3+116468957m^4 \\
&\quad +87187760m^5+38333768m^6+9157280m^7+915728m^8),
\end{aligned}
$$

and

$$
g(k,m) = \frac{-(k-1)\big[p_0(k,m)h(k,m)+p_1(k,m)h(k+1,m)+p_2(k,m)h(k+2,m)\big]}{(k+1)^2\prod_{i=1}^{5}(2m-k+i)\prod_{i=3}^{6}(2m+k+i)}\binom{2m+k-1}{2m}
$$

where the polynomials $p_i(k,m)$ are given in Appendix D. ♢

Hence identity (47) follows together with Section 5. Moreover, identity (11) is immediate with the relation (70) and the fact that (11) holds for $m=1$.

Finally, a proof of identity (16) is given in

Remark 16. Define

$$
p(s,r,m) := \frac{(-1)^s}{4^s}\frac{(m)_r(-3m-1)_r}{r!(1/2-2m)_r}.
$$

Then observe that

$$
h_2(0,m) = \sum_{r=0}^{m}\frac{m-r}{m}p(m,r,m)
$$

and

$$
h_2(2,m) = \sum_{r=0}^{m-1}\frac{m-r}{m}p(m-1,r,m) + \frac{1}{2}\sum_{r=0}^{m-2}\frac{m-r}{m}p(m-2,r,m)
$$

which gives

$$
4h_2(0,m) - h_2(2,m) = \frac{2(2m+1)}{(m+1)(2m-1)}p(m,m,m).
$$

Therefore with Lemma 2 we get

$$3(6m+1)(6m+5)T_2(m)-4(4m+1)(4m+3)T_2(m+1)=-\frac{12(2m+1)(4m+3)}{m+1}p(m,m,m). \quad (133)$$

Now note that [PS95] or Sigma gives the same recurrence (133) for $h_2(0,m)$. Since $t_1(2m)$ is a solution of the homogeneous version of (133) and (16) holds for $m=1$, identity (16) holds for all $m \geq 1$. $\Diamond$

9. Conclusion

There are two essential observations arising from this work. First, as noted in the introduction, the challenge of proving equation (6) has led to significant new discoveries in methods of summation [Sch04a] and [PS04].

Second, and most tantalizing, Okada's theorem [Oka89, Sec. 4] as stated in the introduction was, in fact, originally given for the general q case not just $q=1$ as presented here. So one would very much like to produce the q-analog of (6) which would then complete the proof of all the classical finite plane partition product formulas [Sta86b]. Unfortunately, preliminary study suggests that the q-analog of the matrix $W(n)$ in (6) is substantially more intricate than the already very elaborate $W(n)$ constructed for Theorem 1 in this paper. In addition, it is also reasonable to suppose that the solution of the full Andrews-Robbins conjecture [Sta86b, p. 106] as formulated by Okada for general q along the lines suggested would again lead to further development and refinement of current summations methods.

Appendix A. The certificates needed in Remark 5

$$
\begin{aligned}
g_2(k,m) :=\ & \frac{1}{2\ (1+k)\ m\ (-1+2\ m)} (-2\ (3\ k^2+k^3+3\ (1-2\ m)\ m+k\ (2+m-2\ m^2))\ h_1(k,m)+(9\ k^2+3\ k^3+2\ (1-2\ m)\ m+\\
& k\ (6+4\ m-8\ m^2))\ h_1(k+1,m)-k\ (2+3\ k+k^2+2\ m-4\ m^2)\ h_1(k+2,m))
\end{aligned}
$$

$$
\begin{aligned}
g_3(k,m) :=\ & (-2\ (2\ k^5\ m\ (-1+2\ m)+k^4\ (3-8\ m+16\ m^2)+12\ m\ (1-m-4\ m^2+4\ m^3)+\\
& \quad k^2\ (15+16\ m-40\ m^2+32\ m^3-32\ m^4)+k^3\ (12-4\ m+6\ m^2+8\ m^3-8\ m^4)+k\ (6+26\ m-50\ m^2-8\ m^3+8\ m^4))\ h_1(k,m)+\\
& (6\ k^5\ m\ (-1+2\ m)+k^4\ (9-20\ m+40\ m^2)+16\ m\ (1-m-4\ m^2+4\ m^3)+k^2\ (45+68\ m-144\ m^2+32\ m^3-32\ m^4)+\\
& \quad 2\ k^3\ (18+m-6\ m^2+16\ m^3-16\ m^4)+2\ k\ (9+34\ m-58\ m^2-40\ m^3+40\ m^4))\ h_1(k+1,m)+\\
& (2\ k^5\ (1-2\ m)\ m-3\ k^4\ (1-2\ m+4\ m^2)+k^2\ (-15-26\ m+52\ m^2)-\\
& \quad 4\ m\ (1-m-4\ m^2+4\ m^3)-6\ k\ (1+4\ m-6\ m^2-8\ m^3+8\ m^4)+2\ k^3\ (-6-m+4\ m^2-8\ m^3+8\ m^4))\ h_1(k+2,m))/\\
& (4\ (1+k)^2\ (-1+m)\ m\ (-1+2\ m)\ (1+2\ m))
\end{aligned}
$$

$$
\begin{aligned}
g_4(k,m) :=\ & (-2\ (k^6\ m\ (1-m-4\ m^2+4\ m^3)+k^5\ m\ (1+3\ m-20\ m^2+20\ m^3)+4\ m\ (9-10\ m-33\ m^2+38\ m^3-12\ m^4+8\ m^5)+\\
& \quad k^3\ (36-14\ m+11\ m^2+73\ m^3-98\ m^4+60\ m^5-40\ m^6)+k^4\ (9-11\ m+25\ m^2-11\ m^3+6\ m^4+12\ m^5-8\ m^6)+\\
& \quad k^2\ (45+28\ m-92\ m^2+143\ m^3-138\ m^4-12\ m^5+8\ m^6)+k\ (18+67\ m-134\ m^2-13\ m^3+78\ m^4-156\ m^5+104\ m^6))\ h_1(k,m)+\\
& (48\ m\ (1-m-4\ m^2+4\ m^3)+3\ k^6\ m\ (1-m-4\ m^2+4\ m^3)+\\
& \quad k^5\ m\ (-1+13\ m-44\ m^2+44\ m^3)+k^3\ (108-9\ m-41\ m^2+242\ m^3-272\ m^4+72\ m^5-48\ m^6)+\\
& \quad k^4\ (27-38\ m+73\ m^2+16\ m^3-36\ m^4+48\ m^5-32\ m^6)+2\ k\ (27+92\ m-158\ m^2-113\ m^3+158\ m^4-108\ m^5+72\ m^6)+\\
& \quad k^2\ (135+161\ m-378\ m^2+204\ m^3-104\ m^4-240\ m^5+160\ m^6))\ h_1(k+1,m)+\\
& (k^6\ m\ (-1+m+4\ m^2-4\ m^3)+k^5\ (m-5\ m^2+12\ m^3-12\ m^4)-\\
& \quad 6\ k\ (3+11\ m-16\ m^2-24\ m^3+24\ m^4)+k^3\ (-36-m+25\ m^2-92\ m^3+92\ m^4)+4\ m\ (-3+2\ m+15\ m^2-10\ m^3-12\ m^4+8\ m^5)+\\
& \quad k^4\ (-9+12\ m-23\ m^2-6\ m^3+16\ m^4-24\ m^5+16\ m^6)-k^2\ (45+65\ m-142\ m^2+38\ m^3+12\ m^4-120\ m^5+80\ m^6))\ h_1(k+2,m))/\\
& (4\ (1+k)^2\ (-1+m)\ m\ (1+m)\ (-3+2\ m)\ (-1+2\ m)\ (1+2\ m))
\end{aligned}
$$

$$
\begin{aligned}
g_5(k,m) :=\ & (-2\ (2\ k^7\ m\ (-3+2\ m+15\ m^2-10\ m^3-12\ m^4+8\ m^5)-144\ m\ (-9+7\ m+42\ m^2-32\ m^3-24\ m^4+16\ m^5)+\\
& \quad k^6\ m\ (-33+19\ m+174\ m^2-104\ m^3-168\ m^4+112\ m^5)+k^5\ m\ (-231+379\ m+310\ m^2-202\ m^3-248\ m^4+128\ m^5+64\ m^6-32\ m^7)+\\
& \quad k^4\ (324-609\ m+1261\ m^2-134\ m^3-70\ m^4+568\ m^5-640\ m^6+448\ m^7-224\ m^8)+\\
& \quad k^3\ (1296+21\ m-47\ m^2+104\ m^3-534\ m^4+1088\ m^5-912\ m^6+320\ m^7-160\ m^8)+\\
& \quad 24\ k\ (27+117\ m-203\ m^2-116\ m^3+80\ m^4+64\ m^5+32\ m^6-128\ m^7+64\ m^8)+\\
& \quad 2\ k^2\ (810+969\ m-2188\ m^2+1006\ m^3-999\ m^4-224\ m^5+840\ m^6-1184\ m^7+592\ m^8))\ h_1(k,m)+\\
& (6\ k^7\ m\ (-3+2\ m+15\ m^2-10\ m^3-12\ m^4+8\ m^5)+3\ k^6\ m\ (-21+11\ m+114\ m^2-64\ m^3-120\ m^4+80\ m^5)+\\
& \quad k^5\ m\ (-585+1071\ m+370\ m^2-248\ m^3-248\ m^4+16\ m^5+256\ m^6-128\ m^7)-\\
& \quad 72\ m\ (-24+21\ m+106\ m^2-95\ m^3-32\ m^4+40\ m^5-32\ m^6+16\ m^7)-\\
& \quad 8\ k\ (-243-954\ m+1491\ m^2+1648\ m^3-1547\ m^4-248\ m^5+184\ m^6-32\ m^7+16\ m^8)-\\
& \quad 3\ k^4\ (-324+549\ m-1091\ m^2-74\ m^3+312\ m^4-616\ m^5+560\ m^6-256\ m^7+128\ m^8)+\\
& \quad 6\ k^2\ (810+1365\ m-2627\ m^2-404\ m^3+390\ m^4-112\ m^5+560\ m^6-832\ m^7+416\ m^8)+\\
& \quad k^3\ (3888+1179\ m-2691\ m^2+1576\ m^3-2744\ m^4+2512\ m^5-704\ m^6-1664\ m^7+832\ m^8))\ h_1(k+1,m)+\\
& (k^6\ m\ (15-7\ m-84\ m^2+44\ m^3+96\ m^4-64\ m^5)+2\ k^7\ m\ (3-2\ m-15\ m^2+10\ m^3+12\ m^4-8\ m^5)+\\
& \quad 24\ m\ (-18+15\ m+82\ m^2-71\ m^3-32\ m^4+40\ m^5-32\ m^6+16\ m^7)+k^5\ m\ (195-355\ m-128\ m^2+72\ m^3+112\ m^4-128\ m^6+64\ m^7)+\\
& \quad 36\ k\ (-18-75\ m+102\ m^2+188\ m^3-167\ m^4-56\ m^5+56\ m^6-32\ m^7+16\ m^8)+\\
& \quad k^4\ (-324+501\ m-997\ m^2-80\ m^3+384\ m^4-752\ m^5+576\ m^6-128\ m^7+64\ m^8)-\\
& \quad 4\ k^2\ (405+777\ m-1421\ m^2-530\ m^3+530\ m^4-56\ m^5+224\ m^6-320\ m^7+160\ m^8)-\\
& \quad k^3\ (1296+525\ m-1295\ m^2+1030\ m^3-1252\ m^4+376\ m^5+272\ m^6-896\ m^7+448\ m^8))\ h_1(k+2,m))/\\
& (24\ (1+k)^2\ (-2+m)\ (-1+m)\ m\ (1+m)\ (-3+2\ m)\ (-1+2\ m)\ (1+2\ m)\ (3+2\ m))
\end{aligned}
$$

$$
\begin{aligned}
g_6(k,m) :=\ & (-2\ (-2\ k^5\ (1-2\ m)^2\ m^2\ (192-19\ m+43\ m^2-20\ m^3+20\ m^4)+10\ k^7\ m\ (-3+2\ m+15\ m^2-10\ m^3-12\ m^4+8\ m^5)+\\
& \quad k^8\ m\ (-3+2\ m+15\ m^2-10\ m^3-12\ m^4+8\ m^5)+k^6\ m\ (-192+167\ m+842\ m^2-713\ m^3-304\ m^4+184\ m^5+32\ m^6-16\ m^7)-\\
& \quad 288\ m\ (15-14\ m-62\ m^2+53\ m^3+16\ m^4+8\ m^5-32\ m^6+16\ m^7)+\\
& \quad k^4\ (-1080+1521\ m-2790\ m^2-1003\ m^3+956\ m^4+236\ m^5-568\ m^6+704\ m^7-352\ m^8)-\\
& \quad 6\ k\ (360+1434\ m-2575\ m^2-1282\ m^3+1855\ m^4-1504\ m^5+1432\ m^6-736\ m^7+368\ m^8)+\\
& \quad 2\ k^3\ (-2160+501\ m-103\ m^2-3808\ m^3+4906\ m^4-2848\ m^5+2608\ m^6-1216\ m^7+608\ m^8)+\\
& \quad k^2\ (-5400-4638\ m+12029\ m^2-10774\ m^3+9775\ m^4+1328\ m^5+2680\ m^6-6112\ m^7+3056\ m^8))\ h_1(k,m)+\\
& (22\ k^7\ m\ (-3+2\ m+15\ m^2-10\ m^3-12\ m^4+8\ m^5)+\\
& \quad 3\ k^8\ m\ (-3+2\ m+15\ m^2-10\ m^3-12\ m^4+8\ m^5)-2\ k^6\ m\ (219-203\ m-922\ m^2+834\ m^3+200\ m^4-96\ m^5-64\ m^6+32\ m^7)-\\
& \quad 48\ m\ (120-101\ m-530\ m^2+407\ m^3+256\ m^4-40\ m^5-224\ m^6+112\ m^7)-\\
& \quad 2\ k^5\ m\ (-192+776\ m-1621\ m^2+1897\ m^3-724\ m^4+688\ m^5-352\ m^6+176\ m^7)+\\
& \quad 2\ k^2\ (-8100-11346\ m+23215\ m^2-1664\ m^3-469\ m^4+5080\ m^5-3256\ m^6-224\ m^7+112\ m^8)+\\
& \quad k^4\ (-3240+4563\ m-7674\ m^2-5881\ m^3+6260\ m^4-988\ m^5+920\ m^6-448\ m^7+224\ m^8)+\\
& \quad 2\ k^3\ (-6480-579\ m+3932\ m^2-11298\ m^3+12425\ m^4-3360\ m^5+4424\ m^6-3744\ m^7+1872\ m^8)-\\
& \quad 4\ k\ (1620+5874\ m-9055\ m^2-10570\ m^3+9703\ m^4+1280\ m^5+1816\ m^6-4576\ m^7+2288\ m^8))\ h_1(k+1,m)+\\
& (k^8\ m\ (3-2\ m-15\ m^2+10\ m^3+12\ m^4-8\ m^5)-\\
& \quad 6\ k^7\ m\ (-3+2\ m+15\ m^2-10\ m^3-12\ m^4+8\ m^5)+48\ m\ (30-17\ m-158\ m^2+89\ m^3+160\ m^4-88\ m^5-32\ m^6+16\ m^7)+\\
& \quad 2\ k^6\ m\ (69-64\ m-290\ m^2+261\ m^3+64\ m^4-24\ m^5-32\ m^6+16\ m^7)+
\end{aligned}
$$

$$2\,k^5\,m\,(-96+292\,m-417\,m^2+505\,m^3-228\,m^4+208\,m^5-96\,m^6+48\,m^7)+$$
$$k^4\,(1080-1401\,m+2374\,m^2+1807\,m^3-2296\,m^4+1252\,m^5-1096\,m^6+448\,m^7-224\,m^8)+$$
$$12\,k^2\,(450+741\,m-1427\,m^2-251\,m^3+413\,m^4-428\,m^5+416\,m^6-224\,m^7+112\,m^8)+$$
$$8\,k\,(270+1047\,m-1364\,m^2-2774\,m^3+2075\,m^4+1504\,m^5-424\,m^6-992\,m^7+496\,m^8)-$$
$$2\,k^3\,(-2160-435\,m+1958\,m^2-4298\,m^3+4071\,m^4+304\,m^5+600\,m^6-1376\,m^7+688\,m^8))\;h_1(k+2,m))/$$
$$(48\,(1+k)^2\,(-1+m)\,m\,(1+m)\,(2+m)\,(-5+2\,m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m))$$

$$g_7(k,m):=(-2\,(2\,k^9\,m\,(-180+72\,m+1025\,m^2-410\,m^3-1365\,m^4+546\,m^5+600\,m^6-240\,m^7-80\,m^8+32\,m^9)+$$
$$k^8\,m\,(-5490+2241\,m+31150\,m^2-12705\,m^3-41020\,m^4+16688\,m^5+17600\,m^6-7120\,m^7-2240\,m^8+896\,m^9)-$$
$$2\,k^7\,m\,(22050-10515\,m-121197\,m^2+$$
$$57164\,m^3+144335\,m^4-65525\,m^5-50836\,m^6+20972\,m^7+5840\,m^8-2160\,m^9-192\,m^{10}+64\,m^{11})-$$
$$960\,m\,(-4050+2610\,m+20505\,m^2-13223\,m^3-17493\,m^4+11579\,m^5+684\,m^6-1668\,m^7+1680\,m^8-496\,m^9-192\,m^{10}+64\,m^{11})-$$
$$4\,k^6\,m\,(28980-15477\,m-155139\,m^2+$$
$$82558\,m^3+166545\,m^4-86131\,m^5-40332\,m^6+14484\,m^7+6960\,m^8-1552\,m^9-1344\,m^{10}+448\,m^{11})-$$
$$2\,k^5\,m\,(288270-501843\,m-298671\,m^2+301047\,m^3-$$
$$21275\,m^4+60096\,m^5-55088\,m^6-29864\,m^7+71440\,m^8-18368\,m^9-11136\,m^{10}+3712\,m^{11})-$$
$$4\,k^2\,(-1215000-1571130\,m+3426987\,m^2-1003736\,m^3+295207\,m^4+$$
$$1880310\,m^5-1834214\,m^6+919272\,m^7-264744\,m^8-273760\,m^9+137312\,m^{10}-30336\,m^{11}+10112\,m^{12})+$$
$$k^4\,(972000-1497510\,m+3038199\,m^2-6810\,m^3-843115\,m^4+$$
$$2152120\,m^5-1797428\,m^6+574000\,m^7-162560\,m^8-181760\,m^9+104384\,m^{10}-34560\,m^{11}+11520\,m^{12})-$$
$$8\,k\,(-243000-1074600\,m+1740870\,m^2+1571949\,m^3-1269413\,m^4-$$
$$697425\,m^5+342995\,m^6+312012\,m^7-401124\,m^8+288720\,m^9-39280\,m^{10}-83136\,m^{11}+27712\,m^{12})+$$
$$2\,k^3\,(1944000+288900\,m-752550\,m^2+708823\,m^3-739111\,m^4-$$
$$24555\,m^5+292005\,m^6-282156\,m^7-318108\,m^8+560240\,m^9-120080\,m^{10}-113472\,m^{11}+37824\,m^{12}))$$
$$h_1(k,m)+$$
$$(6\,k^9\,m\,(-180+72\,m+1025\,m^2-410\,m^3-1365\,m^4+546\,m^5+600\,m^6-240\,m^7-80\,m^8+32\,m^9)+k^8\,m\,(-12870+5283\,m+72950\,m^2-$$
$$29915\,m^3-95760\,m^4+39144\,m^5+40800\,m^6-16560\,m^7-5120\,m^8+2048\,m^9)+2\,k^7\,m\,(-49050+24345\,m+$$
$$267098\,m^2-130861\,m^3-308240\,m^4+144020\,m^5+101584\,m^6-41408\,m^7-12160\,m^8+4160\,m^9+768\,m^{10}-256\,m^{11})+480\,m$$
$$(10800-7740\,m-52599\,m^2+38137\,m^3+36111\,m^4-28525\,m^5+6732\,m^6-1092\,m^7-3120\,m^8+1424\,m^9-192\,m^{10}+64\,m^{11})-$$
$$4\,k^6\,m\,(50040-26631\,m-268276\,m^2+$$
$$143007\,m^3+287830\,m^4-150578\,m^5-65288\,m^6+19336\,m^7+16160\,m^8-3296\,m^9-3456\,m^{10}+1152\,m^{11})-$$
$$2\,k^5\,m\,(774810-1452609\,m-428789\,m^2+635873\,m^3-$$
$$514425\,m^4+392748\,m^5-52512\,m^6-54696\,m^7+104240\,m^8-34304\,m^9-8064\,m^{10}+2688\,m^{11})-$$
$$48\,k\,(-121500-490050\,m+707805\,m^2+1058171\,m^3-896532\,m^4-$$
$$423225\,m^5+390055\,m^6-146532\,m^7-20676\,m^8+118480\,m^9-33840\,m^{10}-14784\,m^{11}+4928\,m^{12})+$$
$$4\,k^3\,(2916000+1270800\,m-2899125\,m^2+1372531\,m^3-1047282\,m^4-$$
$$983105\,m^5+1288925\,m^6-921452\,m^7+55684\,m^8+520080\,m^9-168080\,m^{10}-43584\,m^{11}+14528\,m^{12})+$$
$$k^4\,(2916000-4226130\,m+7816437\,m^2+2925370\,m^3-4814165\,m^4+$$
$$4420240\,m^5-2618024\,m^6-380000\,m^7-242000\,m^8+491520\,m^9-27648\,m^{10}-184320\,m^{11}+61440\,m^{12})-$$
$$4\,k^2\,(-3645000-6552990\,m+12031461\,m^2+4504816\,m^3-5512867\,m^4+$$
$$2288850\,m^5-1168662\,m^6-309912\,m^7-564456\,m^8+844640\,m^9-108704\,m^{10}-249984\,m^{11}+83328\,m^{12}))$$
$$h_1(k+1,m)+$$
$$(k^8\,m\,(3690-1521\,m-20900\,m^2+8605\,m^3+27370\,m^4-11228\,m^5-11600\,m^6+4720\,m^7+1440\,m^8-576\,m^9)-$$
$$2\,k^9\,m\,(-180+72\,m+1025\,m^2-410\,m^3-1365\,m^4+546\,m^5+600\,m^6-240\,m^7-80\,m^8+32\,m^9)-2\,k^5\,m\,(-264870+482583\,m+$$
$$191515\,m^2-209565\,m^3+63095\,m^4-105166\,m^5+96920\,m^6-20320\,m^7-37520\,m^8+15008\,m^9)-960\,m\,(1350-855\,m-$$
$$6849\,m^2+4240\,m^3+6015\,m^4-3253\,m^5-1044\,m^6-84\,m^7+720\,m^8-112\,m^9-192\,m^{10}+64\,m^{11})+2\,k^7\,m\,(14850-7365\,m-$$
$$80854\,m^2+39483\,m^3+93420\,m^4-43070\,m^5-31512\,m^6+12264\,m^7+4480\,m^8-1440\,m^9-384\,m^{10}+128\,m^{11})+$$
$$4\,k^6\,m\,(11880-6027\,m-64472\,m^2+$$
$$32859\,m^3+72110\,m^4-36046\,m^5-18616\,m^6+4472\,m^7+5920\,m^8-1312\,m^9-1152\,m^{10}+384\,m^{11})+$$
$$k^4\,(-972000+1283310\,m-2362119\,m^2-991880\,m^3+1827455\,m^4-$$
$$1822110\,m^5+639648\,m^6+845280\,m^7-83040\,m^8-415840\,m^9+92416\,m^{10}+80640\,m^{11}-26880\,m^{12})-$$
$$40\,k\,(48600+207900\,m-257706\,m^2-602583\,m^3+$$
$$452603\,m^4+365379\,m^5-260129\,m^6+22140\,m^7+4620\,m^8-21360\,m^9+9424\,m^{10}-960\,m^{11}+320\,m^{12})+$$
$$4\,k^3\,(-972000-523350\,m+1248150\,m^2-742213\,m^3+416581\,m^4+$$
$$823705\,m^5-684225\,m^6+209116\,m^7-47012\,m^8-81040\,m^9+41040\,m^{10}-9408\,m^{11}+3136\,m^{12})+$$
$$12\,k^2\,(-405000-824010\,m+1431979\,m^2+881209\,m^3-955718\,m^4+$$
$$121605\,m^5+93087\,m^6-233508\,m^7-40404\,m^8+192560\,m^9-43056\,m^{10}-37056\,m^{11}+12352\,m^{12}))$$
$$h_1(k+2,m))/$$
$$(480\,(1+k)^2\,(-3+m)\,(-2+m)\,(-1+m)\,m\,(1+m)\,(2+m)\,(-5+2\,m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\,(5+2\,m))$$

$$
\begin{aligned}
g_8(k,m) := \big(&-2\,\big(k^{10}\,m\,(2700-900\,m-15807\,m^2+5269\,m^3+22935\,m^4-7645\,m^5-12276\,m^6+4092\,m^7+2640\,m^8-880\,m^9-192\,m^{10}+64\,m^{11})+\\
&k^9\,m\,(57780-19692\,m-337233\,m^2+114871\,m^3+\\
&\quad 484905\,m^4-164911\,m^5-254844\,m^6+86388\,m^7+53040\,m^8-17872\,m^9-3648\,m^{10}+1216\,m^{11})+\\
&k^6\,m\,(10534860-7160184\,m-51898347\,m^2+33606621\,m^3+41665687\,m^4-20601929\,m^5-\\
&\quad 11234536\,m^6+3069668\,m^7+3037056\,m^8-509520\,m^9-733312\,m^{10}+184256\,m^{11}+55552\,m^{12}-15872\,m^{13})+\\
&k^5\,m\,(-6032880+23045202\,m-45259965\,m^2+51887997\,m^3-15305633\,m^4+7385209\,m^5+8023910\,m^6-\\
&\quad 11721784\,m^7+9439416\,m^8-2098944\,m^9-1819360\,m^{10}+589952\,m^{11}+15232\,m^{12}-4352\,m^{13})+\\
&k^7\,m\,(2835000-1211490\,m-15935949\,m^2+6738150\,m^3+20454622\,m^4-8348042\,m^5-\\
&\quad 8622517\,m^6+3223310\,m^7+1388316\,m^8-432936\,m^9-127984\,m^{10}+33440\,m^{11}+8512\,m^{12}-2432\,m^{13})+\\
&k^8\,m\,(605070-225423\,m-3484494\,m^2+1292931\,m^3+4815763\,m^4-1764782\,m^5-\\
&\quad 2342407\,m^6+835538\,m^7+434244\,m^8-147192\,m^9-28624\,m^{10}+9056\,m^{11}+448\,m^{12}-128\,m^{13})+\\
&2880\,m\,(85050-54000\,m-422775\,m^2+226854\,m^3+403550\,m^4-\\
&\quad 56872\,m^5-323873\,m^6+63838\,m^7+126156\,m^8-41064\,m^9-10544\,m^{10}+4000\,m^{11}-448\,m^{12}+128\,m^{13})-\\
&432\,k\,(-283500-1221750\,m+1862535\,m^2+2281381\,m^3-2101781\,m^4-259369\,m^5-328406\,m^6+\\
&\quad 581798\,m^7+375932\,m^8-805112\,m^9+156112\,m^{10}+175776\,m^{11}-49856\,m^{12}-8064\,m^{13}+2304\,m^{14})+\\
&k^3\,(244944000+17720100\,m-130295700\,m^2+386039019\,m^3-418990570\,m^4+77121898\,m^5-37682848\,m^6-51734773\,m^7+\\
&\quad 93771830\,m^8-82525956\,m^9+17055736\,m^{10}+17221904\,m^{11}-5435360\,m^{12}-281792\,m^{13}+80512\,m^{14})+\\
&k^4\,(61236000-75573270\,m+107776503\,m^2+168572874\,m^3-142484821\,m^4-71219273\,m^5+75657692\,m^6-46082863\,m^7+\\
&\quad 16761962\,m^8+10577796\,m^9-8482328\,m^{10}+4232624\,m^{11}-735776\,m^{12}-623168\,m^{13}+178048\,m^{14})-\\
&6\,k^2\,(-51030000-60310440\,m+129386466\,m^2-28728909\,m^3-901280\,m^4+59825492\,m^5-3004244\,m^6-60104627\,m^7+\\
&\quad 21629050\,m^8+13146756\,m^9-8928760\,m^{10}+3592816\,m^{11}-512800\,m^{12}-632128\,m^{13}+180608\,m^{14})\big)\\
&h_1(k,m)+\\
&(3\,k^{10}\,m\,(2700-900\,m-15807\,m^2+5269\,m^3+22935\,m^4-7645\,m^5-12276\,m^6+4092\,m^7+2640\,m^8-880\,m^9-192\,m^{10}+64\,m^{11})+\\
&9\,k^9\,m\,(15660-5364\,m-91335\,m^2+\\
&\quad 31265\,m^3+131055\,m^4-44777\,m^5-68580\,m^6+23340\,m^7+14160\,m^8-4784\,m^9-960\,m^{10}+320\,m^{11})+\\
&k^8\,m\,(1353510-515349\,m-7767153\,m^2+2942643\,m^3+10621327\,m^4-3961574\,m^5-\\
&\quad 5060164\,m^6+1821704\,m^7+914496\,m^8-306240\,m^9-63808\,m^{10}+19328\,m^{11}+1792\,m^{12}-512\,m^{13})+\\
&1440\,m\,(226800-127980\,m-1178055\,m^2+583878\,m^3+1271810\,m^4-\\
&\quad 347644\,m^5-801911\,m^6+260386\,m^7+191892\,m^8-84888\,m^9+6832\,m^{10}+1120\,m^{11}-3136\,m^{12}+896\,m^{13})-\\
&6\,k^7\,m\,(-930150+410265\,m+5195127\,m^2-2261046\,m^3-6538432\,m^4+2717667\,m^5+\\
&\quad 2669171\,m^6-981838\,m^7-444436\,m^8+127016\,m^9+52752\,m^{10}-13216\,m^{11}-4032\,m^{12}+1152\,m^{13})-\\
&3\,k^6\,m\,(-7964460+5913864\,m+37832307\,m^2-26897589\,m^3-25126175\,m^4+13204977\,m^5+\\
&\quad 5382856\,m^6-1025732\,m^7-1994240\,m^8+351184\,m^9+469632\,m^{10}-127424\,m^{11}-26880\,m^{12}+7680\,m^{13})+\\
&9\,k^5\,m\,(-3585960+9305334\,m-8950481\,m^2+10956671\,m^3-4368411\,m^4+1363743\,m^5+\\
&\quad 2898132\,m^6-2066292\,m^7+846992\,m^8-378608\,m^9+22464\,m^{10}+27456\,m^{11}-32256\,m^{12}+9216\,m^{13})+\\
&48\,k\,(7654500+30176550\,m-39256245\,m^2-79013457\,m^3+53454327\,m^4+54189803\,m^5-13406458\,m^6-34505876\,m^7+\\
&\quad 6319096\,m^8+14035584\,m^9-4514304\,m^{10}-1236032\,m^{11}+472192\,m^{12}-55552\,m^{13}+15872\,m^{14})+\\
&3\,k^4\,(61236000-72530370\,m+92772063\,m^2+203199981\,m^3-173332591\,m^4-72403839\,m^5+51830688\,m^6-10547932\,m^7+\\
&\quad 15596672\,m^8-11349152\,m^9-203680\,m^{10}+4956736\,m^{11}-1289216\,m^{12}-335104\,m^{13}+95744\,m^{14})-6\,k^3\\
&\quad(-122472000-47808900\,m+134524530\,m^2-151585827\,m^3+132081006\,m^4+43098422\,m^5-17623682\,m^6-13933611\,m^7-\\
&\quad 5959902\,m^8+17675236\,m^9-7824536\,m^{10}+505328\,m^{11}+361056\,m^{12}-488768\,m^{13}+139648\,m^{14})-4\,k^2\\
&\quad(-229635000-393630030\,m+691905537\,m^2+379143432\,m^3-369473559\,m^4-14899475\,m^5-10513724\,m^6+6928661\,m^7+\\
&\quad 62967458\,m^8-76638540\,m^9+7210632\,m^{10}+24482672\,m^{11}-6338464\,m^{12}-1682240\,m^{13}+480640\,m^{14}))\\
&h_1(k+1,m)+\\
&(k^9\,m\,(-41580+14292\,m+242391\,m^2-\\
&\quad 83257\,m^3-347295\,m^4+119041\,m^5+181188\,m^6-61836\,m^7-37200\,m^8+12592\,m^9+2496\,m^{10}-832\,m^{11})+\\
&k^{10}\,m\,(-2700+900\,m+15807\,m^2-5269\,m^3-22935\,m^4+7645\,m^5+12276\,m^6-4092\,m^7-2640\,m^8+880\,m^9+192\,m^{10}-64\,m^{11})+\\
&k^5\,m\,(13744080-30529602\,m+13926465\,m^2-22297959\,m^3+17708731\,m^4-4073867\,m^5-12454060\,m^6+\\
&\quad 4415828\,m^7+1833168\,m^8+390000\,m^9-1122880\,m^{10}+203456\,m^{11}+157696\,m^{12}-45056\,m^{13})-\\
&2880\,m\,(28350-9045\,m-167688\,m^2+56703\,m^3+245807\,m^4-\\
&\quad 95512\,m^5-116369\,m^6+55414\,m^7+5916\,m^8-6696\,m^9+4432\,m^{10}-992\,m^{11}-448\,m^{12}+128\,m^{13})+\\
&k^8\,m\,(-402570+153063\,m+2310453\,m^2-873177\,m^3-3162041\,m^4+\\
&\quad 1172470\,m^5+1514534\,m^6-537556\,m^7-282168\,m^8+91536\,m^9+22688\,m^{10}-6592\,m^{11}-896\,m^{12}+256\,m^{13})+\\
&2\,k^7\,m\,(-757350+332775\,m+4231512\,m^2-1827834\,m^3-5342140\,m^4+2174675\,m^5+\\
&\quad 2237266\,m^6-778112\,m^7-424680\,m^8+112320\,m^9+59872\,m^{10}-15104\,m^{11}-4480\,m^{12}+1280\,m^{13})+\\
&k^6\,m\,(-7586460+5549904\,m+36211239\,m^2-25089345\,m^3-25152707\,m^4+11876869\,m^5+\\
&\quad 7437992\,m^6-1426900\,m^7-2818176\,m^8+656400\,m^9+500864\,m^{10}-145600\,m^{11}-19712\,m^{12}+5632\,m^{13})+\\
&k^4\,(-61236000+65083770\,m-83576943\,m^2-185166723\,m^3+178657387\,m^4+18810991\,m^5-24435720\,m^6+25312466\,m^7-\\
&\quad 24308884\,m^8+13525848\,m^9-1425136\,m^{10}-4253728\,m^{11}+1261632\,m^{12}+144256\,m^{13}-41216\,m^{14})-\\
&24\,k\,(5103000+21392100\,m-23125230\,m^2-71672661\,m^3+38307336\,m^4+72077684\,m^5-22139884\,m^6-41410823\,m^7+\\
&\quad 13171378\,m^8+10068852\,m^9-4046232\,m^{10}-51536\,m^{11}+137056\,m^{12}-110656\,m^{13}+31616\,m^{14})+\\
&12\,k^2\,(-25515000-49955670\,m+82807623\,m^2+66962142\,m^3-60379133\,m^4-12211229\,m^5-2495052\,m^6+13041881\,m^7+\\
&\quad 5984186\,m^8-15211932\,m^9+3191720\,m^{10}+3079472\,m^{11}-898848\,m^{12}-117824\,m^{13}+33664\,m^{14})+\\
&4\,k^3\,(-61236000-30700350\,m+83766060\,m^2-82981056\,m^3+51700237\,m^4+68862175\,m^5-25613295\,m^6-33066313\,m^7+\\
&\quad 11865626\,m^8+7302540\,m^9-4982680\,m^{10}+2003024\,m^{11}-257568\,m^{12}-378560\,m^{13}+108160\,m^{14}))\\
&h_1(k+2,m)\big)\big/\\
&(1440\,(1+k)^2\,(-3+m)\,(-2+m)\,(-1+m)\,m\,(1+m)\,(2+m)\,(3+m)\,(-7+2\,m)\,(-5+2\,m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)\\
&(3+2\,m)\,(5+2\,m))
\end{aligned}
$$

$$
\begin{aligned}
g_9(k,m) := \Big(& -2\,\big(2\,k^{11}\,m\,(18900-11700\,m-96249\,m^2+\\
&\quad 56497\,m^3+89841\,m^4-41665\,m^5-29988\,m^6+11868\,m^7+4080\,m^8-1456\,m^9-192\,m^{10}+64\,m^{11})+\\
& k^{10}\,m\,(1077300-673200\,m-5469393\,m^2+3243212\,m^3+\\
&\quad 5038449\,m^4-2361080\,m^5-1644048\,m^6+656904\,m^7+215760\,m^8-77504\,m^9-9600\,m^{10}+3200\,m^{11})+\\
& k^9\,m\,(14556780-9351792\,m-73215663\,m^2+44715468\,m^3+64753085\,m^4-31196092\,m^5-\\
&\quad 19786718\,m^6+8064412\,m^7+2390328\,m^8-859824\,m^9-104480\,m^{10}+33856\,m^{11}+896\,m^{12}-256\,m^{13})+\\
& 181440\,m\,(-176400+159600\,m+756490\,m^2-674843\,m^3-196933\,m^4+\\
&\quad 137133\,m^5-20045\,m^6+33826\,m^7-27844\,m^8+3464\,m^9+8080\,m^{10}-2208\,m^{11}-448\,m^{12}+128\,m^{13})-\\
& 2\,k^8\,m\,(-49809060+33611724\,m+246104421\,m^2-158300597\,m^3-200718040\,m^4+100919767\,m^5+\\
&\quad 54485683\,m^6-22542926\,m^7-6103380\,m^8+2117864\,m^9+345616\,m^{10}-103072\,m^{11}-11200\,m^{12}+3200\,m^{13})-\\
& 7\,k^6\,m\,(-119338380+110040912\,m+503865255\,m^2-452143120\,m^3-115091327\,m^4+48386372\,m^5+41482376\,m^6-\\
&\quad 3389272\,m^7-21882768\,m^8+6264256\,m^9+2670848\,m^{10}-816512\,m^{11}-68096\,m^{12}+19456\,m^{13})-\\
& 2\,k^7\,m\,(-226497600+173548620\,m+1060263528\,m^2-777501095\,m^3-651751710\,m^4+346636967\,m^5+143787042\,m^6-\\
&\quad 55163856\,m^7-21182856\,m^8+6253184\,m^9+2353632\,m^{10}-670976\,m^{11}-104832\,m^{12}+29952\,m^{13})+\\
& k^5\,m\,(4125790620-9084595248\,m+3226307565\,m^2-2640652672\,m^3-1719540003\,m^4+2164447336\,m^5-1631269962\,m^6+\\
&\quad 530028948\,m^7+382377480\,m^8-155585360\,m^9+1589664\,m^{10}+1610432\,m^{11}-1975680\,m^{12}+564480\,m^{13})+\\
& 864\,k\,(-18522000-76734000\,m+141953070\,m^2+43835777\,m^3-31475444\,m^4-36458663\,m^5+46031644\,m^6-33570880\,m^7+\\
&\quad 7638200\,m^8+12078928\,m^9-4582144\,m^{10}-309376\,m^{11}+165248\,m^{12}-57344\,m^{13}+16384\,m^{14})-96\,k^2\\
&\ (416745000+423084060\,m-1057960674\,m^2+854405667\,m^3-891391718\,m^4+92082328\,m^5-65241249\,m^6-21685090\,m^7+\\
&\quad 79288328\,m^8-84204984\,m^9+19703264\,m^{10}+14909984\,m^{11}-4404096\,m^{12}-522368\,m^{13}+149248\,m^{14})+\\
& 2\,k^4\,(-4000752000+6885186840\,m-15915276036\,m^2+8782225686\,m^3-\\
&\quad 9771692147\,m^4+2234585635\,m^5-975802383\,m^6-1064879977\,m^7+977030714\,m^8-546038580\,m^9+\\
&\quad 125618024\,m^{10}+99449936\,m^{11}-30177312\,m^{12}-2744000\,m^{13}+784000\,m^{14})-4\,k^3\,(8001504000-\\
&\quad 947286900\,m+1171099800\,m^2+2814339039\,m^3-2311045009\,m^4-1790099602\,m^5+3081940119\,m^6-3028083485\,m^7+\\
&\quad 986138242\,m^8+717030348\,m^9-311584280\,m^{10}+23089168\,m^{11}-1138080\,m^{12}-6053824\,m^{13}+1729664\,m^{14})\big)\\
& h_1(k,m)+\\
& (6\,k^{11}\,m\,(18900-11700\,m-96249\,m^2+\\
&\quad 56497\,m^3+89841\,m^4-41665\,m^5-29988\,m^6+11868\,m^7+4080\,m^8-1456\,m^9-192\,m^{10}+64\,m^{11})+\\
& k^{10}\,m\,(2702700-1692000\,m-13713207\,m^2+8147720\,m^3+\\
&\quad 12599799\,m^4-5916620\,m^5-4092480\,m^6+1638408\,m^7+533040\,m^8-191744\,m^9-23424\,m^{10}+7808\,m^{11})+\\
& k^9\,m\,(33086340-21389976\,m-166048149\,m^2+102087790\,m^3+145436141\,m^4-70468082\,m^5-\\
&\quad 43792784\,m^6+17888704\,m^7+5231472\,m^8-1869280\,m^9-241280\,m^{10}+76544\,m^{11}+3584\,m^{12}-1024\,m^{13})+\\
& 201600\,m\,(-211680+203238\,m+870321\,m^2-822115\,m^3-101938\,m^4+\\
&\quad 27837\,m^5+46243\,m^6+31450\,m^7-67284\,m^8+16072\,m^9+11536\,m^{10}-3360\,m^{11}-448\,m^{12}+128\,m^{13})+\\
& 7\,k^6\,m\,(205076340-197359056\,m-841426953\,m^2+795151448\,m^3+97718365\,m^4-23329084\,m^5-61131748\,m^6+\\
&\quad 4564592\,m^7+33475008\,m^8-10995488\,m^9-2632000\,m^{10}+910336\,m^{11}-30464\,m^{12}+8704\,m^{13})-\\
& 2\,k^8\,m\,(-102592980+70001712\,m+504744525\,m^2-328345206\,m^3-403571965\,m^4+204087494\,m^5+107413852\,m^6-\\
&\quad 43972952\,m^7-12565824\,m^8+4209408\,m^9+869056\,m^{10}-252800\,m^{11}-34048\,m^{12}+9728\,m^{13})-\\
& 2\,k^7\,m\,(-478396800+378286020\,m+2205519672\,m^2-1671960829\,m^3-1230795030\,m^4+658415605\,m^5+263263266\,m^6-\\
&\quad 95810712\,m^7-45296040\,m^8+13239520\,m^9+5191392\,m^{10}-1512064\,m^{11}-201600\,m^{12}+57600\,m^{13})+k^5\,m\\
&\ (11651309460-26455840344\,m+12268772895\,m^2-10491426358\,m^3-4585047555\,m^4+4129502362\,m^5-1868188644\,m^6+\\
&\quad 992327304\,m^7-32902752\,m^8-74908736\,m^9+92480448\,m^{10}-24833920\,m^{11}-5531904\,m^{12}+1580544\,m^{13})-\\
& 4\,k^4\,(6001128000-9946359360\,m+21247603884\,m^2-\\
&\quad 5491676520\,m^3+5890208793\,m^4-1159111915\,m^5+1392385163\,m^6-840592991\,m^7-126776954\,m^8+\\
&\quad 707344212\,m^9-238330344\,m^{10}-49765328\,m^{11}+18426400\,m^{12}-1696576\,m^{13}+484736\,m^{14})+\\
& 8\,k^2\,(-15002820000-22896223560\,m+47280581244\,m^2-5670984414\,m^3+\\
&\quad 4498759153\,m^4+1529148847\,m^5+3053404701\,m^6-6157266913\,m^7+1045919066\,m^8+2619504108\,m^9-\\
&\quad 865737880\,m^{10}-199888432\,m^{11}+68751840\,m^{12}-1959104\,m^{13}+559744\,m^{14})+48\,k\,(-1000188000-\\
&\quad 3760495200\,m+6213977280\,m^2+5201196198\,m^3-5048261729\,m^4-473279927\,m^5+680014935\,m^6-664395319\,m^7+\\
&\quad 422399174\,m^8-93562380\,m^9-15077608\,m^{10}+54959024\,m^{11}-14458848\,m^{12}-3563840\,m^{13}+1018240\,m^{14})-\\
& 4\,k^3\,(24004512000+4138287300\,m-12749524380\,m^2+\\
&\quad 17770245393\,m^3-17086592950\,m^4-2079619261\,m^5+2930960370\,m^6-2963823800\,m^7+2223702448\,m^8-\\
&\quad 840053520\,m^9+42349216\,m^{10}+309350656\,m^{11}-84751872\,m^{12}-16952320\,m^{13}+4843520\,m^{14}))\\
& h_1(k+1,m)+(-2\,k^{11}\,m\,(18900-\\
&\quad 11700\,m-96249\,m^2+56497\,m^3+89841\,m^4-41665\,m^5-29988\,m^6+11868\,m^7+4080\,m^8-1456\,m^9-192\,m^{10}+64\,m^{11})-\\
& 3\,k^{10}\,m\,(270900-169800\,m-1373969\,m^2+817418\,m^3+\\
&\quad 1260225\,m^4-592590\,m^5-408072\,m^6+163584\,m^7+52880\,m^8-19040\,m^9-2304\,m^{10}+768\,m^{11})-\\
& 20160\,m\,(-529200+452340\,m+2319636\,m^2-1828315\,m^3-1005763\,m^4+\\
&\quad 116973\,m^5+851923\,m^6-187790\,m^7-304164\,m^8+94984\,m^9+28816\,m^{10}-9120\,m^{11}-448\,m^{12}+128\,m^{13})+\\
& k^9\,m\,(-9793980+6327792\,m+49160715\,m^2-30195228\,m^3-43106071\,m^4+20825768\,m^5+\\
&\quad 13064044\,m^6-5297792\,m^7-1606656\,m^8+565152\,m^9+83392\,m^{10}-25856\,m^{11}-1792\,m^{12}+512\,m^{13})+\\
& 4\,k^8\,m\,(-14386680+9792072\,m+70835088\,m^2-45906553\,m^3-56933703\,m^4+28468921\,m^5+\\
&\quad 15619317\,m^6-6225210\,m^7-2034300\,m^8+658840\,m^9+164592\,m^{10}-47584\,m^{11}-6720\,m^{12}+1920\,m^{13})+\\
& 14\,k^7\,m\,(-20412000+16035300\,m+94350168\,m^2-70788573\,m^3-53998148\,m^4+27708865\,m^5+\\
&\quad 13384352\,m^6-4666300\,m^7-2573040\,m^8+775248\,m^9+271232\,m^{10}-80704\,m^{11}-8960\,m^{12}+2560\,m^{13})-\\
& 7\,k^6\,m\,(48160980-46746072\,m-196225569\,m^2+186594834\,m^3+18958949\,m^4+630914\,m^5-20434796\,m^6+4444120\,m^7+\\
&\quad 7656960\,m^8-2969856\,m^9-130496\,m^{10}+92032\,m^{11}-44800\,m^{12}+12800\,m^{13})-k^5\,m\,(4055407020-\\
&\quad 8856333648\,m+3003539649\,m^2-3090872792\,m^3+96038503\,m^4+324583044\,m^5-705198424\,m^6+444411416\,m^7-\\
&\quad 104735664\,m^8+6923840\,m^9+37154816\,m^{10}-10727040\,m^{11}-1530368\,m^{12}+437248\,m^{13})-336\,k\,(-47628000-\\
&\quad 190776600\,m+273493980\,m^2+423168306\,m^3-379514261\,m^4-95734799\,m^5+36215019\,m^6+37464113\,m^7+5172902\,m^8-\\
&\quad 29536812\,m^9+7192472\,m^{10}+4913456\,m^{11}-1422816\,m^{12}-198464\,m^{13}+56704\,m^{14})+12\,k^3\,(2667168000+
\end{aligned}
$$

$$
\begin{aligned}
&736199100\,m - 2255122680\,m^{2} + 3070362415\,m^{3} - 2780249927\,m^{4} - 630189578\,m^{5} + 225283925\,m^{6} + 234052763\,m^{7} +\\
&125082218\,m^{8} - 298684660\,m^{9} + 71708168\,m^{10} + 50866064\,m^{11} - 14885408\,m^{12} - 1910720\,m^{13} + 545920\,m^{14}\big) +\\
&2\,k^{4}\,\big(4000752000 - 6138882540\,m + 13077117216\,m^{2} - 2992877121\,m^{3} +\\
&2044864416\,m^{4} + 1864333301\,m^{5} + 84135488\,m^{6} - 2104600064\,m^{7} + 627641320\,m^{8} + 562629840\,m^{9} -\\
&217183680\,m^{10} - 10377344\,m^{11} + 6448768\,m^{12} - 2759680\,m^{13} + 788480\,m^{14}\big) - 8\,k^{2}\,\big(- 5000940000 -\\
&8823612420\,m + 17242362828\,m^{2} + 1844965095\,m^{3} - 2853747344\,m^{4} + 1766306629\,m^{5} + 941916744\,m^{6} - 3381158284\,m^{7} +\\
&1061356976\,m^{8} + 834760800\,m^{9} - 324468256\,m^{10} - 12920512\,m^{11} + 8684544\,m^{12} - 4040960\,m^{13} + 1154560\,m^{14}\big)\big)\\
&h_1(k+2,m)\big)\Big/\\
&\big(20160\,(1+k)^{2}\,(-4+m)\,(-3+m)\,(-2+m)\,(-1+m)\,m\,(2+m)\,(3+m)\,(-7+2\,m)\,(-5+2\,m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\\
&(5+2\,m)\,(7+2\,m)\big)
\end{aligned}
$$

$$
\begin{aligned}
g_{10}(k,m) := \Big(& - 2\,\big(32\,k^{11}\,m\,\big(1814400 - 461700\,m - 10953504\,m^{2} + 2786697\,m^{3} + 17365712\,m^{4} - 4415502\,m^{5} - 11147136\,m^{6} + 2831257\,m^{7} +\\
&3402256\,m^{8} - 862576\,m^{9} - 518336\,m^{10} + 131040\,m^{11} + 37632\,m^{12} - 9472\,m^{13} - 1024\,m^{14} + 256\,m^{15}\big) +\\
&k^{12}\,m\,\big(1587600 - 396900\,m - 9600516\,m^{2} + 2400129\,m^{3} + 15291640\,m^{4} - 3822910\,m^{5} - 9901892\,m^{6} + 2475473\,m^{7} +\\
&3065920\,m^{8} - 766480\,m^{9} - 477568\,m^{10} + 119392\,m^{11} + 35840\,m^{12} - 8960\,m^{13} - 1024\,m^{14} + 256\,m^{15}\big) +\\
&k^{8}\,m\,\big(54285986160 - 17849254284\,m - 318185975088\,m^{2} + 103772190339\,m^{3} + 464461703229\,m^{4} - 147820162076\,m^{5} -\\
&256570600762\,m^{6} + 77522360799\,m^{7} + 63601516717\,m^{8} - 17462805946\,m^{9} - 8232326352\,m^{10} + 1975344320\,m^{11} +\\
&675853024\,m^{12} - 145416384\,m^{13} - 37090048\,m^{14} + 7950592\,m^{15} + 933120\,m^{16} - 207360\,m^{17}\big) +\\
&k^{10}\,m\,\big(992703600 - 263349900\,m - 5968170576\,m^{2} + 1581703059\,m^{3} + 9354610805\,m^{4} -\\
&2472344232\,m^{5} - 5879073326\,m^{6} + 1545570835\,m^{7} + 1735311369\,m^{8} - 452125938\,m^{9} - 252301520\,m^{10} +\\
&64801088\,m^{11} + 17394272\,m^{12} - 4370368\,m^{13} - 476928\,m^{14} + 115968\,m^{15} + 2304\,m^{16} - 512\,m^{17}\big) +\\
&241920\,m\,\big(57153600 - 32281200\,m - 294688800\,m^{2} + 137297502\,m^{3} +\\
&324649721\,m^{4} - 46340124\,m^{5} - 264094168\,m^{6} + 63312892\,m^{7} + 91480559\,m^{8} - 35679470\,m^{9} -\\
&2455184\,m^{10} + 3179680\,m^{11} - 2026976\,m^{12} + 362432\,m^{13} + 170752\,m^{14} - 39424\,m^{15} - 2304\,m^{16} + 512\,m^{17}\big) -\\
&2\,k^{9}\,m\,\big(- 4584399120 + 1306148868\,m + 27351546732\,m^{2} - 7770795873\,m^{3} - 41971575298\,m^{4} + 11828793524\,m^{5} +\\
&25369616622\,m^{6} - 7037035319\,m^{7} - 7061880488\,m^{8} + 1905164480\,m^{9} + 959258080\,m^{10} - 247464928\,m^{11} -\\
&64740480\,m^{12} + 15681536\,m^{13} + 2210816\,m^{14} - 500480\,m^{15} - 36864\,m^{16} + 8192\,m^{17}\big) - 4\,k^{7}\,m\,\big(- 45298990800 +\\
&18950806980\,m + 255431488176\,m^{2} - 104891331465\,m^{3} - 332389359087\,m^{4} + 128021127306\,m^{5} + 149487158408\,m^{6} -\\
&48408757047\,m^{7} - 32159038409\,m^{8} + 7174027442\,m^{9} + 5657756976\,m^{10} - 992293376\,m^{11} - 778858976\,m^{12} +\\
&157418688\,m^{13} + 50896640\,m^{14} - 11232512\,m^{15} - 1052928\,m^{16} + 233984\,m^{17}\big) + k^{6}\,m\,\big(533201492880 -\\
&384728957172\,m - 2565670008816\,m^{2} + 1776166820109\,m^{3} + 1810845092651\,m^{4} - 948157686224\,m^{5} - 364034204954\,m^{6} -\\
&4316501483\,m^{7} + 205985694687\,m^{8} - 192070878\,m^{9} - 82806119984\,m^{10} + 16287251904\,m^{11} + 10156246688\,m^{12} -\\
&2488865344\,m^{13} - 332656896\,m^{14} + 86065920\,m^{15} - 2048256\,m^{16} + 455168\,m^{17}\big) + 2\,k^{5}\,m\,\big(- 275677985520 +\\
&671127368508\,m - 572407062300\,m^{2} + 1034645953473\,m^{3} - 1080332948282\,m^{4} + 596209401228\,m^{5} + 319848426674\,m^{6} -\\
&420811240273\,m^{7} + 324813734580\,m^{8} - 82176994968\,m^{9} - 56227379488\,m^{10} + 25247347552\,m^{11} -\\
&5165882624\,m^{12} + 292299520\,m^{13} + 920100864\,m^{14} - 196484352\,m^{15} - 23675904\,m^{16} + 5261312\,m^{17}\big) +\\
&k^{4}\,\big(3456649728000 - 4210994511360\,m + 5654349571824\,m^{2} + 10808435865924\,m^{3} - 9376988512548\,m^{4} -\\
&4113282068729\,m^{5} + 5004974554266\,m^{6} - 3872338578626\,m^{7} + 1828695439448\,m^{8} + 419279808583\,m^{9} -\\
&697815557854\,m^{10} + 522827691920\,m^{11} - 82360945312\,m^{12} - 83674371424\,m^{13} + 22163495104\,m^{14} +\\
&1528716032\,m^{15} - 549605888\,m^{16} + 118183680\,m^{17} - 26263040\,m^{18}\big) + 144\,k\,\big(48009024000 +\\
&210928536000\,m - 309097736640\,m^{2} - 435552994404\,m^{3} + 366102616512\,m^{4} + 124553980857\,m^{5} + 42483316598\,m^{6} -\\
&156646319046\,m^{7} - 63099062628\,m^{8} + 173231852961\,m^{9} - 38353158066\,m^{10} - 34403043216\,m^{11} + 12387575840\,m^{12} -\\
&497769888\,m^{13} - 350922432\,m^{14} + 378076416\,m^{15} - 73972224\,m^{16} - 14503680\,m^{17} + 3223040\,m^{18}\big) +\\
&4\,k^{2}\,\big(4320812160000 + 5466485465280\,m - 11224225600992\,m^{2} +\\
&636839647308\,m^{3} + 1834214361408\,m^{4} - 4972908681879\,m^{5} + 89370376814\,m^{6} + 5632907943810\,m^{7} -\\
&2929189446348\,m^{8} - 287772852039\,m^{9} + 953052255774\,m^{10} - 817376931984\,m^{11} + 112943120672\,m^{12} +\\
&143507222880\,m^{13} - 35587630272\,m^{14} - 4470000384\,m^{15} + 1290567168\,m^{16} - 122164992\,m^{17} + 27147776\,m^{18}\big) -\\
&4\,k^{3}\,\big(- 3456649728000 - 514312999200\,m + 2204369839800\,m^{2} -\\
&4807556573508\,m^{3} + 5327882415426\,m^{4} - 1217926648777\,m^{5} + 611239487018\,m^{6} + 731779284194\,m^{7} -\\
&1317602907422\,m^{8} + 1210916107547\,m^{9} - 356052576230\,m^{10} - 170317436144\,m^{11} + 104864253216\,m^{12} -\\
&39104723168\,m^{13} + 3716881856\,m^{14} + 5781899008\,m^{15} - 1216952320\,m^{16} - 161309952\,m^{17} + 35846656\,m^{18}\big)\big)\\
&h_1(k,m) +\\
&\big(3\,k^{12}\,m\,\big(1587600 - 396900\,m - 9600516\,m^{2} + 2400129\,m^{3} + 15291640\,m^{4} - 3822910\,m^{5} - 9901892\,m^{6} + 2475473\,m^{7} + 3065920\,m^{8} -\\
&766480\,m^{9} - 477568\,m^{10} + 119392\,m^{11} + 35840\,m^{12} - 8960\,m^{13} - 1024\,m^{14} + 256\,m^{15}\big) + 16\,k^{11}\,m\,\big(9298800 -\\
&2373300\,m - 56120508\,m^{2} + 14320053\,m^{3} + 88902632\,m^{4} - 22670102\,m^{5} - 56980924\,m^{6} + 14512069\,m^{7} + 17347616\,m^{8} -\\
&4408976\,m^{9} - 2632448\,m^{10} + 666848\,m^{11} + 189952\,m^{12} - 47872\,m^{13} - 5120\,m^{14} + 1280\,m^{15}\big) + k^{8}\,m\,\big(112658046480 -\\
&38248923492\,m - 657375259788\,m^{2} + 221041616337\,m^{3} + 947395904466\,m^{4} - 309272300810\,m^{5} - 511731431960\,m^{6} +\\
&156502928049\,m^{7} + 124196892194\,m^{8} - 33524588852\,m^{9} - 16567450656\,m^{10} + 3801023136\,m^{11} +\\
&1509120704\,m^{12} - 318284928\,m^{13} - 87927296\,m^{14} + 18998528\,m^{15} + 2105856\,m^{16} - 467968\,m^{17}\big) +\\
&k^{10}\,m\,\big(2298618000 - 614960100\,m - 13807357380\,m^{2} + 3689473041\,m^{3} + 21590092360\,m^{4} -\\
&5749520858\,m^{5} - 13509468168\,m^{6} + 3573993029\,m^{7} + 3961539956\,m^{8} - 1036213112\,m^{9} - 571704064\,m^{10} +\\
&146781664\,m^{11} + 39408768\,m^{12} - 9823232\,m^{13} - 1138688\,m^{14} + 271616\,m^{15} + 9216\,m^{16} - 2048\,m^{17}\big) -\\
&161280\,m\,\big(- 114307200 + 58038120\,m + 611712810\,m^{2} - 273874293\,m^{3} -\\
&726792144\,m^{4} + 208615451\,m^{5} + 455486907\,m^{6} - 180067848\,m^{7} - 75354261\,m^{8} + 52361370\,m^{9} -\\
&20820864\,m^{10} + 1325120\,m^{11} + 5004384\,m^{12} - 1172928\,m^{13} - 208128\,m^{14} + 55296\,m^{15} - 2304\,m^{16} + 512\,m^{17}\big) -\\
&2\,k^{9}\,m\,\big(- 9720693360 + 2809394604\,m + 57900776256\,m^{2} - 16674948099\,m^{3} - 88450486441\,m^{4} +
\end{aligned}
$$

$$25215442730\, m^5 + 53050223884\, m^6 - 14818109177\, m^7 - 14634006627\, m^8 + 3941167590\, m^9 + 1989789904\, m^{10} -$$
$$504813824\, m^{11} - 141143968\, m^{12} + 33088064\, m^{13} + 5657856\, m^{14} - 1248000\, m^{15} - 117504\, m^{16} + 26112\, m^{17}\big) -$$
$$4\, k^7\, m\, \big(-88059409200 + 38930639580\, m + 491233767048\, m^2 - 212787310095\, m^3 - 617861726991\, m^4 + 248733027842\, m^5 +$$
$$260708170492\, m^6 - 84630969465\, m^7 - 55391069845\, m^8 + 11406686122\, m^9 + 10728787632\, m^{10} - 1945169600\, m^{11} -$$
$$1435699552\, m^{12} + 310941120\, m^{13} + 78224128\, m^{14} - 18077440\, m^{15} - 1043712\, m^{16} + 231936\, m^{17}\big) + 2\, k^5\, m$$
$$\big(-1104540086160 + 2368385341524\, m - 839078167680\, m^2 + 1824805873395\, m^3 - 2232408668287\, m^4 + 1034858416830\, m^5 +$$
$$816395906508\, m^6 - 520378284095\, m^7 + 192215811187\, m^8 - 154534652806\, m^9 + 65513136304\, m^{10} + 9426359040\, m^{11} -$$
$$27297795168\, m^{12} + 5575718336\, m^{13} + 1766144768\, m^{14} - 425565440\, m^{15} - 11273472\, m^{16} + 2505216\, m^{17}\big) + k^6\, m$$
$$\big(1243670909040 - 969617575836\, m - 5785269760908\, m^2 + 4373091666351\, m^3 + 3304856724876\, m^4 - 1951339535894\, m^5 -$$
$$318391671752\, m^6 - 106868555325\, m^7 + 300734632088\, m^8 - 14035597664\, m^9 - 111344100672\, m^{10} + 29344869792\, m^{11} +$$
$$7482062336\, m^{12} - 2565970176\, m^{13} + 315538432\, m^{14} - 45069568\, m^{15} - 23869440\, m^{16} + 5304320\, m^{17}\big) +$$
$$96\, k\, \big(216040608000 + 869039539200\, m - 1087590732480\, m^2 -$$
$$2418078313188\, m^3 + 1539487432692\, m^4 + 1884537711087\, m^5 - 535448999410\, m^6 - 1130841411798\, m^7 +$$
$$299084482116\, m^8 + 365887511547\, m^9 - 163852976838\, m^{10} + 8955256080\, m^{11} + 13400495072\, m^{12} -$$
$$14045710560\, m^{13} + 2483845056\, m^{14} + 1199398656\, m^{15} - 269882880\, m^{16} - 21153024\, m^{17} + 4700672\, m^{18}\big) -$$
$$8\, k^3\, \big(-5184974592000 - 2403550875600\, m + 6152222887380\, m^2 -$$
$$5245110485820\, m^3 + 4580902412271\, m^4 + 1522533681862\, m^5 - 774665962480\, m^6 - 406017445694\, m^7 +$$
$$129727687013\, m^8 + 272289151012\, m^9 - 434627587288\, m^{10} + 296358559328\, m^{11} - 2694564704\, m^{12} -$$
$$84013646080\, m^{13} + 17337753856\, m^{14} + 5279553536\, m^{15} - 1253185280\, m^{16} - 47084544\, m^{17} + 10463232\, m^{18}\big) +$$
$$24\, k^2\, \big(2160406080000 + 3873942858240\, m - 6565700748816\, m^2 -$$
$$4627997415108\, m^3 + 4148970362172\, m^4 + 1025549084561\, m^5 - 353289753762\, m^6 - 242092487350\, m^7 -$$
$$649660492616\, m^8 + 902938360409\, m^9 - 112275583586\, m^{10} - 270012198416\, m^{11} + 86315364256\, m^{12} +$$
$$5881232224\, m^{13} - 5138615488\, m^{14} + 2618631424\, m^{15} - 469683712\, m^{16} - 130569984\, m^{17} + 29015552\, m^{18}\big) -$$
$$2\, k^4\, \big(-5184974592000 + 5974887875040\, m - 7142860128456\, m^2 -$$
$$18682595644068\, m^3 + 15905159497242\, m^4 + 6970974994643\, m^5 - 5742408764850\, m^6 + 2175775869022\, m^7 -$$
$$1773097725386\, m^8 + 723200459427\, m^9 + 93850174794\, m^{10} - 422927021360\, m^{11} + 146569511456\, m^{12} +$$
$$551681312\, m^{13} - 8153955904\, m^{14} + 6082957056\, m^{15} - 1128102912\, m^{16} - 277155072\, m^{17} + 61590016\, m^{18}\big)\big)$$
$$h_1(k+1, m) +$$
$$\big(241920\, m\, \big(3 + m - 2\, m^2\big)^2\, \big(-2116800 + 2002560\, m + 8828684\, m^2 - 8331868\, m^3 - 1300561\, m^4 + 1223848\, m^5 -$$
$$595985\, m^6 + 257830\, m^7 + 56476\, m^8 - 28328\, m^9 + 5968\, m^{10} - 1504\, m^{11} - 448\, m^{12} + 128\, m^{13}\big) + k^{12}\, m$$
$$\big(-1587600 + 396900\, m + 9600516\, m^2 - 2400129\, m^3 - 15291640\, m^4 + 3822910\, m^5 + 9901892\, m^6 - 2475473\, m^7 - 3065920\, m^8 +$$
$$766480\, m^9 + 477568\, m^{10} - 119392\, m^{11} - 35840\, m^{12} + 8960\, m^{13} + 1024\, m^{14} - 256\, m^{15}\big) - 8\, k^{11}\, m\, \big(5670000 -$$
$$1449900\, m - 34213500\, m^2 + 8746659\, m^3 + 54171208\, m^4 - 13839098\, m^5 - 34686652\, m^6 + 8849555\, m^7 + 10543104\, m^8 -$$
$$2683824\, m^9 - 1595776\, m^{10} + 404768\, m^{11} + 114688\, m^{12} - 28928\, m^{13} - 3072\, m^{14} + 768\, m^{15}\big) + k^6\, m\, \big(-401043318480 +$$
$$302624000532\, m + 1889676202860\, m^2 - 1360657162413\, m^3 - 1205544127214\, m^4 + 603488327302\, m^5 + 290464486452\, m^6 -$$
$$18343029313\, m^7 - 153695348866\, m^8 + 30548625652\, m^9 + 33577740896\, m^{10} - 10819642400\, m^{11} -$$
$$558269376\, m^{12} + 459743872\, m^{13} - 229918208\, m^{14} + 44638976\, m^{15} + 9063936\, m^{16} - 2014208\, m^{17}\big) +$$
$$k^{10}\, m\, \big(-681534000 + 182155500\, m + 4094188740\, m^2 - 1092683955\, m^3 - 6403790878\, m^4 +$$
$$1702155002\, m^5 + 4010634572\, m^6 - 1057641575\, m^7 - 1179440482\, m^8 + 306782740\, m^9 + 171746656\, m^{10} -$$
$$43667040\, m^{11} - 12189632\, m^{12} + 2989184\, m^{13} + 389632\, m^{14} - 90880\, m^{15} - 4608\, m^{16} + 1024\, m^{17}\big) + 2\, k^9\, m$$
$$\big(-2768184720 + 796075668\, m + 16496782128\, m^2 - 4723668861\, m^3 - 25241806275\, m^4 + 7138291426\, m^5 + 15208363688\, m^6 -$$
$$4194550275\, m^7 - 4250346389\, m^8 + 1121175626\, m^9 + 599583600\, m^{10} - 147411712\, m^{11} - 46478432\, m^{12} +$$
$$10542528\, m^{13} + 2134784\, m^{14} - 465152\, m^{15} - 48384\, m^{16} + 10752\, m^{17}\big) + 4\, k^7\, m\, \big(-24066428400 +$$
$$10586853900\, m + 134313101208\, m^2 - 57595091931\, m^3 - 169624893399\, m^4 + 66331219034\, m^5 + 73981598236\, m^6 -$$
$$22193571141\, m^7 - 17850677437\, m^8 + 3501809722\, m^9 + 3667781424\, m^{10} - 728052800\, m^{11} - 439564384\, m^{12} +$$
$$101459904\, m^{13} + 19186432\, m^{14} - 4649728\, m^{15} - 103680\, m^{16} + 23040\, m^{17}\big) + k^8\, m\, \big(-32700976560 +$$
$$10980933804\, m + 191069868924\, m^2 - 63406985571\, m^3 - 276651262980\, m^4 + 88544772114\, m^5 + 151619185172\, m^6 -$$
$$44832232419\, m^7 - 38455808924\, m^8 + 9843807272\, m^9 + 5649144384\, m^{10} - 1242079904\, m^{11} - 562067072\, m^{12} +$$
$$118752768\, m^{13} + 32645120\, m^{14} - 7129856\, m^{15} - 728064\, m^{16} + 161792\, m^{17}\big) + 2\, k^5\, m\, \big(442236276720 -$$
$$854807925708\, m - 36545711184\, m^2 - 357328567989\, m^3 + 854364607789\, m^4 - 281174896198\, m^5 - 438434087840\, m^6 +$$
$$92041377005\, m^7 + 130834343107\, m^8 + 21837633050\, m^9 - 72590416528\, m^{10} + 10514973952\, m^{11} + 12216415904\, m^{12} -$$
$$2897367104\, m^{13} - 480187648\, m^{14} + 127244032\, m^{15} - 5080320\, m^{16} + 1128960\, m^{17}\big) + 576\, k\, \big(-12002256000 -$$
$$51266779200\, m + 53153762760\, m^2 + 178911501372\, m^3 - 89605370010\, m^4 - 193655867483\, m^5 + 61526788044\, m^6 +$$
$$110306988064\, m^7 - 40733532700\, m^8 - 20935646561\, m^9 + 11548937138\, m^{10} - 2849372944\, m^{11} - 56545120\, m^{12} +$$
$$892568096\, m^{13} - 202922048\, m^{14} - 41972992\, m^{15} + 10862080\, m^{16} - 260352\, m^{17} + 57856\, m^{18}\big) - 24\, k^3\, \big(576108288000 +$$
$$330493942800\, m - 835620004980\, m^2 + 639973822848\, m^3 - 356475413279\, m^4 - 627902921835\, m^5 + 240637266730\, m^6 +$$
$$314935359060\, m^7 - 172818007401\, m^8 - 7691761545\, m^9 + 56815752530\, m^{10} - 52104954384\, m^{11} + 6636391104\, m^{12} +$$
$$9616538400\, m^{13} - 2329439040\, m^{14} - 342100224\, m^{15} + 95006976\, m^{16} - 6693120\, m^{17} + 1487360\, m^{18}\big) -$$
$$32\, k^2\, \big(540101520000 + 1099567496160\, m - 1758066850944\, m^2 - 1705005060660\, m^3 +$$
$$1425604055346\, m^4 + 580216777614\, m^5 - 112475768053\, m^6 - 364153546323\, m^7 - 78034540539\, m^8 +$$
$$329883106761\, m^9 - 95734758354\, m^{10} - 45318743424\, m^{11} + 24231589280\, m^{12} - 7239796704\, m^{13} +$$
$$403036608\, m^{14} + 1285016832\, m^{15} - 262048512\, m^{16} - 41792256\, m^{17} + 9287168\, m^{18}\big) + 4\, k^4\, \big(-864162432000 +$$
$$887639815440\, m - 1061208482436\, m^2 - 2832604381260\, m^3 + 2695946299905\, m^4 + 430718247250\, m^5 - 632676984636\, m^6 +$$
$$605024718354\, m^7 - 335825962177\, m^8 + 48209769656\, m^9 - 22653218528\, m^{10} - 6780706528\, m^{11} + 18881129184\, m^{12} -$$
$$13619340672\, m^{13} + 1464828928\, m^{14} + 1875702784\, m^{15} - 387142912\, m^{16} - 57729024\, m^{17} + 12828672\, m^{18}\big)\big)$$
$$h_1(k+2, m)\big) \Big/$$
$$\big(80640\, (1+k)^2\, (-4+m)\, (-3+m)\, (-2+m)\, (-1+m)\, m\, (1+m)\, (2+m)\, (3+m)\, (4+m)\, (-9+2\, m)\, (-7+2\, m)\, (-5+2\, m)$$
$$(-3+2\, m)\, (-1+2\, m)\, (1+2\, m)\, (3+2\, m)\, (5+2\, m)\, (7+2\, m)\big)$$

Appendix B. The certificates needed in Remark 10

$$g_2(k,m) := \frac{1}{2\,(1+k)\,m\,(1+2\,m)}\big(-2\,(3\,k^2+k^3-3\,m\,(1+2\,m)-k\,(-2+m+2\,m^2))\,\mathrm{h2}(k,m)+(9\,k^2+3\,k^3-2\,m\,(1+2\,m)+$$
$$k\,(6-4\,m-8\,m^2))\,h_2(k+1,m)-k\,(2+3\,k+k^2-2\,m-4\,m^2)\,h_2(k+2,m)\big)$$

$$g_3(k,m) := \big(-2\,(2\,k^5\,m\,(1+2\,m)+k^4\,(3+8\,m+16\,m^2)+12\,m\,(-1-m+4\,m^2+4\,m^3)+$$
$$k^3\,(12+4\,m+6\,m^2-8\,m^3-8\,m^4)+k\,(6-26\,m-50\,m^2+8\,m^3+8\,m^4)-k^2\,(-15+16\,m+40\,m^2+32\,m^3+32\,m^4))\,\mathrm{h2}(k,m)+$$
$$(6\,k^5\,m\,(1+2\,m)+k^4\,(9+20\,m+40\,m^2)+16\,m\,(-1-m+4\,m^2+4\,m^3)-2\,k^3\,(-18+m+6\,m^2+16\,m^3+16\,m^4)-$$
$$k^2\,(-45+68\,m+144\,m^2+32\,m^3+32\,m^4)+2\,k\,(9-34\,m-58\,m^2+40\,m^3+40\,m^4))\,h_2(k+1,m)-$$
$$(2\,k^5\,m\,(1+2\,m)+k^2\,(15-26\,m-52\,m^2)+3\,k^4\,(1+2\,m+4\,m^2)+$$
$$4\,m\,(-1-m+4\,m^2+4\,m^3)+6\,k\,(1-4\,m-6\,m^2+8\,m^3+8\,m^4)-2\,k^3\,(-6+m+4\,m^2+8\,m^3+8\,m^4))\,h_2(k+2,m)\big)/$$
$$(4\,(1+k)^2\,m\,(1+m)\,(-1+2\,m)\,(1+2\,m))$$

$$g_4(k,m) := \big(-2\,(k^6\,m\,(-1-m+4\,m^2+4\,m^3)+k^5\,m\,(-1+3\,m+20\,m^2+20\,m^3)+4\,m\,(-9-10\,m+33\,m^2+38\,m^3+12\,m^4+8\,m^5)+$$
$$k^3\,(36+14\,m+11\,m^2-73\,m^3-98\,m^4-60\,m^5-40\,m^6)+k^4\,(9+11\,m+25\,m^2+11\,m^3+6\,m^4-12\,m^5-8\,m^6)+$$
$$k^2\,(45-28\,m-92\,m^2-143\,m^3-138\,m^4+12\,m^5+8\,m^6)+k\,(18-67\,m-134\,m^2+13\,m^3+78\,m^4+156\,m^5+104\,m^6))\,\mathrm{h2}(k,m)+$$
$$(48\,m\,(-1-m+4\,m^2+4\,m^3)+3\,k^6\,m\,(-1-m+4\,m^2+4\,m^3)+$$
$$k^5\,m\,(1+13\,m+44\,m^2+44\,m^3)+k^4\,(27+38\,m+73\,m^2-16\,m^3-36\,m^4-48\,m^5-32\,m^6)-$$
$$k^3\,(-108-9\,m+41\,m^2+242\,m^3+272\,m^4+72\,m^5+48\,m^6)+2\,k\,(27-92\,m-158\,m^2+113\,m^3+158\,m^4+108\,m^5+72\,m^6)+$$
$$k^2\,(135-161\,m-378\,m^2-204\,m^3-104\,m^4+240\,m^5+160\,m^6))\,h_2(k+1,m)+$$
$$(-k^5\,m\,(1+5\,m+12\,m^2+12\,m^3)+k^6\,(m+m^2-4\,m^3-4\,m^4)-$$
$$6\,k\,(3-11\,m-16\,m^2+24\,m^3+24\,m^4)+k^3\,(-36+m+25\,m^2+92\,m^3+92\,m^4)+4\,m\,(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5)+$$
$$k^2\,(-45+65\,m+142\,m^2+38\,m^3-12\,m^4-120\,m^5-80\,m^6)+k^4\,(-9-12\,m-23\,m^2+6\,m^3+16\,m^4+24\,m^5+16\,m^6))\,h_2(k+2,m)\big)/$$
$$(4\,(1+k)^2\,(-1+m)\,m\,(1+m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m))$$

$$g_5(k,m) := \big(-2\,(2\,k^7\,m\,(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5)-144\,m\,(9+7\,m-42\,m^2-32\,m^3+24\,m^4+16\,m^5)+$$
$$k^6\,m\,(33+19\,m-174\,m^2-104\,m^3+168\,m^4+112\,m^5)+k^5\,m\,(231+379\,m-310\,m^2-202\,m^3+248\,m^4+128\,m^5-64\,m^6-32\,m^7)+$$
$$k^4\,(324+609\,m+1261\,m^2+134\,m^3-70\,m^4-568\,m^5-640\,m^6-448\,m^7-224\,m^8)+$$
$$24\,k\,(27-117\,m-203\,m^2+116\,m^3+80\,m^4-64\,m^5+32\,m^6+128\,m^7+64\,m^8)-$$
$$k^3\,(-1296+21\,m+47\,m^2+104\,m^3+534\,m^4+1088\,m^5+912\,m^6+320\,m^7+160\,m^8)+$$
$$2\,k^2\,(810-969\,m-2188\,m^2-1006\,m^3-999\,m^4+224\,m^5+840\,m^6+1184\,m^7+592\,m^8))\,\mathrm{h2}(k,m)+$$
$$(6\,k^7\,m\,(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5)+$$
$$3\,k^6\,m\,(21+11\,m-114\,m^2-64\,m^3+120\,m^4+80\,m^5)+k^5\,m\,(585+1071\,m-370\,m^2-248\,m^3+248\,m^4+16\,m^5-256\,m^6-128\,m^7)-$$
$$72\,m\,(24+21\,m-106\,m^2-95\,m^3+32\,m^4+40\,m^5+32\,m^6+16\,m^7)-$$
$$8\,k\,(-243+954\,m+1491\,m^2-1648\,m^3-1547\,m^4+248\,m^5+184\,m^6+32\,m^7+16\,m^8)-$$
$$3\,k^4\,(-324-549\,m-1091\,m^2+74\,m^3+312\,m^4+616\,m^5+560\,m^6+256\,m^7+128\,m^8)+$$
$$6\,k^2\,(810-1365\,m-2627\,m^2+404\,m^3+390\,m^4+112\,m^5+560\,m^6+832\,m^7+416\,m^8)+$$
$$k^3\,(3888-1179\,m-2691\,m^2-1576\,m^3-2744\,m^4-2512\,m^5-704\,m^6+1664\,m^7+832\,m^8))\,h_2(k+1,m)+$$
$$(-2\,k^7\,m\,(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5)-k^6\,m\,(15+7\,m-84\,m^2-44\,m^3+96\,m^4+64\,m^5)+$$
$$24\,m\,(18+15\,m-82\,m^2-71\,m^3+32\,m^4+40\,m^5+32\,m^6+16\,m^7)+k^5\,m\,(-195-355\,m+128\,m^2+72\,m^3-112\,m^4+128\,m^6+64\,m^7)+$$
$$k^3\,(-1296+525\,m+1295\,m^2+1030\,m^3+1252\,m^4+376\,m^5-272\,m^6-896\,m^7-448\,m^8)+$$
$$36\,k\,(-18+75\,m+102\,m^2-188\,m^3-167\,m^4+56\,m^5+56\,m^6+32\,m^7+16\,m^8)+$$
$$k^4\,(-324-501\,m-997\,m^2+80\,m^3+384\,m^4+752\,m^5+576\,m^6+128\,m^7+64\,m^8)-$$
$$4\,k^2\,(405-777\,m-1421\,m^2+530\,m^3+530\,m^4+56\,m^5+224\,m^6+320\,m^7+160\,m^8))\,h_2(k+2,m)\big)/$$
$$(24\,(1+k)^2\,(-1+m)\,m\,(1+m)\,(2+m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m))$$

$$g_6(k,m) := \big(-2\,(-2\,k^5\,m^2\,(1+2\,m)^2\,(192+19\,m+43\,m^2+20\,m^3+20\,m^4)+10\,k^7\,m\,(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5)+$$
$$k^8\,m\,(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5)+k^6\,m\,(192+167\,m-842\,m^2-713\,m^3+304\,m^4+184\,m^5-32\,m^6-16\,m^7)-$$
$$288\,m\,(-15-14\,m+62\,m^2+53\,m^3-16\,m^4+8\,m^5+32\,m^6+16\,m^7)-$$
$$k^4\,(1080+1521\,m+2790\,m^2-1003\,m^3-956\,m^4+236\,m^5+568\,m^6+704\,m^7+352\,m^8)-$$
$$6\,k\,(360-1434\,m-2575\,m^2+1282\,m^3+1855\,m^4+1504\,m^5+1432\,m^6+736\,m^7+368\,m^8)+$$
$$2\,k^3\,(-2160-501\,m-103\,m^2+3808\,m^3+4906\,m^4+2848\,m^5+2608\,m^6+1216\,m^7+608\,m^8)+$$
$$k^2\,(-5400+4638\,m+12029\,m^2+10774\,m^3+9775\,m^4-1328\,m^5+2680\,m^6+6112\,m^7+3056\,m^8))\,\mathrm{h2}(k,m)+$$
$$(22\,k^7\,m\,(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5)+$$
$$3\,k^8\,m\,(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5)-2\,k^6\,m\,(-219-203\,m+922\,m^2+834\,m^3-200\,m^4-96\,m^5+64\,m^6+32\,m^7)-$$
$$48\,m\,(-120-101\,m+530\,m^2+407\,m^3-256\,m^4-40\,m^5+224\,m^6+112\,m^7)-$$
$$2\,k^5\,m\,(192+776\,m+1621\,m^2+1897\,m^3+724\,m^4+688\,m^5+352\,m^6+176\,m^7)+$$
$$2\,k^2\,(-8100+11346\,m+23215\,m^2+1664\,m^3-469\,m^4-5080\,m^5-3256\,m^6+224\,m^7+112\,m^8)+$$
$$k^4\,(-3240-4563\,m-7674\,m^2+5881\,m^3+6260\,m^4+988\,m^5+920\,m^6+448\,m^7+224\,m^8)+$$
$$2\,k^3\,(-6480+579\,m+3932\,m^2+11298\,m^3+12425\,m^4+3360\,m^5+4424\,m^6+3744\,m^7+1872\,m^8)-$$

$$
\begin{aligned}
&\quad 4\,k\,(1620 - 5874\,m - 9055\,m^2 + 10570\,m^3 + 9703\,m^4 - 1280\,m^5 + 1816\,m^6 + 4576\,m^7 + 2288\,m^8))\ h_2(k+1,m) + \\
&(-6\,k^7\,m\,(3 + 2\,m - 15\,m^2 - 10\,m^3 + 12\,m^4 + 8\,m^5) - \\
&\quad k^8\,m\,(3 + 2\,m - 15\,m^2 - 10\,m^3 + 12\,m^4 + 8\,m^5) + 48\,m\,(-30 - 17\,m + 158\,m^2 + 89\,m^3 - 160\,m^4 - 88\,m^5 + 32\,m^6 + 16\,m^7) + \\
&\quad 2\,k^6\,m\,(-69 - 64\,m + 290\,m^2 + 261\,m^3 - 64\,m^4 - 24\,m^5 + 32\,m^6 + 16\,m^7) + \\
&\quad 2\,k^5\,m\,(96 + 292\,m + 417\,m^2 + 505\,m^3 + 228\,m^4 + 208\,m^5 + 96\,m^6 + 48\,m^7) + \\
&\quad 12\,k^2\,(450 - 741\,m - 1427\,m^2 + 251\,m^3 + 413\,m^4 + 428\,m^5 + 416\,m^6 + 224\,m^7 + 112\,m^8) - \\
&\quad k^4\,(-1080 - 1401\,m - 2374\,m^2 + 1807\,m^3 + 2296\,m^4 + 1252\,m^5 + 1096\,m^6 + 448\,m^7 + 224\,m^8) + \\
&\quad 8\,k\,(270 - 1047\,m - 1364\,m^2 + 2774\,m^3 + 2075\,m^4 - 1504\,m^5 - 424\,m^6 + 992\,m^7 + 496\,m^8) - \\
&\quad 2\,k^3\,(-2160 + 435\,m + 1958\,m^2 + 4298\,m^3 + 4071\,m^4 - 304\,m^5 + 600\,m^6 + 1376\,m^7 + 688\,m^8))\ h_2(k+2,m))/ \\
&(48\,(1+k)^2\,(-2+m)\,(-1+m)\,m\,(1+m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\,(5+2\,m))
\end{aligned}
$$

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\begin{aligned}
g_7(k,m) := (&-2\,(2\,k^9\,m\,(180 + 72\,m - 1025\,m^2 - 410\,m^3 + 1365\,m^4 + 546\,m^5 - 600\,m^6 - 240\,m^7 + 80\,m^8 + 32\,m^9) + \\
&\quad k^8\,m\,(5490 + 2241\,m - 31150\,m^2 - 12705\,m^3 + 41020\,m^4 + 16688\,m^5 - 17600\,m^6 - 7120\,m^7 + 2240\,m^8 + 896\,m^9) - \\
&\quad 2\,k^7\,m\,(-22050 - 10515\,m + 121197\,m^2 + \\
&\qquad 57164\,m^3 - 144335\,m^4 - 65525\,m^5 + 50836\,m^6 + 20972\,m^7 - 5840\,m^8 - 2160\,m^9 + 192\,m^{10} + 64\,m^{11}) - \\
&\quad 960\,m\,(4050 + 2610\,m - 20505\,m^2 - 13223\,m^3 + 17493\,m^4 + 11579\,m^5 - 684\,m^6 - 1668\,m^7 - 1680\,m^8 - 496\,m^9 + 192\,m^{10} + 64\,m^{11}) - \\
&\quad 4\,k^6\,m\,(-28980 - 15477\,m + 155139\,m^2 + \\
&\qquad 82558\,m^3 - 166545\,m^4 - 86131\,m^5 + 40332\,m^6 + 14484\,m^7 - 6960\,m^8 - 1552\,m^9 + 1344\,m^{10} + 448\,m^{11}) - \\
&\quad 2\,k^5\,m\,(-288270 - 501843\,m + 298671\,m^2 + 301047\,m^3 + \\
&\qquad 21275\,m^4 + 60096\,m^5 + 55088\,m^6 - 29864\,m^7 - 71440\,m^8 - 18368\,m^9 + 11136\,m^{10} + 3712\,m^{11}) - \\
&\quad 4\,k^2\,(-1215000 + 1571130\,m + 3426987\,m^2 + 1003736\,m^3 + 295207\,m^4 - \\
&\qquad 1880310\,m^5 - 1834214\,m^6 - 919272\,m^7 - 264744\,m^8 + 273760\,m^9 + 137312\,m^{10} + 30336\,m^{11} + 10112\,m^{12}) + \\
&\quad k^4\,(972000 + 1497510\,m + 3038199\,m^2 + 6810\,m^3 - 843115\,m^4 - \\
&\qquad 2152120\,m^5 - 1797428\,m^6 - 574000\,m^7 - 162560\,m^8 + 181760\,m^9 + 104384\,m^{10} + 34560\,m^{11} + 11520\,m^{12}) - \\
&\quad 8\,k\,(-243000 + 1074600\,m + 1740870\,m^2 - 1571949\,m^3 - 1269413\,m^4 + \\
&\qquad 697425\,m^5 + 342995\,m^6 - 312012\,m^7 - 401124\,m^8 - 288720\,m^9 - 39280\,m^{10} + 83136\,m^{11} + 27712\,m^{12}) + \\
&\quad 2\,k^3\,(1944000 - 288900\,m - 752550\,m^2 - 708823\,m^3 - 739111\,m^4 + \\
&\qquad 24555\,m^5 + 292005\,m^6 + 282156\,m^7 - 318108\,m^8 - 560240\,m^9 - 120080\,m^{10} + 113472\,m^{11} + 37824\,m^{12})) \\
&\ h2(k,m) + \\
&(6\,k^9\,m\,(180 + 72\,m - 1025\,m^2 - 410\,m^3 + 1365\,m^4 + 546\,m^5 - 600\,m^6 - 240\,m^7 + 80\,m^8 + 32\,m^9) + k^8\,m\,(12870 + 5283\,m - 72950\,m^2 - \\
&\qquad 29915\,m^3 + 95760\,m^4 + 39144\,m^5 - 40800\,m^6 - 16560\,m^7 + 5120\,m^8 + 2048\,m^9) + 2\,k^7\,m\,(49050 + 24345\,m - \\
&\qquad 267098\,m^2 - 130861\,m^3 + 308240\,m^4 + 144020\,m^5 - 101584\,m^6 - 41408\,m^7 + 12160\,m^8 + 4160\,m^9 - 768\,m^{10} - 256\,m^{11}) + 480\,m \\
&\quad (-10800 - 7740\,m + 52599\,m^2 + 38137\,m^3 - 36111\,m^4 - 28525\,m^5 - 6732\,m^6 - 1092\,m^7 + 3120\,m^8 + 1424\,m^9 + 192\,m^{10} + 64\,m^{11}) - \\
&\quad 4\,k^6\,m\,(-50040 - 26631\,m + 268276\,m^2 + \\
&\qquad 143007\,m^3 - 287830\,m^4 - 150578\,m^5 + 65288\,m^6 + 19336\,m^7 - 16160\,m^8 - 3296\,m^9 + 3456\,m^{10} + 1152\,m^{11}) - \\
&\quad 2\,k^5\,m\,(-774810 - 1452609\,m + 428789\,m^2 + 635873\,m^3 + \\
&\qquad 514425\,m^4 + 392748\,m^5 + 52512\,m^6 - 54696\,m^7 - 104240\,m^8 - 34304\,m^9 + 8064\,m^{10} + 2688\,m^{11}) - \\
&\quad 48\,k\,(-121500 + 490050\,m + 707805\,m^2 - 1058171\,m^3 - 896532\,m^4 + \\
&\qquad 423225\,m^5 + 390055\,m^6 + 146532\,m^7 - 20676\,m^8 - 118480\,m^9 - 33840\,m^{10} + 14784\,m^{11} + 4928\,m^{12}) + \\
&\quad 4\,k^3\,(2916000 - 1270800\,m - 2899125\,m^2 - 1372531\,m^3 - 1047282\,m^4 + \\
&\qquad 983105\,m^5 + 1288925\,m^6 + 921452\,m^7 + 55684\,m^8 - 520080\,m^9 - 168080\,m^{10} + 43584\,m^{11} + 14528\,m^{12}) + \\
&\quad k^4\,(2916000 + 4226130\,m + 7816437\,m^2 - 2925370\,m^3 - 4814165\,m^4 - \\
&\qquad 4420240\,m^5 - 2618024\,m^6 + 380000\,m^7 - 242000\,m^8 - 491520\,m^9 - 27648\,m^{10} + 184320\,m^{11} + 61440\,m^{12}) - \\
&\quad 4\,k^2\,(-3645000 + 6552990\,m + 12031461\,m^2 - 4504816\,m^3 - 5512867\,m^4 - \\
&\qquad 2288850\,m^5 - 1168662\,m^6 + 309912\,m^7 - 564456\,m^8 - 844640\,m^9 - 108704\,m^{10} + 249984\,m^{11} + 83328\,m^{12})) \\
&\ h_2(k+1,m) - \\
&(2\,k^9\,m\,(180 + 72\,m - 1025\,m^2 - 410\,m^3 + 1365\,m^4 + 546\,m^5 - 600\,m^6 - 240\,m^7 + 80\,m^8 + 32\,m^9) + k^8\,m\,(3690 + 1521\,m - 20900\,m^2 - \\
&\qquad 8605\,m^3 + 27370\,m^4 + 11228\,m^5 - 11600\,m^6 - 4720\,m^7 + 1440\,m^8 + 576\,m^9) + 2\,k^5\,m\,(264870 + 482583\,m - \\
&\qquad 191515\,m^2 - 209565\,m^3 - 63095\,m^4 - 105166\,m^5 - 96920\,m^6 - 20320\,m^7 + 37520\,m^8 + 15008\,m^9) + 960\,m\,(-1350 - 855\,m + \\
&\qquad 6849\,m^2 + 4240\,m^3 - 6015\,m^4 - 3253\,m^5 + 1044\,m^6 - 84\,m^7 - 720\,m^8 - 112\,m^9 + 192\,m^{10} + 64\,m^{11}) - 2\,k^7\,m\,(-14850 - 7365\,m + \\
&\qquad 80854\,m^2 + 39483\,m^3 - 93420\,m^4 - 43070\,m^5 + 31512\,m^6 + 12264\,m^7 - 4480\,m^8 - 1440\,m^9 + 384\,m^{10} + 128\,m^{11}) - \\
&\quad 4\,k^6\,m\,(-11880 - 6027\,m + 64472\,m^2 + \\
&\qquad 32859\,m^3 - 72110\,m^4 - 36046\,m^5 + 18616\,m^6 + 4472\,m^7 - 5920\,m^8 - 1312\,m^9 + 1152\,m^{10} + 384\,m^{11}) + \\
&\quad 40\,k\,(48600 - 207900\,m - 257706\,m^2 + 602583\,m^3 + \\
&\qquad 452603\,m^4 - 365379\,m^5 - 260129\,m^6 - 22140\,m^7 + 4620\,m^8 + 21360\,m^9 + 9424\,m^{10} + 960\,m^{11} + 320\,m^{12}) - \\
&\quad 4\,k^3\,(-972000 + 523350\,m + 1248150\,m^2 + 742213\,m^3 + 416581\,m^4 - \\
&\qquad 823705\,m^5 - 684225\,m^6 - 209116\,m^7 - 47012\,m^8 + 81040\,m^9 + 41040\,m^{10} + 9408\,m^{11} + 3136\,m^{12}) - \\
&\quad 12\,k^2\,(-405000 + 824010\,m + 1431979\,m^2 - 881209\,m^3 - 955718\,m^4 - \\
&\qquad 121605\,m^5 + 93087\,m^6 + 233508\,m^7 - 40404\,m^8 - 192560\,m^9 - 43056\,m^{10} + 37056\,m^{11} + 12352\,m^{12}) + \\
&\quad k^4\,(972000 + 1283310\,m + 2362119\,m^2 - 991880\,m^3 - 1827455\,m^4 - \\
&\qquad 1822110\,m^5 - 639648\,m^6 + 845280\,m^7 + 83040\,m^8 - 415840\,m^9 - 92416\,m^{10} + 80640\,m^{11} + 26880\,m^{12})) \\
&\ h_2(k+2,m))/ \\
(480\,&(1+k)^2\,(-2+m)\,(-1+m)\,m\,(1+m)\,(2+m)\,(3+m)\,(-5+2\,m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\,(5+2\,m))
\end{aligned}
$$

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\begin{aligned}
g_8(k,m) := \Big(&-2\,\big(k^{10}\,m\,(-2700-900\,m+15807\,m^2+5269\,m^3-22935\,m^4-7645\,m^5+12276\,m^6+4092\,m^7-2640\,m^8-880\,m^9+192\,m^{10}+64\,m^{11})+\\
&k^9\,m\,(-57780-19692\,m+337233\,m^2+114871\,m^3-\\
&\quad 484905\,m^4-164911\,m^5+254844\,m^6+86388\,m^7-53040\,m^8-17872\,m^9+3648\,m^{10}+1216\,m^{11})+\\
&k^5\,m\,(6032880+23045202\,m+45259965\,m^2+51887997\,m^3+15305633\,m^4+7385209\,m^5-8023910\,m^6-\\
&\quad 11721784\,m^7-9439416\,m^8-2098944\,m^9+1819360\,m^{10}+589952\,m^{11}-15232\,m^{12}-4352\,m^{13})-\\
&k^8\,m\,(605070+225423\,m-3484494\,m^2-1292931\,m^3+4815763\,m^4+1764782\,m^5-\\
&\quad 2342407\,m^6-835538\,m^7+434244\,m^8+147192\,m^9-28624\,m^{10}-9056\,m^{11}+448\,m^{12}+128\,m^{13})+\\
&2880\,m\,(-85050-54000\,m+422775\,m^2+226854\,m^3-403550\,m^4-\\
&\quad 56872\,m^5+323873\,m^6+63838\,m^7-126156\,m^8-41064\,m^9+10544\,m^{10}+4000\,m^{11}+448\,m^{12}+128\,m^{13})-\\
&k^7\,m\,(2835000+1211490\,m-15935949\,m^2-6738150\,m^3+20454622\,m^4+8348042\,m^5-\\
&\quad 8622517\,m^6-3223310\,m^7+1388316\,m^8+432936\,m^9-127984\,m^{10}-33440\,m^{11}+8512\,m^{12}+2432\,m^{13})-\\
&k^6\,m\,(10534860+7160184\,m-51898347\,m^2-33606621\,m^3+41665687\,m^4+20601929\,m^5-\\
&\quad 11234536\,m^6-3069668\,m^7+3037056\,m^8+509520\,m^9-733312\,m^{10}-184256\,m^{11}+55552\,m^{12}+15872\,m^{13})-\\
&432\,k\,(-283500+1221750\,m+1862535\,m^2-2281381\,m^3-2101781\,m^4+259369\,m^5-328406\,m^6-\\
&\quad 581798\,m^7+375932\,m^8+805112\,m^9+156112\,m^{10}-175776\,m^{11}-49856\,m^{12}+8064\,m^{13}+2304\,m^{14})+\\
&k^3\,(244944000-17720100\,m-130295700\,m^2-386039019\,m^3-418990570\,m^4-77121898\,m^5-37682848\,m^6+51734773\,m^7+\\
&\quad 93771830\,m^8+82525956\,m^9+17055736\,m^{10}-17221904\,m^{11}-5435360\,m^{12}+281792\,m^{13}+80512\,m^{14})+\\
&k^4\,(61236000+75573270\,m+107776503\,m^2-168572874\,m^3-142484821\,m^4+71219273\,m^5+75657692\,m^6+46082863\,m^7+\\
&\quad 16761962\,m^8-10577796\,m^9-8482328\,m^{10}-4232624\,m^{11}-735776\,m^{12}+623168\,m^{13}+178048\,m^{14})-\\
&6\,k^2\,(-51030000+60310440\,m+129386466\,m^2+28728909\,m^3-901280\,m^4-59825492\,m^5-3004244\,m^6+60104627\,m^7+\\
&\quad 21629050\,m^8-13146756\,m^9-8928760\,m^{10}-3592816\,m^{11}-512800\,m^{12}+632128\,m^{13}+180608\,m^{14})\big)\\
&h2(k,m)+\\
&\big(3\,k^{10}\,m\,(-2700-900\,m+15807\,m^2+5269\,m^3-22935\,m^4-7645\,m^5+12276\,m^6+4092\,m^7-2640\,m^8-880\,m^9+192\,m^{10}+64\,m^{11})+\\
&9\,k^9\,m\,(-15660-5364\,m+91335\,m^2+\\
&\quad 31265\,m^3-131055\,m^4-44777\,m^5+68580\,m^6+23340\,m^7-14160\,m^8-4784\,m^9+960\,m^{10}+320\,m^{11})-\\
&k^8\,m\,(1353510+515349\,m-7767153\,m^2-2942643\,m^3+10621327\,m^4+3961574\,m^5-\\
&\quad 5060164\,m^6-1821704\,m^7+914496\,m^8+306240\,m^9-63808\,m^{10}-19328\,m^{11}+1792\,m^{12}+512\,m^{13})+\\
&1440\,m\,(-226800-127980\,m+1178055\,m^2+583878\,m^3-1271810\,m^4-\\
&\quad 347644\,m^5+801911\,m^6+260386\,m^7-191892\,m^8-84888\,m^9-6832\,m^{10}+1120\,m^{11}+3136\,m^{12}+896\,m^{13})-\\
&6\,k^7\,m\,(930150+410265\,m-5195127\,m^2-2261046\,m^3+6538432\,m^4+2717667\,m^5-\\
&\quad 2669171\,m^6-981838\,m^7+444436\,m^8+127016\,m^9-52752\,m^{10}-13216\,m^{11}+4032\,m^{12}+1152\,m^{13})-\\
&3\,k^6\,m\,(7964460+5913864\,m-37832307\,m^2-26897589\,m^3+25126175\,m^4+13204977\,m^5-\\
&\quad 5382856\,m^6-1025732\,m^7+1994240\,m^8+351184\,m^9-469632\,m^{10}-127424\,m^{11}+26880\,m^{12}+7680\,m^{13})+\\
&9\,k^5\,m\,(3585960+9305334\,m+8950481\,m^2+10956671\,m^3+4368411\,m^4+1363743\,m^5-\\
&\quad 2898132\,m^6-2066292\,m^7-846992\,m^8-378608\,m^9-22464\,m^{10}+27456\,m^{11}+32256\,m^{12}+9216\,m^{13})+\\
&48\,k\,(7654500-30176550\,m-39256245\,m^2+79013457\,m^3+53454327\,m^4-54189803\,m^5-13406458\,m^6+34505876\,m^7+\\
&\quad 6319096\,m^8-14035584\,m^9-4514304\,m^{10}+1236032\,m^{11}+472192\,m^{12}+55552\,m^{13}+15872\,m^{14})+\\
&3\,k^4\,(61236000+72530370\,m+92772063\,m^2-203199981\,m^3-173332591\,m^4+72403839\,m^5+51830688\,m^6+10547932\,m^7+\\
&\quad 15596672\,m^8+11349152\,m^9-203680\,m^{10}-4956736\,m^{11}-1289216\,m^{12}+335104\,m^{13}+95744\,m^{14})-6\,k^3\\
&\ (-122472000+47808900\,m+134524530\,m^2+151585827\,m^3+132081006\,m^4-43098422\,m^5-17623682\,m^6+13933611\,m^7-\\
&\quad 5959902\,m^8-17675236\,m^9-7824536\,m^{10}-505328\,m^{11}+361056\,m^{12}+488768\,m^{13}+139648\,m^{14})-4\,k^2\\
&\ (-229635000+393630030\,m+691905537\,m^2-379143432\,m^3-369473559\,m^4+14899475\,m^5-10513724\,m^6-6928661\,m^7+\\
&\quad 62967458\,m^8+76638540\,m^9+7210632\,m^{10}-24482672\,m^{11}-6338464\,m^{12}+1682240\,m^{13}+480640\,m^{14})\big)\\
&h_2(k+1,m)-\\
&\big(k^{10}\,m\,(-2700-900\,m+15807\,m^2+5269\,m^3-22935\,m^4-7645\,m^5+12276\,m^6+4092\,m^7-2640\,m^8-880\,m^9+192\,m^{10}+64\,m^{11})+\\
&k^9\,m\,(-41580-14292\,m+242391\,m^2+\\
&\quad 83257\,m^3-347295\,m^4-119041\,m^5+181188\,m^6+61836\,m^7-37200\,m^8-12592\,m^9+2496\,m^{10}+832\,m^{11})+\\
&2880\,m\,(-28350-9045\,m+167688\,m^2+56703\,m^3-245807\,m^4-\\
&\quad 95512\,m^5+116369\,m^6+55414\,m^7-5916\,m^8-6696\,m^9-4432\,m^{10}-992\,m^{11}+448\,m^{12}+128\,m^{13})-\\
&k^8\,m\,(402570+153063\,m-2310453\,m^2-873177\,m^3+3162041\,m^4+\\
&\quad 1172470\,m^5-1514534\,m^6-537556\,m^7+282168\,m^8+91536\,m^9-22688\,m^{10}-6592\,m^{11}+896\,m^{12}+256\,m^{13})-\\
&2\,k^7\,m\,(757350+332775\,m-4231512\,m^2-1827834\,m^3+5342140\,m^4+2174675\,m^5-\\
&\quad 2237266\,m^6-778112\,m^7+424680\,m^8+112320\,m^9-59872\,m^{10}-15104\,m^{11}+4480\,m^{12}+1280\,m^{13})-\\
&k^6\,m\,(7586460+5549904\,m-36211239\,m^2-25089345\,m^3+25152707\,m^4+11876869\,m^5-\\
&\quad 7437992\,m^6-1426900\,m^7+2818176\,m^8+656400\,m^9-500864\,m^{10}-145600\,m^{11}+19712\,m^{12}+5632\,m^{13})+\\
&k^5\,m\,(13744080+30529602\,m+13926465\,m^2+22297959\,m^3+17708731\,m^4+4073867\,m^5-12454060\,m^6-\\
&\quad 4415828\,m^7+1833168\,m^8-390000\,m^9-1122880\,m^{10}-203456\,m^{11}+157696\,m^{12}+45056\,m^{13})+\\
&24\,k\,(5103000-21392100\,m-23125230\,m^2+71672661\,m^3+38307336\,m^4-72077684\,m^5-22139884\,m^6+41410823\,m^7+\\
&\quad 13171378\,m^8-10068852\,m^9-4046232\,m^{10}+51536\,m^{11}+137056\,m^{12}+110656\,m^{13}+31616\,m^{14})-\\
&12\,k^2\,(-25515000+49955670\,m+82807623\,m^2-66962142\,m^3-60379133\,m^4+12211229\,m^5-2495052\,m^6-13041881\,m^7+\\
&\quad 5984186\,m^8+15211932\,m^9+3191720\,m^{10}-3079472\,m^{11}-898848\,m^{12}+117824\,m^{13}+33664\,m^{14})+\\
&k^4\,(61236000+65083770\,m+83576943\,m^2-185166723\,m^3-178657387\,m^4+18810991\,m^5+24435720\,m^6+25312466\,m^7+\\
&\quad 24308884\,m^8+13525848\,m^9+1425136\,m^{10}-4253728\,m^{11}-1261632\,m^{12}+144256\,m^{13}+41216\,m^{14})-\\
&4\,k^3\,(-61236000+30700350\,m+83766060\,m^2+82981056\,m^3+51700237\,m^4-68862175\,m^5-25613295\,m^6+33066313\,m^7+\\
&\quad 11865626\,m^8-7302540\,m^9-4982680\,m^{10}-2003024\,m^{11}-257568\,m^{12}+378560\,m^{13}+108160\,m^{14})\big)\\
&h_2(k+2,m)\Big)\Big/\\
\big(1440\,&(1+k)^2\,(-3+m)\,(-2+m)\,(-1+m)\,m\,(1+m)\,(2+m)\,(3+m)\,(-5+2\,m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\\
&(5+2\,m)\,(7+2\,m)\big)
\end{aligned}
$$

$$
\begin{aligned}
&g_9(k,m) := (-2\,(2\,k^{11}\,m\,(-18900 - 11700\,m + 96249\,m^2 + \\
&\quad 56497\,m^3 - 89841\,m^4 - 41665\,m^5 + 29988\,m^6 + 11868\,m^7 - 4080\,m^8 - 1456\,m^9 + 192\,m^{10} + 64\,m^{11}) + \\
&\quad k^{10}\,m\,(-1077300 - 673200\,m + 5469393\,m^2 + 3243212\,m^3 - \\
&\quad 5038449\,m^4 - 2361080\,m^5 + 1644048\,m^6 + 656904\,m^7 - 215760\,m^8 - 77504\,m^9 + 9600\,m^{10} + 3200\,m^{11}) + \\
&\quad 181440\,m\,(176400 + 159600\,m - 756490\,m^2 - 674843\,m^3 + 196933\,m^4 + \\
&\quad 137133\,m^5 + 20045\,m^6 + 33826\,m^7 + 27844\,m^8 + 3464\,m^9 - 8080\,m^{10} - 2208\,m^{11} + 448\,m^{12} + 128\,m^{13}) - \\
&\quad k^9\,m\,(14556780 + 9351792\,m - 73215663\,m^2 - 44715468\,m^3 + 64753085\,m^4 + 31196092\,m^5 - \\
&\quad 19786718\,m^6 - 8064412\,m^7 + 2390328\,m^8 + 859824\,m^9 - 104480\,m^{10} - 33856\,m^{11} + 896\,m^{12} + 256\,m^{13}) - \\
&\quad 2\,k^8\,m\,(49809060 + 33611724\,m - 246104421\,m^2 - 158300597\,m^3 + 200718040\,m^4 + 100919767\,m^5 - \\
&\quad 54485683\,m^6 - 22542926\,m^7 + 6103380\,m^8 + 2117864\,m^9 - 345616\,m^{10} - 103072\,m^{11} + 11200\,m^{12} + 3200\,m^{13}) - \\
&\quad 7\,k^6\,m\,(119338380 + 110040912\,m - 503865255\,m^2 - 452143120\,m^3 + 115091327\,m^4 + 48386372\,m^5 - 41482376\,m^6 - \\
&\quad 3389272\,m^7 + 21882768\,m^8 + 6264256\,m^9 - 2670848\,m^{10} - 816512\,m^{11} + 68096\,m^{12} + 19456\,m^{13}) - \\
&\quad 2\,k^7\,m\,(226497600 + 173548620\,m - 1060263528\,m^2 - 777501095\,m^3 + 651751710\,m^4 + 346636967\,m^5 - 143787042\,m^6 - \\
&\quad 55163856\,m^7 + 21182856\,m^8 + 6253184\,m^9 - 2353632\,m^{10} - 670976\,m^{11} + 104832\,m^{12} + 29952\,m^{13}) + \\
&\quad k^5\,m\,(-4125790620 - 9084595248\,m - 3226307565\,m^2 - 2640652672\,m^3 + 1719540003\,m^4 + 2164447336\,m^5 + 1631269962\,m^6 + \\
&\quad 530028948\,m^7 - 382377480\,m^8 - 155585360\,m^9 - 1589664\,m^{10} + 1610432\,m^{11} + 1975680\,m^{12} + 564480\,m^{13}) + \\
&\quad 864\,k\,(-18522000 + 76734000\,m + 141953070\,m^2 - 43835777\,m^3 - 31475444\,m^4 + 36458663\,m^5 + 46031644\,m^6 + 33570880\,m^7 + \\
&\quad 7638200\,m^8 - 12078928\,m^9 - 4582144\,m^{10} + 309376\,m^{11} + 165248\,m^{12} + 57344\,m^{13} + 16384\,m^{14}) - 96\,k^2 \\
&\quad (416745000 - 423084060\,m - 1057960674\,m^2 - 854405667\,m^3 - 891391718\,m^4 - 92082328\,m^5 - 65241249\,m^6 + 21685090\,m^7 + \\
&\quad 79288328\,m^8 + 84204984\,m^9 + 19703264\,m^{10} - 14909984\,m^{11} - 4404096\,m^{12} + 522368\,m^{13} + 149248\,m^{14}) + \\
&\quad 2\,k^4\,(-4000752000 - 6885186840\,m - 15915276036\,m^2 - 8782225686\,m^3 - \\
&\quad 9771692147\,m^4 - 2234585635\,m^5 - 975802383\,m^6 + 1064879977\,m^7 + 977030714\,m^8 + 546038580\,m^9 + \\
&\quad 125618024\,m^{10} - 99449936\,m^{11} - 30177312\,m^{12} + 2744000\,m^{13} + 784000\,m^{14}) - 4\,k^3\,(8001504000 + \\
&\quad 947286900\,m + 1171099800\,m^2 - 2814339039\,m^3 - 2311045009\,m^4 + 1790099602\,m^5 + 3081940119\,m^6 + 3028083485\,m^7 + \\
&\quad 986138242\,m^8 - 717030348\,m^9 - 311584280\,m^{10} - 23089168\,m^{11} - 1138080\,m^{12} + 6053824\,m^{13} + 1729664\,m^{14})) \\
&\quad h2(k,m) + \\
&\quad (6\,k^{11}\,m\,(-18900 - 11700\,m + 96249\,m^2 + \\
&\quad 56497\,m^3 - 89841\,m^4 - 41665\,m^5 + 29988\,m^6 + 11868\,m^7 - 4080\,m^8 - 1456\,m^9 + 192\,m^{10} + 64\,m^{11}) + \\
&\quad k^{10}\,m\,(-2702700 - 1692000\,m + 13713207\,m^2 + 8147720\,m^3 - \\
&\quad 12599799\,m^4 - 5916620\,m^5 + 4092480\,m^6 + 1638408\,m^7 - 533040\,m^8 - 191744\,m^9 + 23424\,m^{10} + 7808\,m^{11}) + \\
&\quad 201600\,m\,(211680 + 203238\,m - 870321\,m^2 - 822115\,m^3 + 101938\,m^4 + \\
&\quad 27837\,m^5 - 46243\,m^6 + 31450\,m^7 + 67284\,m^8 + 16072\,m^9 - 11536\,m^{10} - 3360\,m^{11} + 448\,m^{12} + 128\,m^{13}) - \\
&\quad k^9\,m\,(33086340 + 21389976\,m - 166048149\,m^2 - 102087790\,m^3 + 145436141\,m^4 + 70468082\,m^5 - \\
&\quad 43792784\,m^6 - 17888704\,m^7 + 5231472\,m^8 + 1869280\,m^9 - 241280\,m^{10} - 76544\,m^{11} + 3584\,m^{12} + 1024\,m^{13}) + \\
&\quad 7\,k^6\,m\,(-205076340 - 197359056\,m + 841426953\,m^2 + 795151448\,m^3 - 97718365\,m^4 - 23329084\,m^5 + 61131748\,m^6 + \\
&\quad 4564592\,m^7 - 33475008\,m^8 - 10995488\,m^9 + 2632000\,m^{10} + 910336\,m^{11} + 30464\,m^{12} + 8704\,m^{13}) - 2\,k^8\,m \\
&\quad (102592980 + 70001712\,m - 504744525\,m^2 - 328345206\,m^3 + 403571965\,m^4 + 204087494\,m^5 - 107413852\,m^6 - 43972952\,m^7 + \\
&\quad 12565824\,m^8 + 4209408\,m^9 - 869056\,m^{10} - 252800\,m^{11} + 34048\,m^{12} + 9728\,m^{13}) - 2\,k^7\,m\,(478396800 + \\
&\quad 378286020\,m - 2205519672\,m^2 - 1671960829\,m^3 + 1230795030\,m^4 + 658415605\,m^5 - 263263266\,m^6 - 95810712\,m^7 + \\
&\quad 45296040\,m^8 + 13239520\,m^9 - 5191392\,m^{10} - 1512064\,m^{11} + 201600\,m^{12} + 57600\,m^{13}) + k^5\,m\,(-11651309460 - \\
&\quad 26455840344\,m - 12268772895\,m^2 - 10491426358\,m^3 + 4585047555\,m^4 + 4129502362\,m^5 + 1868188644\,m^6 + \\
&\quad 992327304\,m^7 + 32902752\,m^8 - 74908736\,m^9 - 92480448\,m^{10} - 24833920\,m^{11} + 5531904\,m^{12} + 1580544\,m^{13}) - \\
&\quad 4\,k^4\,(6001128000 + 9946359360\,m + 21247603884\,m^2 + \\
&\quad 5491676520\,m^3 + 5890208793\,m^4 + 1159111915\,m^5 + 1392385163\,m^6 + 840592991\,m^7 - 126776954\,m^8 - \\
&\quad 707344212\,m^9 - 238330344\,m^{10} + 49765328\,m^{11} + 18426400\,m^{12} + 1696576\,m^{13} + 484736\,m^{14}) + \\
&\quad 8\,k^2\,(-15002820000 + 22896223560\,m + 47280581244\,m^2 + 5670984414\,m^3 + \\
&\quad 4498759153\,m^4 - 1529148847\,m^5 + 3053404701\,m^6 + 6157266913\,m^7 + 1045919066\,m^8 - 2619504108\,m^9 - \\
&\quad 865737880\,m^{10} + 199888432\,m^{11} + 68751840\,m^{12} + 1959104\,m^{13} + 559744\,m^{14}) + 48\,k\,(-1000188000 + \\
&\quad 3760495200\,m + 6213977280\,m^2 - 5201196198\,m^3 - 5048261729\,m^4 + 473279927\,m^5 + 680014935\,m^6 + 664395319\,m^7 + \\
&\quad 422399174\,m^8 + 93562380\,m^9 - 15077608\,m^{10} - 54959024\,m^{11} - 14458848\,m^{12} + 3563840\,m^{13} + 1018240\,m^{14}) - \\
&\quad 4\,k^3\,(24004512000 - 4138287300\,m - 12749524380\,m^2 - \\
&\quad 17770245393\,m^3 - 17086592950\,m^4 + 2079619261\,m^5 + 2930960370\,m^6 + 2963823800\,m^7 + 2223702448\,m^8 + \\
&\quad 840053520\,m^9 + 42349216\,m^{10} - 309350656\,m^{11} - 84751872\,m^{12} + 16952320\,m^{13} + 4843520\,m^{14})) \\
&\quad h_2(k+1,m) + (-2\,k^{11}\,m\,(-18900 - \\
&\quad 11700\,m + 96249\,m^2 + 56497\,m^3 - 89841\,m^4 - 41665\,m^5 + 29988\,m^6 + 11868\,m^7 - 4080\,m^8 - 1456\,m^9 + 192\,m^{10} + 64\,m^{11}) - \\
&\quad 3\,k^{10}\,m\,(-270900 - 169800\,m + 1373969\,m^2 + 817418\,m^3 - \\
&\quad 1260225\,m^4 - 592590\,m^5 + 408072\,m^6 + 163584\,m^7 - 52880\,m^8 - 19040\,m^9 + 2304\,m^{10} + 768\,m^{11}) + \\
&\quad k^5\,m\,(4055407020 + 8856333648\,m + 3003539649\,m^2 + 3090872792\,m^3 + 96038503\,m^4 - 324583044\,m^5 - 705198424\,m^6 - \\
&\quad 444411416\,m^7 - 104735664\,m^8 - 6923840\,m^9 + 37154816\,m^{10} + 10727040\,m^{11} - 1530368\,m^{12} - 437248\,m^{13}) - \\
&\quad 20160\,m\,(529200 + 452340\,m - 2319636\,m^2 - 1828315\,m^3 + 1005763\,m^4 + \\
&\quad 116973\,m^5 - 851923\,m^6 - 187790\,m^7 + 304164\,m^8 + 94984\,m^9 - 28816\,m^{10} - 9120\,m^{11} + 448\,m^{12} + 128\,m^{13}) + \\
&\quad k^9\,m\,(9793980 + 6327792\,m - 49160715\,m^2 - 30195228\,m^3 + 43106071\,m^4 + 20825768\,m^5 - \\
&\quad 13064044\,m^6 - 5297792\,m^7 + 1606656\,m^8 + 565152\,m^9 - 83392\,m^{10} - 25856\,m^{11} + 1792\,m^{12} + 512\,m^{13}) + \\
&\quad 4\,k^8\,m\,(14386680 + 9792072\,m - 70835088\,m^2 - 45906553\,m^3 + 56933703\,m^4 + 28468921\,m^5 - \\
&\quad 15619317\,m^6 - 6225210\,m^7 + 2034300\,m^8 + 658840\,m^9 - 164592\,m^{10} - 47584\,m^{11} + 6720\,m^{12} + 1920\,m^{13}) + \\
&\quad 14\,k^7\,m\,(20412000 + 16035300\,m - 94350168\,m^2 - 70788573\,m^3 + 53998148\,m^4 + 27708865\,m^5 - \\
&\quad 13384352\,m^6 - 4666300\,m^7 + 2573040\,m^8 + 775248\,m^9 - 271232\,m^{10} - 80704\,m^{11} + 8960\,m^{12} + 2560\,m^{13}) - \\
&\quad 7\,k^6\,m\,(-48160980 - 46746072\,m + 196225569\,m^2 + 186594834\,m^3 - 18958949\,m^4 + 630914\,m^5 + 20434796\,m^6 + 4444120\,m^7 - \\
&\quad 7656960\,m^8 - 2969856\,m^9 + 130496\,m^{10} + 92032\,m^{11} + 44800\,m^{12} + 12800\,m^{13}) - 336\,k\,(-47628000 +
\end{aligned}
$$

$$190776600\,m + 273493980\,m^2 - 423168306\,m^3 - 379514261\,m^4 + 95734799\,m^5 + 36215019\,m^6 - 37464113\,m^7 + 5172902\,m^8 +$$
$$29536812\,m^9 + 7192472\,m^{10} - 4913456\,m^{11} - 1422816\,m^{12} + 198464\,m^{13} + 56704\,m^{14}\big) + 12\,k^3\,\big(2667168000 -$$
$$736199100\,m - 2255122680\,m^2 - 3070362415\,m^3 - 2780249927\,m^4 + 630189578\,m^5 + 225283925\,m^6 - 234052763\,m^7 +$$
$$125082218\,m^8 + 298684660\,m^9 + 71708168\,m^{10} - 50866064\,m^{11} - 14885408\,m^{12} + 1910720\,m^{13} + 545920\,m^{14}\big) +$$
$$2\,k^4\,\big(4000752000 + 6138882540\,m + 13077117216\,m^2 + 2992877121\,m^3 +$$
$$2044864416\,m^4 - 1864333301\,m^5 + 84135488\,m^6 + 2104600064\,m^7 + 627641320\,m^8 - 562629840\,m^9 -$$
$$217183680\,m^{10} + 10377344\,m^{11} + 6448768\,m^{12} + 2759680\,m^{13} + 788480\,m^{14}\big) - 8\,k^2\,\big(-5000940000 +$$
$$8823612420\,m + 17242362828\,m^2 - 1844965095\,m^3 - 2853747344\,m^4 - 1766306629\,m^5 + 941916744\,m^6 + 3381158284\,m^7 +$$
$$1061356976\,m^8 - 834760800\,m^9 - 324468256\,m^{10} + 12920512\,m^{11} + 8684544\,m^{12} + 4040960\,m^{13} + 1154560\,m^{14}\big)\big)$$
$$h_2(k+2,m)\big)\Big/$$
$$\big(20160\,(1+k)^2\,(-3+m)\,(-2+m)\,m\,(1+m)\,(2+m)\,(3+m)\,(4+m)\,(-7+2\,m)\,(-5+2\,m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)$$
$$(5+2\,m)\,(7+2\,m)\big)$$

$$g_{10}(k,m) := \big(-2\,\big(32\,k^{11}\,m\,\big(-1814400 - 461700\,m + 10953504\,m^2 + 2786697\,m^3 - 17365712\,m^4 - 4415502\,m^5 + 11147136\,m^6 + 2831257\,m^7 -$$
$$3402256\,m^8 - 862576\,m^9 + 518336\,m^{10} + 131040\,m^{11} - 37632\,m^{12} - 9472\,m^{13} + 1024\,m^{14} + 256\,m^{15}\big) +$$
$$k^{12}\,m\,\big(-1587600 - 396900\,m + 9600516\,m^2 + 2400129\,m^3 - 15291640\,m^4 - 3822910\,m^5 + 9901892\,m^6 + 2475473\,m^7 -$$
$$3065920\,m^8 - 766480\,m^9 + 477568\,m^{10} + 119392\,m^{11} - 35840\,m^{12} - 8960\,m^{13} + 1024\,m^{14} + 256\,m^{15}\big) -$$
$$k^{10}\,m\,\big(992703600 + 263349900\,m - 5968170576\,m^2 - 1581703059\,m^3 + 9354610805\,m^4 +$$
$$2472344232\,m^5 - 5879073326\,m^6 - 1545570835\,m^7 + 1735311369\,m^8 + 452125938\,m^9 - 252301520\,m^{10} -$$
$$64801088\,m^{11} + 17394272\,m^{12} + 4370368\,m^{13} - 476928\,m^{14} - 115968\,m^{15} + 2304\,m^{16} + 512\,m^{17}\big) +$$
$$241920\,m\,\big(-57153600 - 32281200\,m + 294688800\,m^2 + 137297502\,m^3 -$$
$$324649721\,m^4 - 46340124\,m^5 + 264094168\,m^6 + 63312892\,m^7 - 91480559\,m^8 - 35679470\,m^9 +$$
$$2455184\,m^{10} + 3179680\,m^{11} + 2026976\,m^{12} + 362432\,m^{13} - 170752\,m^{14} - 39424\,m^{15} + 2304\,m^{16} + 512\,m^{17}\big) -$$
$$2\,k^9\,m\,\big(4584399120 + 1306148868\,m - 27351546732\,m^2 - 7770795873\,m^3 + 41971575298\,m^4 +$$
$$11828793524\,m^5 - 25369616622\,m^6 - 7037035319\,m^7 + 7061880488\,m^8 + 1905164480\,m^9 - 959258080\,m^{10} -$$
$$247464928\,m^{11} + 64740480\,m^{12} + 15681536\,m^{13} - 2210816\,m^{14} - 500480\,m^{15} + 36864\,m^{16} + 8192\,m^{17}\big) -$$
$$k^8\,m\,\big(54285986160 + 17849254284\,m - 318185975088\,m^2 - 103772190339\,m^3 + 464461703229\,m^4 + 147820162076\,m^5 -$$
$$256570600762\,m^6 - 77522360799\,m^7 + 63601516717\,m^8 + 17462805946\,m^9 - 8232326352\,m^{10} - 1975344320\,m^{11} +$$
$$675853024\,m^{12} + 145416384\,m^{13} - 37090048\,m^{14} - 7950592\,m^{15} + 933120\,m^{16} + 207360\,m^{17}\big) -$$
$$4\,k^7\,m\,\big(45298990800 + 18950806980\,m - 255431488176\,m^2 - 104891331465\,m^3 + 332389359087\,m^4 + 128021127306\,m^5 -$$
$$149487158408\,m^6 - 48408757047\,m^7 + 32159038409\,m^8 + 7174027442\,m^9 - 5657756976\,m^{10} - 992293376\,m^{11} +$$
$$778858976\,m^{12} + 157418688\,m^{13} - 50896640\,m^{14} - 11232512\,m^{15} + 1052928\,m^{16} + 233984\,m^{17}\big) +$$
$$k^6\,m\,\big(-533201492880 - 384728957172\,m + 2565670008816\,m^2 + 1776166820109\,m^3 - 1810845092651\,m^4 - 948157686224\,m^5 +$$
$$364034204954\,m^6 - 4316501483\,m^7 - 205985694687\,m^8 - 192070878\,m^9 + 82806119984\,m^{10} + 16287251904\,m^{11} -$$
$$10156246688\,m^{12} - 2488865344\,m^{13} + 332656896\,m^{14} + 86065920\,m^{15} + 2048256\,m^{16} + 455168\,m^{17}\big) +$$
$$2\,k^5\,m\,\big(275677985520 + 671127368508\,m + 572407062300\,m^2 + 1034645953473\,m^3 + 1080332948282\,m^4 + 596209401228\,m^5 -$$
$$319848426674\,m^6 - 420811240273\,m^7 - 324813734580\,m^8 - 82176994968\,m^9 + 56227379488\,m^{10} + 25247347552\,m^{11} +$$
$$5165882624\,m^{12} + 292299520\,m^{13} - 920100864\,m^{14} - 196484352\,m^{15} + 23675904\,m^{16} + 5261312\,m^{17}\big) +$$
$$k^4\,\big(3456649728000 + 4210994511360\,m + 5654349571824\,m^2 - 10808435865924\,m^3 - 9376988512548\,m^4 +$$
$$4113282068729\,m^5 + 5004974554266\,m^6 + 3872338578626\,m^7 + 1828695439448\,m^8 - 419279808583\,m^9 -$$
$$697815557854\,m^{10} - 522827691920\,m^{11} - 82360945312\,m^{12} + 83674371424\,m^{13} + 22163495104\,m^{14} -$$
$$1528716032\,m^{15} - 549605888\,m^{16} - 118183680\,m^{17} - 26263040\,m^{18}\big) + 144\,k\,\big(48009024000 -$$
$$210928536000\,m - 309097736640\,m^2 + 435552994404\,m^3 + 366102616512\,m^4 - 124553980857\,m^5 + 42483316598\,m^6 +$$
$$156646319046\,m^7 - 63099062628\,m^8 - 173231852961\,m^9 - 38353158066\,m^{10} + 34403043216\,m^{11} + 12387575840\,m^{12} +$$
$$497769888\,m^{13} - 350922432\,m^{14} - 378076416\,m^{15} - 73972224\,m^{16} + 14503680\,m^{17} + 3223040\,m^{18}\big) +$$
$$4\,k^2\,\big(4320812160000 - 5466485465280\,m - 11224225600992\,m^2 - 636839647308\,m^3 +$$
$$1834214361408\,m^4 + 4972908681879\,m^5 + 89370376814\,m^6 - 5632907943810\,m^7 - 2929189446348\,m^8 +$$
$$287772852039\,m^9 + 953052255774\,m^{10} + 817376931984\,m^{11} + 112943120672\,m^{12} - 143507222880\,m^{13} -$$
$$35587630272\,m^{14} + 4470000384\,m^{15} + 1290567168\,m^{16} + 122164992\,m^{17} + 27147776\,m^{18}\big) -$$
$$4\,k^3\,\big(-3456649728000 + 514312999200\,m + 2204369839800\,m^2 +$$
$$4807556573508\,m^3 + 5327882415426\,m^4 + 1217926648777\,m^5 + 611239487018\,m^6 - 731779284194\,m^7 -$$
$$1317602907422\,m^8 - 1210916107547\,m^9 - 356052576230\,m^{10} + 170317436144\,m^{11} + 104864253216\,m^{12} +$$
$$39104723168\,m^{13} + 3716881856\,m^{14} - 5781899008\,m^{15} - 1216952320\,m^{16} + 161309952\,m^{17} + 35846656\,m^{18}\big)\big)$$
$$h2(k,m) +$$
$$\big(3\,k^{12}\,m\,\big(-1587600 - 396900\,m + 9600516\,m^2 + 2400129\,m^3 - 15291640\,m^4 - 3822910\,m^5 + 9901892\,m^6 + 2475473\,m^7 -$$
$$3065920\,m^8 - 766480\,m^9 + 477568\,m^{10} + 119392\,m^{11} - 35840\,m^{12} - 8960\,m^{13} + 1024\,m^{14} + 256\,m^{15}\big) +$$
$$16\,k^{11}\,m\,\big(-9298800 - 2373300\,m + 56120508\,m^2 + 14320053\,m^3 - 88902632\,m^4 - 22670102\,m^5 + 56980924\,m^6 + 14512069\,m^7 -$$
$$17347616\,m^8 - 4408976\,m^9 + 2632448\,m^{10} + 666848\,m^{11} - 189952\,m^{12} - 47872\,m^{13} + 5120\,m^{14} + 1280\,m^{15}\big) -$$
$$161280\,m\,\big(114307200 + 58038120\,m - 611712810\,m^2 - 273874293\,m^3 +$$
$$726792144\,m^4 + 208615451\,m^5 - 455486907\,m^6 - 180067848\,m^7 + 75354261\,m^8 + 52361370\,m^9 +$$
$$20820864\,m^{10} + 1325120\,m^{11} - 5004384\,m^{12} - 1172928\,m^{13} + 208128\,m^{14} + 55296\,m^{15} + 2304\,m^{16} + 512\,m^{17}\big) -$$
$$k^{10}\,m\,\big(2298618000 + 614960100\,m - 13807357380\,m^2 - 3689473041\,m^3 + 21590092360\,m^4 + 5749520858\,m^5 -$$
$$13509468168\,m^6 - 3573993029\,m^7 + 3961539956\,m^8 + 1036213112\,m^9 - 571704064\,m^{10} - 146781664\,m^{11} +$$
$$39408768\,m^{12} + 9823232\,m^{13} - 1138688\,m^{14} - 271616\,m^{15} + 9216\,m^{16} + 2048\,m^{17}\big) - 2\,k^9\,m\,\big(9720693360 +$$
$$2809394604\,m - 57900776256\,m^2 - 16674948099\,m^3 + 88450486441\,m^4 + 25215442730\,m^5 - 53050223884\,m^6 -$$
$$14818109177\,m^7 + 14634006627\,m^8 + 3941167590\,m^9 - 1989789904\,m^{10} - 504813824\,m^{11} + 141143968\,m^{12} +$$

$$
\begin{aligned}
&33088064\,m^{13} - 5657856\,m^{14} - 1248000\,m^{15} + 117504\,m^{16} + 26112\,m^{17}\big) - 4\,k^{7}\,m\,\big(88059409200 + \\
&38930639580\,m - 491233767048\,m^{2} - 212787310095\,m^{3} + 617861726991\,m^{4} + 248733027842\,m^{5} - 260708170492\,m^{6} - \\
&84630969465\,m^{7} + 55391069845\,m^{8} + 11406686122\,m^{9} - 10728787632\,m^{10} - 1945169600\,m^{11} + 1435699552\,m^{12} + \\
&310941120\,m^{13} - 78224128\,m^{14} - 18077440\,m^{15} + 1043712\,m^{16} + 231936\,m^{17}\big) - k^{8}\,m\,\big(112658046480 + \\
&38248923492\,m - 657375259788\,m^{2} - 221041616337\,m^{3} + 947395904466\,m^{4} + 309272300810\,m^{5} - 511731431960\,m^{6} - \\
&156502928049\,m^{7} + 124196892194\,m^{8} + 33524588852\,m^{9} - 16567450656\,m^{10} - 3801023136\,m^{11} + \\
&1509120704\,m^{12} + 318284928\,m^{13} - 87927296\,m^{14} - 18998528\,m^{15} + 2105856\,m^{16} + 467968\,m^{17}\big) + 2\,k^{5}\,m \\
&\big(1104540086160 + 2368385341524\,m + 839078167680\,m^{2} + 1824805873395\,m^{3} + 2232408668287\,m^{4} + 1034858416830\,m^{5} - \\
&816395906508\,m^{6} - 520378284095\,m^{7} - 192215811187\,m^{8} - 154534652806\,m^{9} - 65513136304\,m^{10} + 9426359040\,m^{11} + \\
&27297795168\,m^{12} + 5575718336\,m^{13} - 1766144768\,m^{14} - 425565440\,m^{15} + 11273472\,m^{16} + 2505216\,m^{17}\big) + k^{6}\,m \\
&\big(-1243670909040 - 969617575836\,m + 5785269760908\,m^{2} + 4373091666351\,m^{3} - 3304856724876\,m^{4} - 1951339535894\,m^{5} + \\
&318391671752\,m^{6} - 106868555325\,m^{7} - 300734632088\,m^{8} - 14035597664\,m^{9} + 111344100672\,m^{10} + 29344869792\,m^{11} - \\
&7482062336\,m^{12} - 2565970176\,m^{13} - 315538432\,m^{14} - 45069568\,m^{15} + 23869440\,m^{16} + 5304320\,m^{17}\big) + \\
&96\,k\,\big(216040608000 - 869039539200\,m - 1087590732480\,m^{2} + \\
&2418078313188\,m^{3} + 1539487432692\,m^{4} - 1884537711087\,m^{5} - 535448999410\,m^{6} + 1130841411798\,m^{7} + \\
&299084482116\,m^{8} - 365887511547\,m^{9} - 163852976838\,m^{10} - 8955256080\,m^{11} + 13400495072\,m^{12} + \\
&14045710560\,m^{13} + 2483845056\,m^{14} - 1199398656\,m^{15} - 269882880\,m^{16} + 21153024\,m^{17} + 4700672\,m^{18}\big) - \\
&8\,k^{3}\,\big(-5184974592000 + 2403550875600\,m + 6152222887380\,m^{2} + \\
&5245110485820\,m^{3} + 4580902412271\,m^{4} - 1522533681862\,m^{5} - 774665962480\,m^{6} + 406017445694\,m^{7} + \\
&129727687013\,m^{8} - 272289151012\,m^{9} - 434627587288\,m^{10} - 296358559328\,m^{11} - 2694564704\,m^{12} + \\
&84013646080\,m^{13} + 17337753856\,m^{14} - 5279553536\,m^{15} - 1253185280\,m^{16} + 47084544\,m^{17} + 10463232\,m^{18}\big) + \\
&24\,k^{2}\,\big(2160406080000 - 3873942858240\,m - 6565700748816\,m^{2} + \\
&4627997415108\,m^{3} + 4148970362172\,m^{4} - 1025549084561\,m^{5} - 353289753762\,m^{6} + 242092487350\,m^{7} - \\
&649660492616\,m^{8} - 902938360409\,m^{9} - 112275583586\,m^{10} + 270012198416\,m^{11} + 86315364256\,m^{12} - \\
&5881232224\,m^{13} - 5138615488\,m^{14} - 2618631424\,m^{15} - 469683712\,m^{16} + 130569984\,m^{17} + 29015552\,m^{18}\big) - \\
&2\,k^{4}\,\big(-5184974592000 - 5974887875040\,m - 7142860128456\,m^{2} + \\
&18682595644068\,m^{3} + 15905159497242\,m^{4} - 6970974994643\,m^{5} - 5742408764850\,m^{6} - 2175775869022\,m^{7} - \\
&1773097725386\,m^{8} - 723200459427\,m^{9} + 93850174794\,m^{10} + 422927021360\,m^{11} + 146569511456\,m^{12} - \\
&551681312\,m^{13} - 8153955904\,m^{14} - 6082957056\,m^{15} - 1128102912\,m^{16} + 277155072\,m^{17} + 61590016\,m^{18}\big)\big) \\
&h_2(k+1,m) + \\
&\big(241920\,m\,\big(-3 + m + 2\,m^{2}\big)^{2}\,\big(2116800 + 2002560\,m - 8828684\,m^{2} - 8331868\,m^{3} + 1300561\,m^{4} + 1223848\,m^{5} + \\
&595985\,m^{6} + 257830\,m^{7} - 56476\,m^{8} - 28328\,m^{9} - 5968\,m^{10} - 1504\,m^{11} + 448\,m^{12} + 128\,m^{13}\big) + k^{12}\,m \\
&\big(1587600 + 396900\,m - 9600516\,m^{2} - 2400129\,m^{3} + 15291640\,m^{4} + 3822910\,m^{5} - 9901892\,m^{6} - 2475473\,m^{7} + 3065920\,m^{8} + \\
&766480\,m^{9} - 477568\,m^{10} - 119392\,m^{11} + 35840\,m^{12} + 8960\,m^{13} - 1024\,m^{14} - 256\,m^{15}\big) - 8\,k^{11}\,m\,\big(-5670000 - \\
&1449900\,m + 34213500\,m^{2} + 8746659\,m^{3} - 54171208\,m^{4} - 13839098\,m^{5} + 34686652\,m^{6} + 8849555\,m^{7} - 10543104\,m^{8} - \\
&2683824\,m^{9} + 1595776\,m^{10} + 404768\,m^{11} - 114688\,m^{12} - 28928\,m^{13} + 3072\,m^{14} + 768\,m^{15}\big) + k^{6}\,m\,\big(401043318480 + \\
&302624000532\,m - 1889676202860\,m^{2} - 1360657162413\,m^{3} + 1205544127214\,m^{4} + 603488327302\,m^{5} - 290464486452\,m^{6} - \\
&18343029313\,m^{7} + 153695348866\,m^{8} + 30548625652\,m^{9} - 33577740896\,m^{10} - 10819642400\,m^{11} + \\
&558269376\,m^{12} + 459743872\,m^{13} + 229918208\,m^{14} + 44638976\,m^{15} - 9063936\,m^{16} - 2014208\,m^{17}\big) + \\
&k^{10}\,m\,\big(681534000 + 182155500\,m - 4094188740\,m^{2} - 1092683955\,m^{3} + 6403790878\,m^{4} + \\
&1702155002\,m^{5} - 4010634572\,m^{6} - 1057641575\,m^{7} + 1179440482\,m^{8} + 306782740\,m^{9} - 171746656\,m^{10} - \\
&43667040\,m^{11} + 12189632\,m^{12} + 2989184\,m^{13} - 389632\,m^{14} - 90880\,m^{15} + 4608\,m^{16} + 1024\,m^{17}\big) + 2\,k^{9}\,m \\
&\big(2768184720 + 796075668\,m - 16496782128\,m^{2} - 4723668861\,m^{3} + 25241806275\,m^{4} + 7138291426\,m^{5} - 15208363688\,m^{6} - \\
&4194550275\,m^{7} + 4250346389\,m^{8} + 1121175626\,m^{9} - 599583600\,m^{10} - 147411712\,m^{11} + 46478432\,m^{12} + \\
&10542528\,m^{13} - 2134784\,m^{14} - 465152\,m^{15} + 48384\,m^{16} + 10752\,m^{17}\big) + 4\,k^{7}\,m\,\big(24066428400 + \\
&10586853900\,m - 134313101208\,m^{2} - 57595091931\,m^{3} + 169624893399\,m^{4} + 66331219034\,m^{5} - 73981598236\,m^{6} - \\
&22193571141\,m^{7} + 17850677437\,m^{8} + 3501809722\,m^{9} - 3667781424\,m^{10} - 728052800\,m^{11} + 439564384\,m^{12} + \\
&101459904\,m^{13} - 19186432\,m^{14} - 4649728\,m^{15} + 103680\,m^{16} + 23040\,m^{17}\big) + k^{8}\,m\,\big(32700976560 + \\
&10980933804\,m - 191069868924\,m^{2} - 63406985571\,m^{3} + 276651262980\,m^{4} + 88544772114\,m^{5} - 151619185172\,m^{6} - \\
&44832232419\,m^{7} + 38455808924\,m^{8} + 9843807272\,m^{9} - 5649144384\,m^{10} - 1242079904\,m^{11} + 562067072\,m^{12} + \\
&118752768\,m^{13} - 32645120\,m^{14} - 7129856\,m^{15} + 728064\,m^{16} + 161792\,m^{17}\big) + 2\,k^{5}\,m\,\big(-442236276720 - \\
&854807925708\,m + 36545711184\,m^{2} - 357328567989\,m^{3} - 854364607789\,m^{4} - 281174896198\,m^{5} + 438434087840\,m^{6} + \\
&92041377005\,m^{7} - 130834343107\,m^{8} + 21837633050\,m^{9} + 72590416528\,m^{10} + 10514973952\,m^{11} - 12216415904\,m^{12} - \\
&2897367104\,m^{13} + 480187648\,m^{14} + 127244032\,m^{15} + 5080320\,m^{16} + 1128960\,m^{17}\big) + 576\,k\,\big(-12002256000 + \\
&51266779200\,m + 53153762760\,m^{2} - 178911501372\,m^{3} - 89605370010\,m^{4} + 193655867483\,m^{5} + 61526788044\,m^{6} - \\
&110306988064\,m^{7} - 40733532700\,m^{8} + 20935646561\,m^{9} + 11548937138\,m^{10} + 2849372944\,m^{11} - 56545120\,m^{12} - \\
&892568096\,m^{13} - 202922048\,m^{14} + 41972992\,m^{15} + 10862080\,m^{16} + 260352\,m^{17} + 57856\,m^{18}\big) - 24\,k^{3}\,\big(576108288000 - \\
&330493942800\,m - 835620004980\,m^{2} - 639973822848\,m^{3} - 356475413279\,m^{4} + 627902921835\,m^{5} + 240637266730\,m^{6} - \\
&314935359060\,m^{7} - 172818007401\,m^{8} + 7691761545\,m^{9} + 56815752530\,m^{10} + 52104954384\,m^{11} + 6636391104\,m^{12} - \\
&9616538400\,m^{13} - 2329439040\,m^{14} + 342100224\,m^{15} + 95006976\,m^{16} + 6693120\,m^{17} + 1487360\,m^{18}\big) - \\
&32\,k^{2}\,\big(540101520000 - 1099567496160\,m - 1758066850944\,m^{2} + 1705005060660\,m^{3} + \\
&1425604055346\,m^{4} - 580216777614\,m^{5} - 112475768053\,m^{6} + 364153546323\,m^{7} - 78034540539\,m^{8} - \\
&329883106761\,m^{9} - 95734758354\,m^{10} + 45318743424\,m^{11} + 24231589280\,m^{12} + 7239796704\,m^{13} + \\
&403036608\,m^{14} - 1285016832\,m^{15} - 262048512\,m^{16} + 41792256\,m^{17} + 9287168\,m^{18}\big) + 4\,k^{4}\,\big(-864162432000 - \\
&887639815440\,m - 1061208482436\,m^{2} + 2832604381260\,m^{3} + 2695946299905\,m^{4} - 430718247250\,m^{5} - 632676984636\,m^{6} - \\
&605024718354\,m^{7} - 335825962177\,m^{8} - 48209769656\,m^{9} - 22653218528\,m^{10} + 6780706528\,m^{11} + 18881129184\,m^{12} + \\
&13619340672\,m^{13} + 1464828928\,m^{14} - 1875702784\,m^{15} - 387142912\,m^{16} + 57729024\,m^{17} + 12828672\,m^{18}\big)\big) \\
&h_2(k+2,m)\big) \Big/ \\
&\big(80640\,(1+k)^{2}\,(-4+m)\,(-3+m)\,(-2+m)\,(-1+m)\,m\,(1+m)\,(2+m)\,(3+m)\,(4+m)\,(-7+2\,m)\,(-5+2\,m)\,(-3+2\,m) \\
&(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\,(5+2\,m)\,(7+2\,m)\,(9+2\,m)\big)
\end{aligned}
$$

$$\begin{aligned}
g_{11}(k,m) := \Big(&-2\big(2k^{13}m\,(57153600+12700800m-349190676m^2-77597928m^3+\\
&572100201m^4+127133378m^5-390874302m^6-86860956m^7+132652377m^8+29478306m^9-\\
&24090768m^{10}-5353504m^{11}+2364768m^{12}+525504m^{13}-117504m^{14}-26112m^{15}+2304m^{16}+512m^{17})+\\
&5k^{12}m\,(1043053200+235361700m-6364692612m^2-1435977297m^3+10392226056m^4+\\
&2343789758m^5-7056042084m^6-1590288609m^7+2370777552m^8+533737776m^9-424135296m^{10}-\\
&95326816m^{11}+40739328m^{12}+9133824m^{13}-1963008m^{14}-438528m^{15}+36864m^{16}+8192m^{17})+\\
&k^{10}m\,(1317237390000+323239077900m-7978547133900m^2-1954750190139m^3+\\
&12769556507640m^4+3114841217978m^5-8362137476044m^6-2022895242347m^7+2657754473136m^8+\\
&634137813312m^9-441091534272m^{10}-103014235744m^{11}+38991323648m^{12}+8824613632m^{13}-\\
&1804637184m^{14}-391545600m^{15}+41496576m^{16}+8572928m^{17}-409600m^{18}-81920m^{19})-\\
&2k^{11}m\,(-55300190400-12875922000m+336540626064m^2+78313889520m^3-545556174696m^4-126753273295m^5+\\
&365596027604m^6+84694143082m^7-120345051216m^8-27746582799m^9+20911867860m^{10}+4786233140m^{11}-\\
&1935269440m^{12}-438141152m^{13}+89876352m^{14}+20025984m^{15}-1717248m^{16}-373504m^{17}+5120m^{18}+1024m^{19})-\\
&2903040m\,(3857868000+2028952800m-20715424380m^2-10801168317m^3+22478775192m^4+10904421720m^5-\\
&5857755932m^6-410451709m^7+2364550032m^8-932848302m^9-1778142936m^{10}-151556088m^{11}+383543680m^{12}+\\
&75961664m^{13}-26510592m^{14}-6134784m^{15}+362496m^{16}+96768m^{17}+10240m^{18}+2048m^{19})-\\
&6k^9m\,(-1676174245200-452423978820m+10057309337124m^2+2703158051373m^3-\\
&15688023347988m^4-4166426692147m^5+9813033686952m^6+2545842269979m^7-2921799405252m^8-\\
&728116737165m^9+453141921564m^{10}+106102726364m^{11}-39533278272m^{12}-8548277664m^{13}+\\
&2114878080m^{14}+426342528m^{15}-70668288m^{16}-13928704m^{17}+1121280m^{18}+224256m^{19})-\\
&3k^8m\,(-16029908974800-5090017697460m+94367654130564m^2+29664378532053m^3-\\
&139603734037692m^4-42574926492986m^5+79534435155548m^6+22723566820477m^7-21120819142884m^8-\\
&5334687073812m^9+3159901020432m^{10}+669473951536m^{11}-331210992256m^{12}-62286624128m^{13}+\\
&24685518336m^{14}+4696815360m^{15}-1018426368m^{16}-201380864m^{17}+15749120m^{18}+3149824m^{19})-\\
&2k^7m\,(-83065584823200-36022119071400m+465011310397032m^2+197098230179502m^3-592313509001736m^4-\\
&231303517582909m^5+259496355649092m^6+79697398545540m^7-59954103622224m^8-\\
&11003055107865m^9+12700399567500m^{10}+1872815971436m^{11}-2030733349824m^{12}-371515849248m^{13}+\\
&160990655616m^{14}+32831387520m^{15}-5170646016m^{16}-1077507328m^{17}+45173760m^{18}+9034752m^{19})-\\
&2k^5m\,(-653934812554800-1289041997951340m+178879252371912m^2+700754763761487m^3+1270019392062849m^4+\\
&874938386486155m^5-70668233783990m^6-326560033803559m^7-367305781072911m^8-102792001658019m^9+\\
&53421002366604m^{10}+26106536208316m^{11}+6595371175840m^{12}+382605628448m^{13}-1162723898496m^{14}-\\
&242142319488m^{15}+34521814272m^{16}+7959552256m^{17}+181918720m^{18}+36383744m^{19})+\\
&k^6m\,(241807922773200+168148007595540m-1185997165177716m^2-810899464766445m^3+878786534825280m^4+\\
&528812181371534m^5-42778387557444m^6+103795560063027m^7+129197924200008m^8-21213159074664m^9-\\
&73091920680864m^{10}-13752052421248m^{11}+9577972932864m^{12}+2418311003136m^{13}-268136441856m^{14}-\\
&83622293760m^{15}-11166308352m^{16}-1878071296m^{17}+381050880m^{18}+76210176m^{19})+\\
&5184k\,(1080203040000-4872915936000m-7353486730800m^2+9125433377460m^3+6872987597292m^4-\\
&5559523820073m^5-4041925268757m^6+127510213874m^7+1802168024002m^8+2249562315663m^9+\\
&725704274355m^{10}-282290059788m^{11}-238894069404m^{12}-114282477472m^{13}-5561148704m^{14}+\\
&20575338624m^{15}+3780476544m^{16}-948442368m^{17}-195289344m^{18}+9774080m^{19}+1954816m^{20})-\\
&24k^3\,(-466647713280000+118875076966800m+308277178524900m^2+250983538312752m^3+90647601156099m^4-\\
&408598297723809m^5-336112708527003m^6-46978953970898m^7+131383807565729m^8+203395223044335m^9+\\
&72304838893887m^{10}-16936311253980m^{11}-16844993877708m^{12}-9127088562848m^{13}-905142351520m^{14}+\\
&1311052799616m^{15}+264050502528m^{16}-44860777728m^{17}-9843627264m^{18}+79221760m^{19}+15844352m^{20})+\\
&144k^2\,(97218273600000-134531234056800m-280746272383560m^2-30757843230516m^3+48233363281146m^4+\\
&185381893995705m^5+114326690191097m^6+1894881253382m^7+40291091190940m^8+36944508135849m^9-\\
&15152179950147m^{10}-29860046834676m^{11}-5310672958372m^{12}+4263030925472m^{13}+1176357898528m^{14}-\\
&50851029120m^{15}-34231632000m^{16}-14540894976m^{17}-2510311168m^{18}+455367680m^{19}+91073536m^{20})-\\
&4k^4\,(-699971569920000-912698304008400m-1750193661877380m^2+353055037231512m^3+634920517530009m^4+\\
&793624145870889m^5+927769647385007m^6+710816610624914m^7+399616947530731m^8-14897476920207m^9-\\
&148703848821291m^{10}-143092064493204m^{11}-26488740151972m^{12}+19895500333472m^{13}+6075847764256m^{14}+\\
&195914909568m^{15}-95347455360m^{16}-98254965504m^{17}-17809614592m^{18}+2541992960m^{19}+508398592m^{20})\big)\\
&h2(k,m)+\\
&\big(6k^{13}m\,(57153600+12700800m-349190676m^2-77597928m^3+572100201m^4+\\
&127133378m^5-390874302m^6-86860956m^7+132652377m^8+29478306m^9-24090768m^{10}-\\
&5353504m^{11}+2364768m^{12}+525504m^{13}-117504m^{14}-26112m^{15}+2304m^{16}+512m^{17})+3k^{12}m\\
&(4529422800+1024398900m-27633174948m^2-6248711349m^3+45095927868m^4+10193348254m^5-30589718796m^6-\\
&6909111573m^7+10262059236m^8+2314949208m^9-1831587264m^{10}-412392032m^{11}+175319424m^{12}+\\
&39363072m^{13}-8404992m^{14}-1879296m^{15}+156672m^{16}+34816m^{17})-2k^{11}m\,(-130408185600-\\
&30540531600m+793222965396m^2+185636961756m^3-1284114383037m^4-299948863422m^5+858423032150m^6+\\
&199802121544m^7-281531770221m^8-65144442246m^9+48690868512m^{10}+11164024560m^{11}-4487529952m^{12}-\\
&1014692672m^{13}+209171712m^{14}+46316544m^{15}-4189440m^{16}-898560m^{17}+20480m^{18}+4096m^{19})-\\
&1451520m\,(10287648000+5866408800m-53843585220m^2-29636753202m^3+54095930253m^4+23990602377m^5-\\
&15875214914m^6+3121545848m^7+11711636157m^8-1153784955m^9-5789033652m^{10}-973804836m^{11}+\\
&858152992m^{12}+209805344m^{13}-29817984m^{14}-9297024m^{15}-1220352m^{16}-182016m^{17}+56320m^{18}+11264m^{19})-
\end{aligned}$$

$$3\,k^{10}\,m\,\big(-953054650800-236126105100\,m+5767441395228\,m^2+1426100444091\,m^3-$$
$$9208284710076\,m^4-2264478048314\,m^5+6004955700644\,m^6+1461389130763\,m^7-1898258080884\,m^8-$$
$$453915264000\,m^9+313831298208\,m^{10}+72990279808\,m^{11}-27950464384\,m^{12}-6240505088\,m^{13}+$$
$$1354767360\,m^{14}+287198976\,m^{15}-35705856\,m^{16}-7221248\,m^{17}+450560\,m^{18}+90112\,m^{19}\big)+$$
$$6\,k^7\,m\,\big(58319161941600+26923065913800\,m-322374518496396\,m^2-145512881733690\,m^3+393796174379805\,m^4+$$
$$163396666778710\,m^5-157820767043422\,m^6-49919399824574\,m^7+34643464241565\,m^8+$$
$$6149363318970\,m^9-7618558072848\,m^{10}-1252266358688\,m^{11}+1124489888480\,m^{12}+231002860480\,m^{13}-$$
$$70760642304\,m^{14}-15854548992\,m^{15}+1306348800\,m^{16}+302103040\,m^{17}+7454720\,m^{18}+1490944\,m^{19}\big)-$$
$$2\,k^9\,m\,\big(-10468986166800-2880583736580\,m+62685667298016\,m^2+17159570589609\,m^3-$$
$$97233277141215\,m^4-26228297381022\,m^5+60245617012318\,m^6+15782810041505\,m^7-17744604606519\,m^8-$$
$$4416441371328\,m^9+2750667059976\,m^{10}+632389319832\,m^{11}-249113425952\,m^{12}-52219159168\,m^{13}+$$
$$14548369920\,m^{14}+2873740800\,m^{15}-526827264\,m^{16}-103729152\,m^{17}+8427520\,m^{18}+1685504\,m^{19}\big)-$$
$$9\,k^8\,m\,\big(-10648783227600-3500908116660\,m+62398603819332\,m^2+20275790466213\,m^3-$$
$$91106469845388\,m^4-28563945022286\,m^5+50748533351980\,m^6+14696165348485\,m^7-13204425657620\,m^8-$$
$$3280112954696\,m^9+2022108226944\,m^{10}+412236703008\,m^{11}-225885968256\,m^{12}-42406737920\,m^{13}+$$
$$16955561984\,m^{14}+3308532992\,m^{15}-644699136\,m^{16}-129906688\,m^{17}+8437760\,m^{18}+1687552\,m^{19}\big)+$$
$$3\,k^6\,m\,\big(138627614235600+108190833003540\,m-648693555742980\,m^2-508663771170381\,m^3+349248823950420\,m^4+$$
$$281434560244742\,m^5+101136177718884\,m^6+91112697387747\,m^7+41667118388124\,m^8-7904025864432\,m^9-$$
$$26222521150368\,m^{10}-7658784611776\,m^{11}+1235690343552\,m^{12}+680569951488\,m^{13}+241803531264\,m^{14}+$$
$$35372056320\,m^{15}-17402397696\,m^{16}-3533529088\,m^{17}+210739200\,m^{18}+42147840\,m^{19}\big)-2\,k^5\,m$$
$$\big(-1909838955704400-3797094058896420\,m+396833152851036\,m^2+1917019692332793\,m^3+3575950905504582\,m^4+$$
$$2120320590809412\,m^5-542154433057256\,m^6-506201326253347\,m^7-341756845776834\,m^8-186880178878734\,m^9-$$
$$38761084563816\,m^{10}+13235740284120\,m^{11}+23111578927552\,m^{12}+5097722864192\,m^{13}-1260500830464\,m^{14}-$$
$$382870588416\,m^{15}-44039846400\,m^{16}-5586163200\,m^{17}+2652928000\,m^{18}+530585600\,m^{19}\big)-$$
$$1152\,k\,\big(-14582741040000+60235464996000\,m+80089522367400\,m^2-152368084977660\,m^3-107263756327356\,m^4+$$
$$104589133396191\,m^5+58463242703379\,m^6-15872783410358\,m^7-804471108406\,m^8+5133362514159\,m^9-$$
$$6155398946685\,m^{10}-7593103444044\,m^{11}-321830355804\,m^{12}+1909244436064\,m^{13}+361121317088\,m^{14}-$$
$$144930154368\,m^{15}-32973889920\,m^{16}+2373102336\,m^{17}+608333568\,m^{18}+51143680\,m^{19}+10228736\,m^{20}\big)+$$
$$288\,k^2\,\big(145827410400000-276752911604400\,m-482131235826540\,m^2+290058174937728\,m^3+309499870867317\,m^4+$$
$$29004451025724\,m^5-32733462789906\,m^6-62725047844700\,m^7+32298731102921\,m^8+76446209794512\,m^9+$$
$$13396707743712\,m^{10}-19473155264064\,m^{11}-7522991785056\,m^{12}-563031297536\,m^{13}+379740011264\,m^{14}+$$
$$403016395776\,m^{15}+64241193216\,m^{16}-25242316800\,m^{17}-5018849280\,m^{18}+372981760\,m^{19}+74596352\,m^{20}\big)-$$
$$12\,k^4\,\big(-699971569920000-874975041939600\,m-1470258132325380\,m^2+1119839242492764\,m^3+1130975612640333\,m^4+$$
$$61346723333148\,m^5+169542208168898\,m^6+271684113867436\,m^7+255376361248613\,m^8+133001441025804\,m^9-$$
$$1011095863536\,m^{10}-62597132962464\,m^{11}-25884302232064\,m^{12}-3055285404032\,m^{13}+1319601920768\,m^{14}+$$
$$1641230429184\,m^{15}+270392017152\,m^{16}-96973624320\,m^{17}-19560448000\,m^{18}+1256366080\,m^{19}+251273216\,m^{20}\big)+$$
$$8\,k^3\,\big(4199829419520000-2302850819883600\,m-5141794159059300\,m^2-1757385883095444\,m^3+157766635350009\,m^4+$$
$$4489820647363062\,m^5+2660598583835832\,m^6-495001340061656\,m^7-125422343554411\,m^8-35687704647714\,m^9-$$
$$387582373725834\,m^{10}-347754396751896\,m^{11}-18602059314264\,m^{12}+85858530196672\,m^{13}+18801525225152\,m^{14}-$$
$$4684896701184\,m^{15}-1292124664320\,m^{16}-75892369920\,m^{17}-6365913600\,m^{18}+6630983680\,m^{19}+1326196736\,m^{20}\big)\big)$$
$$h_2(k+1,m)-$$
$$\big(2\,k^{13}\,m\,\big(57153600+12700800\,m-349190676\,m^2-77597928\,m^3+$$
$$572100201\,m^4+127133378\,m^5-390874302\,m^6-86860956\,m^7+132652377\,m^8+29478306\,m^9-$$
$$24090768\,m^{10}-5353504\,m^{11}+2364768\,m^{12}+525504\,m^{13}-117504\,m^{14}-26112\,m^{15}+2304\,m^{16}+512\,m^{17}\big)+$$
$$k^{12}\,m\,\big(4186501200+948194100\,m-25538030892\,m^2-5783123781\,m^3+41663326662\,m^4+$$
$$9430547986\,m^5-28244472984\,m^6-6387945837\,m^7+9466144974\,m^8+2138079372\,m^9-1687042656\,m^{10}-$$
$$380271008\,m^{11}+161130816\,m^{12}+36210048\,m^{13}-7699968\,m^{14}-1722624\,m^{15}+142848\,m^{16}+31744\,m^{17}\big)+$$
$$k^{10}\,m\,\big(822401521200+202857266700\,m-4978700294292\,m^2-1225137924747\,m^3+7958058577794\,m^4+$$
$$1945378416782\,m^5-5203578360344\,m^6-1256078784451\,m^7+1655269215258\,m^8+391155855444\,m^9-$$
$$277778532576\,m^{10}-63493772320\,m^{11}+25629026752\,m^{12}+5588249216\,m^{13}-1340686848\,m^{14}-$$
$$277196544\,m^{15}+40106496\,m^{16}+8004608\,m^{17}-573440\,m^{18}-114688\,m^{19}\big)-4354560\,m\,\big(857304000+$$
$$387147600\,m-4708887480\,m^2-1867089042\,m^3+5943144699\,m^4+1247642767\,m^5-3836730070\,m^6+28385632\,m^7+$$
$$2303417103\,m^8+346584159\,m^9-601935900\,m^{10}-157744364\,m^{11}+41835872\,m^{12}+14978656\,m^{13}+$$
$$2110080\,m^{14}+144000\,m^{15}-263424\,m^{16}-50432\,m^{17}+5120\,m^{18}+1024\,m^{19}\big)-2\,k^{11}\,m\,\big(-38839953600-$$
$$9089496000\,m+236261463876\,m^2+55245594960\,m^3-382539348333\,m^4-89249843896\,m^5+255843840958\,m^6+$$
$$59440056968\,m^7-84012342765\,m^8-19382726376\,m^9+14578934136\,m^{10}+3326782472\,m^{11}-1356105056\,m^{12}-$$
$$304270720\,m^{13}+64919040\,m^{14}+14199552\,m^{15}-1418496\,m^{16}-299008\,m^{17}+10240\,m^{18}+2048\,m^{19}\big)-$$
$$6\,k^9\,m\,\big(-1002333301200-273003179220\,m+6007698417384\,m^2+1626097779837\,m^3-$$
$$9347212665621\,m^4-2485305180646\,m^5+5835407834162\,m^6+1497744907045\,m^7-1750678820973\,m^8-$$
$$422844077808\,m^9+283121462424\,m^{10}+62596439432\,m^{11}-27697058656\,m^{12}-5618013056\,m^{13}+$$
$$1759127040\,m^{14}+344153088\,m^{15}-66028800\,m^{16}-13035520\,m^{17}+1034240\,m^{18}+206848\,m^{19}\big)-$$
$$3\,k^8\,m\,\big(-8940690738000-2902847403060\,m+52465183440156\,m^2+16791717388581\,m^3-$$
$$76988664289386\,m^4-23584619594966\,m^5+43560361205536\,m^6+12123420363557\,m^7-11858332127538\,m^8-$$
$$2775246537648\,m^9+1979939261040\,m^{10}+388963564912\,m^{11}-234324203456\,m^{12}-44686944256\,m^{13}+$$
$$17117902848\,m^{14}+3419855616\,m^{15}-597189120\,m^{16}-122040320\,m^{17}+6737920\,m^{18}+1347584\,m^{19}\big)+$$
$$2\,k^7\,m\,\big(52369300720800+23658209379000\,m-290617992125028\,m^2-127632581426574\,m^3+360772426660527\,m^4+$$

$$142690225294948\, m^{5} - 154087514472162\, m^{6} - 44451308810406\, m^{7} + 39065965947423\, m^{8} + 7138081703760\, m^{9} -$$
$$8521359206472\, m^{10} - 1630735928504\, m^{11} + 1070842283808\, m^{12} + 240641191296\, m^{13} - 51976078848\, m^{14} -$$
$$12632218368\, m^{15} + 285288192\, m^{16} + 96618496\, m^{17} + 20981760\, m^{18} + 4196352\, m^{19}) + k^{6}\, m\, (95247003070800 +$$
$$79854154328340\, m - 430604909262252\, m^{2} - 367658556019677\, m^{3} + 169133367246390\, m^{4} + 174676429428026\, m^{5} +$$
$$112890178192008\, m^{6} + 72730609617099\, m^{7} + 19728504629982\, m^{8} + 3115898570244\, m^{9} - 7189166828160\, m^{10} -$$
$$5492419795648\, m^{11} - 2424891827904\, m^{12} - 86188749696\, m^{13} + 463121155584\, m^{14} + 88138758912\, m^{15} -$$
$$19356000768\, m^{16} - 4128369664\, m^{17} + 109240320\, m^{18} + 21848064\, m^{19}) - 2\, k^{5}\, m\, (- 672941242234800 -$$
$$1263023864604540\, m + 386068443176412\, m^{2} + 663550762873023\, m^{3} + 689056114695804\, m^{4} + 529909173756616\, m^{5} +$$
$$94177299258092\, m^{6} - 23836566445949\, m^{7} - 106838661790524\, m^{8} - 65966714498070\, m^{9} - 21971201622360\, m^{10} -$$
$$1611429593528\, m^{11} + 5460185497856\, m^{12} + 1797974942272\, m^{13} + 135530063616\, m^{14} - 33480428544\, m^{15} -$$
$$45662094336\, m^{16} - 8049110528\, m^{17} + 1325066240\, m^{18} + 265013248\, m^{19}) - 5184\, k\, (- 1080203040000 +$$
$$4728031560000\, m + 5316424470000\, m^{2} - 15272058440700\, m^{3} - 9062698930776\, m^{4} + 13704815798475\, m^{5} +$$
$$5447665798895\, m^{6} - 5158726160790\, m^{7} + 119930599974\, m^{8} + 2949161853795\, m^{9} - 70279662465\, m^{10} -$$
$$1237692191100\, m^{11} - 205779763276\, m^{12} + 184330011360\, m^{13} + 43210612320\, m^{14} - 7749098880\, m^{15} -$$
$$2160210816\, m^{16} - 143642880\, m^{17} - 16672000\, m^{18} + 9630720\, m^{19} + 1926144\, m^{20}) + 216\, k^{3}\, (51849745920000 -$$
$$33841705942800\, m - 77651998701300\, m^{2} - 32147582999652\, m^{3} + 5248196223213\, m^{4} + 83125341213490\, m^{5} +$$
$$35133499861522\, m^{6} - 27906354511080\, m^{7} + 1750520200333\, m^{8} + 16486705492154\, m^{9} - 1990945011592\, m^{10} -$$
$$8560015112464\, m^{11} - 1516158112208\, m^{12} + 1219701796032\, m^{13} + 325248604928\, m^{14} - 22937252864\, m^{15} -$$
$$11061996800\, m^{16} - 3585701376\, m^{17} - 602359808\, m^{18} + 122818560\, m^{19} + 24563712\, m^{20}) + 144\, k^{2}\, (97218273600000 -$$
$$207739680372000\, m - 341408576793960\, m^{2} + 294673416721596\, m^{3} + 283230453260466\, m^{4} - 45665824155291\, m^{5} -$$
$$86292673522087\, m^{6} - 76488318744306\, m^{7} + 9590019452868\, m^{8} + 62620353980325\, m^{9} + 21902077369533\, m^{10} -$$
$$5480072394516\, m^{11} - 5055043415428\, m^{12} - 2585861007072\, m^{13} - 186701787360\, m^{14} + 421278132096\, m^{15} +$$
$$79924253568\, m^{16} - 17723987712\, m^{17} - 3714117376\, m^{18} + 141818880\, m^{19} + 28363776\, m^{20}) - 4\, k^{4}\, (- 699971569920000 -$$
$$790397201437200\, m - 1317147125655780\, m^{2} + 1098480550109760\, m^{3} + 1298053001192769\, m^{4} + 372846269500515\, m^{5} -$$
$$74043859951909\, m^{6} - 429800732330858\, m^{7} + 86872348543643\, m^{8} + 381671201662347\, m^{9} + 107779340948049\, m^{10} -$$
$$57581803189524\, m^{11} - 32628510470596\, m^{12} - 10443726068384\, m^{13} - 241130560096\, m^{14} + 2094505878912\, m^{15} +$$
$$399385291392\, m^{16} - 87011766528\, m^{17} - 18571627264\, m^{18} + 482728960\, m^{19} + 96545792\, m^{20}))$$
$$h_2(k + 2, m)) /$$
$$(1451520\, (1 + k)^{2}\, (-4 + m)\, (-3 + m)\, (-2 + m)\, (-1 + m)\, m\, (1 + m)\, (2 + m)\, (3 + m)\, (4 + m)\, (5 + m)\, (-9 + 2\, m)\, (-7 + 2\, m)$$
$$(-5 + 2\, m)\, (-3 + 2\, m)\, (-1 + 2\, m)\, (1 + 2\, m)\, (3 + 2\, m)\, (5 + 2\, m)\, (7 + 2\, m)\, (9 + 2\, m))$$

Appendix C. The certificate polynomials of $B_0(m)$

$$p_0(k,m) = 2(- 2124950729932800000\, k^2 - 12586293967887360000\, k^3 - 3405991268203776000\, k^4 +$$
$$30328948861743168000\, k^5 + 15987851881683292800\, k^6 - 25732414446531508800\, k^7 - 15910597606431729600\, k^8 +$$
$$9222734774942640000\, k^9 + 6598812735685333800\, k^{10} - 1205224605696453300\, k^{11} - 1230061332511695600\, k^{12} -$$
$$50556484783570500\, k^{13} + 83244714131161800\, k^{14} + 24601910976745200\, k^{15} + 2168870393894400\, k^{16} -$$
$$1824906284901000\, k^{17} - 494128178296200\, k^{18} + 27243552854700\, k^{19} + 16890164096400\, k^{20} + 1669327663500\, k^{21} -$$
$$22598692200\, k^{22} - 49359277800\, k^{23} - 4200789600\, k^{24} - 8499802919731200000\, k\, m - 80821835972398080000\, k^2\, m -$$
$$31286512055094220800 0\, k^3\, m - 5211054253023782400\, k^4\, m + 79376175175140984960 0\, k^5\, m + 335725729165331202240\, k^6\, m -$$
$$667509786851242976640\, k^7\, m - 373276614320984069280\, k^8\, m + 230702133164152946400\, k^9\, m +$$
$$156433267926110770740\, k^{10}\, m - 28520975904780249540\, k^{11}\, m - 28313287167851417880\, k^{12}\, m -$$
$$1187882260535467800\, k^{13}\, m + 1795334869592781840\, k^{14}\, m + 531178903010928960\, k^{15}\, m + 53366367942220320\, k^{16}\, m -$$
$$37171375237692000\, k^{17}\, m - 10385303649304860\, k^{18}\, m + 496106473177260\, k^{19}\, m + 331273092372120\, k^{20}\, m +$$
$$33036072463800\, k^{21}\, m - 318624069960\, k^{22}\, m - 908851172040\, k^{23}\, m - 75740222880\, k^{24}\, m - 8499802919731200000\, m^2 -$$
$$178626668464742400000\, k\, m^2 - 1201918014236100096000\, k^2\, m^2 - 3445353513513187200000\, k^3\, m^2 +$$
$$950492588768843155200\, k^4\, m^2 + 9371591787713982166080\, k^5\, m^2 + 2940749711112321069120\, k^6\, m^2 -$$
$$7758180648958300212960\, k^7\, m^2 - 3839251334363377793280\, k^8\, m^2 + 2546442921646684624860\, k^9\, m^2 +$$
$$1621134256366067130720\, k^{10}\, m^2 - 290336692361457063150\, k^{11}\, m^2 - 281249143058931544440\, k^{12}\, m^2 -$$
$$12718054737698368410\, k^{13}\, m^2 + 16225683760779006420\, k^{14}\, m^2 + 4906689447227277960\, k^{15}\, m^2 +$$
$$583011960194271600\, k^{16}\, m^2 - 315817644607237080\, k^{17}\, m^2 - 92495076879813480\, k^{18}\, m^2 + 3500322769682370\, k^{19}\, m^2 +$$
$$2680382589861240\, k^{20}\, m^2 + 272848485194550\, k^{21}\, m^2 - 1198342596780\, k^{22}\, m^2 - 6762915824220\, k^{23}\, m^2 -$$
$$546560410320\, k^{24}\, m^2 - 109156936023797760000\, m^3 - 1292215460599848960000\, k\, m^3 - 9619663051364960563200\, k^2\, m^3 -$$
$$22533002739473843727360\, k^3\, m^3 + 13750167985025157722880\, k^4\, m^3 + 67059797679538627625280\, k^5\, m^3 +$$
$$13715844326412269127264\, k^6\, m^3 - 54003599557646261286624\, k^7\, m^3 - 22981742667943594786608\, k^8\, m^3 +$$
$$16450801302886528243740\, k^9\, m^3 + 9709371592827464635674\, k^{10}\, m^3 - 1660088286791448910074\, k^{11}\, m^3 -$$
$$1582774512826367103228\, k^{12}\, m^3 - 83961417829045838100\, k^{13}\, m^3 + 78791351539011102264\, k^{14}\, m^3 +$$
$$25597805330356754496\, k^{15}\, m^3 + 3761173887168808392\, k^{16}\, m^3 - 1454857173942822540\, k^{17}\, m^3 -$$
$$459007798909033206\, k^{18}\, m^3 + 11197364095288326\, k^{19}\, m^3 + 11588283146464932\, k^{20}\, m^3 + 1230358419601620\, k^{21}\, m^3 +$$
$$3319820577204\, k^{22}\, m^3 - 26075102464764\, k^{23}\, m^3 - 2009836648368\, k^{24}\, m^3 + 478144293431556096000\, m^4 -$$
$$2540856270855730176 00\, k\, m^4 - 46533321279579057730560\, k^2\, m^4 - 101201014543895961500160\, k^3\, m^4 +$$
$$97572769103550555864000\, k^4\, m^4 + 329840872991386639989408\, k^5\, m^4 + 33059023372867021009248\, k^6\, m^4 -$$
$$252572214513701815723952\, k^7\, m^4 - 88221579265992728698820\, k^8\, m^4 + 68807036714985557611526\, k^9\, m^4 +$$
$$36744560447619596887628\, k^{10}\, m^4 - 5693154836735139335781\, k^{11}\, m^4 - 5426100019399244954376\, k^{12}\, m^4 -$$
$$380840370127787962151\, k^{13}\, m^4 + 204161287966994341738\, k^{14}\, m^4 + 82015515920805906320\, k^{15}\, m^4 +$$
$$15882969312275681048\, k^{16}\, m^4 - 3794740033483367868\, k^{17}\, m^4 - 1369989527873843772\, k^{18}\, m^4 +$$
$$6665837293269511\, k^{19}\, m^4 + 27687413084294572\, k^{20}\, m^4 + 3226983486788285\, k^{21}\, m^4 + 41689348430918\, k^{22}\, m^4 -$$
$$52046344807938\, k^{23}\, m^4 - 3659751480024\, k^{24}\, m^4 + 1943307807264201830400 0\, m^5 + 62923757660228503572480\, k\, m^5 -$$
$$141319376488855204853760\, k^2\, m^5 - 364379913113477111790336\, k^3\, m^5 + 436624100364335260254912\, k^4\, m^5 +$$
$$1198242991718821899624480\, k^5\, m^5 + 495277410141693207851 2\, k^6\, m^5 - 833716854679384071268808\, k^7\, m^5 -$$
$$214834572856263058876828\, k^8\, m^5 + 189609758195653578015356\, k^9\, m^5 + 85399908538920848143633\, k^{10}\, m^5 -$$
$$10533554549141869440868\, k^{11}\, m^5 - 10322786712306280987330\, k^{12}\, m^5 - 1203807458802335707583\, k^{13}\, m^5 +$$
$$117691327993575867026\, k^{14}\, m^5 + 152709301742882608040\, k^{15}\, m^5 + 44990797810400646964\, k^{16}\, m^5 -$$
$$4266857532043820662\, k^{17}\, m^5 - 2266701937783145479\, k^{18}\, m^5 - 74871181344451784\, k^{19}\, m^5 + 25769234458596434\, k^{20}\, m^5 +$$
$$4236641954679329\, k^{21}\, m^5 + 144105035335948\, k^{22}\, m^5 - 24517555281724\, k^{23}\, m^5 - 684993499712\, k^{24}\, m^5 +$$
$$184960957525279194193920\, m^6 + 536412775742670746096640\, k\, m^6 - 279566261854567264060416\, k^2\, m^6 -$$
$$1290019884197233588215552\, k^3\, m^6 + 1362101158373666983171008\, k^4\, m^6 + 3350540133594446736629648\, k^5\, m^6 -$$
$$299360839484899534657600\, k^6\, m^6 - 1924299865098137757627648\, k^7\, m^6 - 229999453055036481081340\, k^8\, m^6 +$$
$$305202105216833148938123\, k^9\, m^6 + 75458084896444443486364\, k^{10}\, m^6 - 26223975668994971240 0\, k^{11}\, m^6 -$$
$$9031478632436334246 78\, k^{12}\, m^6 - 2406424737093191914026\, k^{13}\, m^6 - 1159253918800171175992\, k^{14}\, m^6 +$$
$$67714876143005559920\, k^{15}\, m^6 + 79514153998055586604\, k^{16}\, m^6 + 6591604314205536295\, k^{17}\, m^6 -$$
$$387403267885524960\, k^{18}\, m^6 - 293222660636970168\, k^{19}\, m^6 - 54019355208672298\, k^{20}\, m^6 - 1824812958474160\, k^{21}\, m^6 +$$
$$213260881052764\, k^{22}\, m^6 + 146507103807728\, k^{23}\, m^6 + 12731027621824\, k^{24}\, m^6 + 95559727496756295647232 0\, m^7 +$$
$$2477754512174098768693248\, k\, m^7 - 604694089908673155095040\, k^2\, m^7 - 4889792632571367903443712\, k^3\, m^7 +$$
$$3198444963443455316976448\, k^4\, m^7 + 7214711468350560840510416\, k^5\, m^7 - 1560098844065188533197984\, k^6\, m^7 -$$
$$2620413743935417569473376\, k^7\, m^7 + 643274623980292488391544\, k^8\, m^7 + 155819090303711826404 81\, k^9\, m^7 -$$
$$285080705416170760554609\, k^{10}\, m^7 + 5743949086526050233722 3\, k^{11}\, m^7 + 6252162218748794793923 2\, k^{12}\, m^7 -$$
$$1373989197963973997672\, k^{13}\, m^7 - 499792691569060619399 2\, k^{14}\, m^7 - 543043176759499054156\, k^{15}\, m^7 +$$
$$47016416462122347362\, k^{16}\, m^7 + 38748191503266405151\, k^{17}\, m^7 + 8767427628058250513\, k^{18}\, m^7 -$$
$$513495812568207523\, k^{19}\, m^7 - 250000379535166378\, k^{20}\, m^7 - 20979637781895424\, k^{21}\, m^7 - 61260075731400\, k^{22}\, m^7 +$$
$$447991620093856\, k^{23}\, m^7 + 33251233684736\, k^{24}\, m^7 + 287543094062824989146419 2\, m^8 + 7139105360024990150619648\, k\, m^8 -$$
$$3487505152249731380485632\, k^2\, m^8 - 17352095668149628989310208\, k^3\, m^8 + 6521476707433310247430496\, k^4\, m^8 +$$
$$10651285381245302427403552\, k^5\, m^8 - 5864886552824850551263240\, k^6\, m^8 + 946785055181835303074012\, k^7\, m^8 +$$
$$4382250198570528845420148\, k^8\, m^8 - 1540024518501084312310551\, k^9\, m^8 - 1553284752258162336273434\, k^{10}\, m^8 +$$
$$170450226658821008147280\, k^{11}\, m^8 + 228701667992915903210790\, k^{12}\, m^8 + 9621939369060272176008\, k^{13}\, m^8 -$$
$$11181885458325260821818\, k^{14}\, m^8 - 2035429404662106909934\, k^{15}\, m^8 - 194205924850385747070\, k^{16}\, m^8 +$$
$$86548158805978682815\, k^{17}\, m^8 + 27074869737693761108\, k^{18}\, m^8 - 333632341738319486\, k^{19}\, m^8 -$$
$$493046124929268660\, k^{20}\, m^8 - 49834135252365936\, k^{21}\, m^8 - 896504895696192\, k^{22}\, m^8 + 694949282636896\, k^{23}\, m^8 +$$

$46410104770688\, k^{24} m^{8} + 370756404725233960036 4544\, m^{9} + 10762842857426543409523200\, k\, m^{9} -$
$2204562293830024941236 5824\, k^{2} m^{9} - 49846478641564347402946304\, k^{3} m^{9} + 15420584714955085247507328\, k^{4} m^{9} +$
$25298259025982266571196 80\, k^{5} m^{9} - 18990245578928519343338704\, k^{6} m^{9} + 17879552528734479711868940\, k^{7} m^{9} +$
$14381035322031703891308720\, k^{8} m^{9} - 5316483776141255018697301\, k^{9} m^{9} - 4292383197538412338751190\, k^{10} m^{9} +$
$2381566852772006988474 14\, k^{11} m^{9} + 4993996246907636487031 06\, k^{12} m^{9} + 420256462213242408185 88\, k^{13} m^{9} -$
$156484824091254991738 82\, k^{14} m^{9} - 428237023060624112116 2\, k^{15} m^{9} - 753253755197720676338\, k^{16} m^{9} +$
$121682062958482506665\, k^{17} m^{9} + 48912350175007047336\, k^{18} m^{9} + 539573317076471128\, k^{19} m^{9} -$
$627038246779043656\, k^{20} m^{9} - 73078179455523456\, k^{21} m^{9} - 1929389248407424\, k^{22} m^{9} + 701222259494208\, k^{23} m^{9} +$
$41904407491584\, k^{24} m^{9} - 7498817213208132666387456\, m^{10} - 10699249183901067045623040\, k\, m^{10} -$
$10494033177414837070525 2096\, k^{2} m^{10} - 10713920767335163958286 8736\, k^{3} m^{10} + 48280446350767500325242656\, k^{4} m^{10} -$
$48142300340259750472774432\, k^{5} m^{10} - 52071585231102316395967872\, k^{6} m^{10} + 62580878037377788915559020\, k^{7} m^{10} +$
$33968443327126482141902132\, k^{8} m^{10} - 11383603395491854164443514\, k^{9} m^{10} - 8431717549937222226795686\, k^{10} m^{10} +$
$2430282828201711168791 8\, k^{11} m^{10} + 773492487585395784026886\, k^{12} m^{10} + 101106675283549445776266\, k^{13} m^{10} -$
$11804426060599833073206\, k^{14} m^{10} - 6412477782766091878674\, k^{15} m^{10} - 153403158524781521936 2\, k^{16} m^{10} +$
$113931632036322620136\, k^{17} m^{10} + 62910881635131636852\, k^{18} m^{10} + 1786609695746372576\, k^{19} m^{10} -$
$558541224668293136\, k^{20} m^{10} - 75771621976066944\, k^{21} m^{10} - 2396653662739520\, k^{22} m^{10} + 489195116534784\, k^{23} m^{10} +$
$25585130452992\, k^{24} m^{10} - 4408390196546323217649868 8\, m^{11} - 12647360792085648018142694 4\, k\, m^{11} -$
$39225823543081932942836 5824\, k^{2} m^{11} - 16024016018533966793616 6144\, k^{3} m^{11} + 15652854384722007804372544 0\, k^{4} m^{11} -$
$19657698054312436344129369 6\, k^{5} m^{11} - 11511754843776840988930131 2\, k^{6} m^{11} + 14873884651366784026518086 0\, k^{7} m^{11} +$
$62088436574927607019291844\, k^{8} m^{11} - 18079674433213122075248168\, k^{9} m^{11} - 1267095569454655508954638 4\, k^{10} m^{11} -$
$689141121240006244558376\, k^{11} m^{11} + 873297057345768670870944\, k^{12} m^{11} + 173476914558967629987296\, k^{13} m^{11} +$
$34768132560945270829 20\, k^{14} m^{11} - 7257187982715715731892\, k^{15} m^{11} - 2197578642644661672676\, k^{16} m^{11} +$
$62521645731427627912\, k^{17} m^{11} + 61031980444223421576\, k^{18} m^{11} + 2616932736803482208\, k^{19} m^{11} -$
$352700439880949984\, k^{20} m^{11} - 57823688640361728\, k^{21} m^{11} - 1978665424996480\, k^{22} m^{11} +$
$238803892771840\, k^{23} m^{11} + 10525174956032\, k^{24} m^{11} - 6127019179244678619188582 4\, m^{12} -$
$46496056446889479786753 3312\, k\, m^{12} - 12023883449771589368771808 00\, k^{2} m^{12} - 14208723930411714744642835 2\, k^{3} m^{12} +$
$43244645038881012359347491 2\, k^{4} m^{12} - 49419906434029622470048782 4\, k^{5} m^{12} - 19495086331536902591822703 2\, k^{6} m^{12} +$
$27614599943033101290801821 6\, k^{7} m^{12} + 88085710533943618245686952\, k^{8} m^{12} - 22473397076017012352325920\, k^{9} m^{12} -$
$14714006738090250039329376\, k^{10} m^{12} - 1831800507859248546977736\, k^{11} m^{12} + 673877035809908338629608\, k^{12} m^{12} +$
$227955267344155944229728\, k^{13} m^{12} + 2485680189984222647159 2\, k^{14} m^{12} - 6281077231062880711184\, k^{15} m^{12} -$
$2387494832070477959776\, k^{16} m^{12} - 292505042497328256\, k^{17} m^{12} + 45575617541059471936\, k^{18} m^{12} +$
$2532956794808898496\, k^{19} m^{12} - 151750656800748736\, k^{20} m^{12} - 32744155349599232\, k^{21} m^{12} -$
$1122552428664832\, k^{22} m^{12} + 80258400673792\, k^{23} m^{12} + 2799584116736\, k^{24} m^{12} + 1836841281814280025130752 00\, m^{13} -$
$11648734286499809815262976 00\, k\, m^{13} - 30767591580779781134127244 80\, k^{2} m^{13} - 51064187968758450237955392\, k^{3} m^{13} +$
$95907572110911171975952384 0\, k^{4} m^{13} - 91557972361047868071151025 6\, k^{5} m^{13} - 22788194611402220461419868 8\, k^{6} m^{13} +$
$41817301108550038459918660 8\, k^{7} m^{13} + 92106465592923379312939360\, k^{8} m^{13} - 22295795280669697493823296\, k^{9} m^{13} -$
$12655962458237795834989200\, k^{10} m^{13} - 2950155106269832281473264\, k^{11} m^{13} + 228132887578593687841552\, k^{12} m^{13} +$
$236074841544157486917888\, k^{13} m^{13} + 41288304809854885825360\, k^{14} m^{13} - 4065339908326290698144\, k^{15} m^{13} -$
$2024980885171810814224\, k^{16} m^{13} - 393626974638242341 60\, k^{17} m^{13} + 26266037732295046912\, k^{18} m^{13} +$
$1765014477925398528\, k^{19} m^{13} - 37752819304990592\, k^{20} m^{13} - 13612834309074176\, k^{21} m^{13} -$
$434143369211904\, k^{22} m^{13} + 17710498758656\, k^{23} m^{13} + 435095339008\, k^{24} m^{13} + 12469235129383480794581345 28\, m^{14} -$
$22021678502272688085436377 60\, k\, m^{14} - 65977846377290487749109173 76\, k^{2} m^{14} - 17797353726934173190170643 2\, k^{3} m^{14} +$
$16699658789042233520472134 72\, k^{4} m^{14} - 12902716932937168036914992 32\, k^{5} m^{14} -$
$10322231662379121046699142 4\, k^{6} m^{14} + 52506165642311177361962731 2\, k^{7} m^{14} + 56487053335439410371872688\, k^{8} m^{14} -$
$17770509280085617096928528\, k^{9} m^{14} - 6627294611409152325302096\, k^{10} m^{14} - 348177567530540043035908 8\, k^{11} m^{14} -$
$249900925140357322255760\, k^{12} m^{14} + 194772802942467340551760\, k^{13} m^{14} + 44804392659408040855920\, k^{14} m^{14} -$
$1799248204354734951712\, k^{15} m^{14} - 1355556072074453126304\, k^{16} m^{14} - 44863973011097963776\, k^{17} m^{14} +$
$11560874281539147904\, k^{18} m^{14} + 907198024021279232\, k^{19} m^{14} - 453416074391808\, k^{20} m^{14} -$
$4036160486916096\, k^{21} m^{14} - 109570390769664\, k^{22} m^{14} + 2310506283008\, k^{23} m^{14} + 30006575104\, k^{24} m^{14} +$
$38268472787958523306469898 24\, m^{15} - 31191293389625471732603143 68\, k\, m^{15} - 11830917718849019569734212 352\, k^{2} m^{15} -$
$12428842330685065552241504 64\, k^{3} m^{15} + 22001811988237730277187962 88\, k^{4} m^{15} -$
$13425860352229906611207326 08\, k^{5} m^{15} + 27153586660444010235499059 2\, k^{6} m^{15} +$
$54983428273329996860680464 0\, k^{7} m^{15} - 17501284125906145055008192\, k^{8} m^{15} - 11482305751343602373861664\, k^{9} m^{15} +$
$888402930161251655869120\, k^{10} m^{15} - 318489706847079296617628 8\, k^{11} m^{15} - 54291728393438196500844 8\, k^{12} m^{15} +$
$127863195744483616573408\, k^{13} m^{15} + 36259003706798199148608\, k^{14} m^{15} - 340832215475576038848\, k^{15} m^{15} -$
$716535962759894436096\, k^{16} m^{15} - 30793590166424501248\, k^{17} m^{15} + 3788833215692997120\, k^{18} m^{15} +$
$342945126280275200\, k^{19} m^{15} + 3468435123957760\, k^{20} m^{15} - 807182459854848\, k^{21} m^{15} - 16280442404864\, k^{22} m^{15} +$
$135029587968\, k^{23} m^{15} + 81527127180767787866031682 56\, m^{16} - 28909292738993326825249443 84\, k\, m^{16} -$
$17631910606713012615679641216\, k^{2} m^{16} - 40904307076320060659706727 68\, k^{3} m^{16} +$
$19423081863913493541328401 92\, k^{4} m^{16} - 87695695522269974139620243 2\, k^{5} m^{16} + 87219624162709806831088876 8\, k^{6} m^{16} +$
$48040163852698983939628160 0\, k^{7} m^{16} - 10363410468649238751696678 4\, k^{8} m^{16} - 6326612659465773969615488\, k^{9} m^{16} +$
$663594827130145829853081 6\, k^{10} m^{16} - 230013873232097723694233 6\, k^{11} m^{16} - 5783510223018417372610 56\, k^{12} m^{16} +$
$65829134160849439752256\, k^{13} m^{16} + 22812401105288927977536\, k^{14} m^{16} + 213596483023975144192\, k^{15} m^{16} -$
$296717400224519356160\, k^{16} m^{16} - 14867071159919658240\, k^{17} m^{16} + 879817313149547520\, k^{18} m^{16} +$
$93019912084898304\, k^{19} m^{16} + 1296632184500224\, k^{20} m^{16} - 97461355937792\, k^{21} m^{16} - 1080236703744\, k^{22} m^{16} +$
$13294723891641200666381168 640\, m^{17} - 29025550898261087561253273 6\, k\, m^{17} - 21567946213267279820289156096\, k^{2} m^{17} -$
$89130432064711803296167027 20\, k^{3} m^{17} + 43790215078527215514823091 2\, k^{4} m^{17} + 23671668713263533891850368\, k^{5} m^{17} +$
$15166905578982189348265108 48\, k^{6} m^{17} + 34912200574061732687797555 2\, k^{7} m^{17} - 16524086965498022321939609 6\, k^{8} m^{17} -$
$35498090365535119735275 52\, k^{9} m^{17} + 87564308074082355346511 36\, k^{10} m^{17} - 1309379017586861999221248\, k^{11} m^{17} -$

$4380296376557657473l4176\ k^{12}\ m^{17} + 2559557837698687552780 8\ k^{13}\ m^{17} + 1130775017393846669926 4\ k^{14}\ m^{17}+$
$243778279901350127616\ k^{15}\ m^{17} - 946470544571858549 76\ k^{16}\ m^{17} - 5205282159247619328\ k^{17}\ m^{17}+$
$130252551246923776\ k^{18}\ m^{17} + 17160138353707008\ k^{19}\ m^{17} + 220573645062144\ k^{20}\ m^{17} - 5359924477952\ k^{21}\ m^{17}+$
$171907530414929972216505 49248\ m^{18} + 519450550916824266128640 0000\ k\ m^{18} - 210639780076779087195789 38880\ k^2\ m^{18}-$
$146125799041294940778872 05632\ k^3\ m^{18} - 212977954745357063084032 0768\ k^4\ m^{18}+$
$994006112874213115465385984\ k^5\ m^{18} + 1948456799452516208655987712\ k^6\ m^{18}+$
$209823732254055258138054912\ k^7\ m^{18} - 179663221339207138819174656\ k^8\ m^{18} - 25607762352396428616829 44\ k^9\ m^{18}+$
$7652738979350021991223552\ k^{10}\ m^{18} - 5743492573615509415651 84\ k^{11}\ m^{18} - 255243291316500418407168\ k^{12}\ m^{18}+$
$678090862534603040051 2\ k^{13}\ m^{18} + 441214627806505502310 4\ k^{14}\ m^{18} + 130152862777749014016\ k^{15}\ m^{18}-$
$22596578053753330176\ k^{16}\ m^{18} - 1304945135889049600\ k^{17}\ m^{18} + 8817987719619584\ k^{18}\ m^{18}+$
$1929174287237120\ k^{19}\ m^{18} + 15374618918912\ k^{20}\ m^{18} + 1777857154864199434347 2188416\ m^{19}+$
$125088993311429449411582 00320\ k\ m^{19} - 152695297314567688423884 41088\ k^2\ m^{19}-$
$190873736596432130850003 25632\ k^3\ m^{19} - 484031172855379807271662 5920\ k^4\ m^{19}+$
$1620705120354580212808748032\ k^5\ m^{19} + 199708000522065740048865 4848\ k^6\ m^{19}+$
$103587752813153521243385344\ k^7\ m^{19} - 151097079187638470421331968\ k^8\ m^{19} - 216118760314200077621 2992\ k^9\ m^{19}+$
$5084169836441535007368192\ k^{10}\ m^{19} - 1817608101825446776192 00\ k^{11}\ m^{19} - 117238954368681104713728\ k^{12}\ m^{19}+$
$7365382209822788884 48\ k^{13}\ m^{19} + 1339623556437329670144\ k^{14}\ m^{19} + 458484979369363077 12\ k^{15}\ m^{19}-$
$3850269494191639552\ k^{16}\ m^{19} - 223177955120609280\ k^{17}\ m^{19} - 367220348909568\ k^{18}\ m^{19}+$
$99726852358144\ k^{19}\ m^{19} + 144920454084055933322831 40096\ m^{20} + 191703326290597652278012 87680\ k\ m^{20}-$
$60707113399080382916687 06304\ k^2\ m^{20} - 204670983097142158071076 88448\ k^3\ m^{20}-$
$6559822622410707172333471744\ k^4\ m^{20} + 1714483171138563649898679296\ k^5\ m^{20}+$
$168306138516585255572892 3648\ k^6\ m^{20} + 41941362402555477267085312\ k^7\ m^{20} - 102417270877269818883434496\ k^8\ m^{20}-$
$1661348407293228277016576\ k^9\ m^{20} + 2673676655117964391794688\ k^{10}\ m^{20} - 325691622509621166510 08\ k^{11}\ m^{20}-$
$42594439200462377613312\ k^{12}\ m^{20} - 3000544190905144913 92\ k^{13}\ m^{20} + 309351092017238126592\ k^{14}\ m^{20}+$
$11167669863962771456\ k^{15}\ m^{20} - 431012455360008192\ k^{16}\ m^{20} - 23386065497726976\ k^{17}\ m^{20} - 77042819284992\ k^{18}\ m^{20}+$
$8775894293434868029455421440\ m^{21} + 22603306359724085337181968384\ k\ m^{21} + 28838508133389980297728 20480\ k^2\ m^{21}-$
$18322427900949974607683727360\ k^3\ m^{21} - 6684733163577315510673080320\ k^4\ m^{21}+$
$1385919861269771483643435008\ k^5\ m^{21} + 118211485684250279281439 1296\ k^6\ m^{21} + 14281550952547230000001024\ k^7\ m^{21}-$
$56843479356925456527159296\ k^8\ m^{21} - 10405442293259833876316 16\ k^9\ m^{21} + 1126370081224037856645120\ k^{10}\ m^{21}+$
$29073948737399511818 24\ k^{11}\ m^{21} - 121285402501830958899 20\ k^{12}\ m^{21} - 188285127426651025408\ k^{13}\ m^{21}+$
$52277086938291556352\ k^{14}\ m^{21} + 1828446915124346880\ k^{15}\ m^{21} - 26886933787242496\ k^{16}\ m^{21}-$
$1134123120824320\ k^{17}\ m^{21} + 30907643146453970929845 78048\ m^{22} + 217110337866342112670332 96896\ k\ m^{22}+$
$8467757558305096419016962048\ k^2\ m^{22} - 1382666942626645021615457 8944\ k^3\ m^{22}-$
$5461304513071819417510027264\ k^4\ m^{22} + 899905023785206948097429504\ k^5\ m^{22} + 695692776907230894151225344\ k^6\ m^{22}+$
$4542296742253542144167936\ k^7\ m^{22} - 25946458535485960203481088\ k^8\ m^{22} - 516890351990301678514176\ k^9\ m^{22}+$
$379516466501749693403136\ k^{10}\ m^{22} + 443105071378431784960 0\ k^{11}\ m^{22} - 265070900764828666675 2\ k^{12}\ m^{22}-$
$53421572437671882752\ k^{13}\ m^{22} + 6055091685659439104\ k^{14}\ m^{22} + 182323089300971520\ k^{15}\ m^{22}-$
$600861505789952\ k^{16}\ m^{22} - 6912790457317143510356 95104\ m^{23} + 174071837686870938891754 78272\ k\ m^{23}+$
$9772356040616664993230733312\ k^2\ m^{23} - 8840113369334566738918440960\ k^3\ m^{23}-$
$3670118106884205729696137216\ k^4\ m^{23} + 478569155165474529278582784\ k^5\ m^{23}+$
$343192573156871265838940160\ k^6\ m^{23} + 16323997558965459270942 72\ k^7\ m^{23} - 9713168674717521469521920\ k^8\ m^{23}-$
$202137286100239184519168\ k^9\ m^{23} + 101107826688660559708160\ k^{10}\ m^{23} + 1691581698559810785 28\ k^{11}\ m^{23}-$
$428848056597679521792\ k^{12}\ m^{23} - 9134839528890408960\ k^{13}\ m^{23} + 425010647666671616\ k^{14}\ m^{23}+$
$8401249336967168\ k^{15}\ m^{23} - 215569690967070109653529 55904\ m^{24} + 117857378624396366719294 34112\ k\ m^{24}+$
$8017666356355686253847494656\ k^2\ m^{24} - 4796875933493490158315274240\ k^3\ m^{24}-$
$2051285025226204489445933056\ k^4\ m^{24} + 209820114109575710134960128\ k^5\ m^{24}+$
$141413227673738189830946816\ k^6\ m^{24} + 684053297068042711269376\ k^7\ m^{24} - 2956057333192605872390144\ k^8\ m^{24}-$
$61659677720240413507584\ k^9\ m^{24} + 20839965095916498780160\ k^{10}\ m^{24} + 388481758008195219456\ k^{11}\ m^{24}-$
$48280814089141551104\ k^{12}\ m^{24} - 910334474721869824\ k^{13}\ m^{24} + 13382468332175360\ k^{14}\ m^{24}-$
$20307475346493298676790 06720\ m^{25} + 677308379393764560907085 4144\ k\ m^{25} + 518776027788438394830854 5536\ k^2\ m^{25}-$
$2206909496621327995479195648\ k^3\ m^{25} - 956812646909948735897665536\ k^4\ m^{25} + 75777732073498863155412992\ k^5\ m^{25}+$
$48319187686916850072682496\ k^6\ m^{25} + 273856176675991269146624\ k^7\ m^{25} - 720241933747645796450304\ k^8\ m^{25}-$
$14408820757133754040320\ k^9\ m^{25} + 3205426713701968183296\ k^{10}\ m^{25} + 57302881635546038272\ k^{11}\ m^{25}-$
$3365322744114839552\ k^{12}\ m^{25} - 41052683675598848\ k^{13}\ m^{25} - 131473551727753064476980 0192\ m^{26}+$
$3307082462211489140867530752\ k\ m^{26} + 2741898510366288710512607232\ k^2\ m^{26} - 857802318996645923202924544\ k^3\ m^{26}-$
$371838431074461178221428736\ k^4\ m^{26} + 22395761722837615760244736\ k^5\ m^{26} + 13533972013961217163264000\ k^6\ m^{26}+$
$89297564382204676472832\ k^7\ m^{26} - 137163886788923670331392\ k^8\ m^{26} - 2496965505980626436096\ k^9\ m^{26}+$
$346149334840579522560\ k^{10}\ m^{26} + 5057229562053853184\ k^{11}\ m^{26} - 108809583835414528\ k^{12}\ m^{26}-$
$657591872426674917846810624\ m^{27} + 1368878526191598717889806336\ k\ m^{27} + 1199532903011380605978673152\ k^2\ m^{27}-$
$279927392820878959744122880\ k^3\ m^{27} - 119667325276582269689004032\ k^4\ m^{27} + 5349981564041516190007296\ k^5\ m^{27}+$
$3054931365467384998526976\ k^6\ m^{27} + 22114040393088505872384\ k^7\ m^{27} - 19667263273359299051520\ k^8\ m^{27}-$
$302875997917183606784\ k^9\ m^{27} + 23390326202591608832\ k^{10}\ m^{27} + 204956663036116992\ k^{11}\ m^{27}-$
$263344391561573685089796096\ m^{28} + 477836949243701479548911616\ k\ m^{28} + 435661231050763294588796928\ k^2\ m^{28}-$

$$
\begin{aligned}
&759749695577735221283389440\,k^3 m^{28} - 315534498313008328605696000\,k^4 m^{28} + 101312458699989789232005120\,k^5 m^{28}+\\
&5419337982108137320611840\,k^6 m^{28} + 398385176600865511833600\,k^7 m^{28} - 19961149632736458178560\,k^8 m^{28}-\\
&22985309079385669632\,k^9 m^{28} + 743197360972627968\,k^{10} m^{28} - 853132281712096518120407040\,m^{29}+\\
&1394816819906554999933501440\,k\,m^{29} + 1308809226809979182996520960\,k^2 m^{29} - 1692125769236065858722201600\,k^3 m^{29}-\\
&670527946580218472300544000\,k^4 m^{29} + 147667632172947762315264000\,k^5 m^{29} + 7272675317657148850176000\,k^6 m^{29}+\\
&4932655129169913446400\,k^7 m^{29} - 1277736288469395701760\,k^8 m^{29} - 822334511671934976\,k^9 m^{29}-\\
&2230590490493843789158809600\,m^{30} + 3362688987855731964562636800\,k\,m^{30} + 3222018729804493893402624000\,k^2 m^{30}-\\
&3034597397952720663478272000\,k^3 m^{30} - 1120393313358231738777600000\,k^4 m^{30} + 158247532703166360453120000\,k^5 m^{30}+\\
&6938150474605233438720000\,k^6 m^{30} + 37644363032193662976000\,k^7 m^{30} - 3876554718842978304000\,k^8 m^{30}-\\
&4651903045732723731726336000\,m^{31} + 6577734892619308046745600000\,k\,m^{31} + 6398571591717348175773696000\,k^2 m^{31}-\\
&4264453762335562719559680000\,k^3 m^{31} - 1417370754236233631662080000\,k^4 m^{31} + 115573322672826325401600000\,k^5 m^{31}+\\
&4191376894541860700160000\,k^6 m^{31} + 1338542904343265280000\,k^7 m^{31} - 7571311708494943066521600000\,m^{32}+\\
&10175039444899047596359680000\,k\,m^{32} + 10005018290281420223938560000\,k^2 m^{32} - 4508757236549846827008000000\,k^3 m^{32}-\\
&1276039272429193789440000000\,k^4 m^{32} + 49863607418785628160000000\,k^5 m^{32} + 1204796853626653900800000\,k^6 m^{32}-\\
&9276940062264278384640000000\,m^{33} + 11975603932000963421798400000\,k\,m^{33} + 11863676388399314160844800000\,k^2 m^{33}-\\
&3359324128388981981184000000\,k^3 m^{33} - 728387604547576528896000000\,k^4 m^{33} + 894474879221366784000000\,k^5 m^{33}-\\
&8053080171541954560000000000\,m^{34} + 10073801563080851718144000000\,k\,m^{34} + 10029000275408241819648000000\,k^2 m^{34}-\\
&1563925996361566126080000000\,k^3 m^{34} - 198122238562617262080000000\,k^4 m^{34} - 4416224037921965998080000000\,m^{35}+\\
&5393295025376707215360000000\,k\,m^{35} + 5385366433646971453440000000\,k^2 m^{35} - 339496791252664320000000\,k^3 m^{35}-\\
&1150478015168249856000000000\,m^{36} + 1380573618201899827200000000\,k\,m^{36} + 1380573618201899827200000000\,k^2 m^{36}\big)
\end{aligned}
$$

$$
\begin{aligned}
p_1(k,m) = -(-1+k+2\,m)\,\big(&6374852189798400000\,k^2 + 44133734093460480000\,k^3 + 54351707898071808000\,k^4 - 36635138687157696000\,k^5-\\
&84598694332207574400\,k^6 - 7401450992613048000\,k^7 + 40330341826682140800\,k^8 + 12662137501854220800\,k^9-\\
&7134300705201780600\,k^{10} - 3518626888112420700\,k^{11} + 171557109422666100\,k^{12} + 323226563773377600\,k^{13}+\\
&73492421379892200\,k^{14} - 313311550343400\,k^{15} - 6819922732026600\,k^{16} - 1345203877323600\,k^{17} + 137180657565000\,k^{18}+\\
&55449999000900\,k^{19} + 4779506711700\,k^{20} - 228476278800\,k^{21} - 160680202200\,k^{22} - 12602368800\,k^{23}+\\
&2124950729932800000\,k\,m + 23711753700601344000\,k^2 m + 135107600192244633600\,k^3 m + 148932489596371557120\,k^4 m-\\
&108468569366409888000\,k^5 m - 224903064347327827872\,k^6 m - 18346594266499580640\,k^7 m+\\
&101147106186915752304\,k^8 m + 31861213381730714904\,k^9 m - 16560052999872847758\,k^{10} m-\\
&8287390025154052596\,k^{11} m + 297155578412605788\,k^{12} m + 696717919178435808\,k^{13} m + 163892394147258576\,k^{14} m+\\
&2870940649741848\,k^{15} m - 13976074326829968\,k^{16} m - 2886317535038328\,k^{17} m + 252491476871610\,k^{18} m+\\
&108690605672652\,k^{19} m + 9591169392756\,k^{20} m - 376051836384\,k^{21} m - 294537260556\,k^{22} m-\\
&22722066864\,k^{23} m + 4249901459865600000\,m^2 - 12977000967813120000\,k\,m^2 + 35855799492590945280000\,k^2 m^2+\\
&190475585025791945472000\,k^3 m^2 + 190302379270770195648000\,k^4 m^2 - 143001580368307152009600\,k^5 m^2-\\
&269367951336173199475200\,k^6 m^2 - 21267344179784859019200\,k^7 m^2 + 111727622832594348168000\,k^8 m^2+\\
&35487193729666202911800\,k^9 m^2 - 16488446009705073639000\,k^{10} m^2 - 8460282310618466056500\,k^{11} m^2+\\
&157372794028063122300\,k^{12} m^2 + 630515903667448941000\,k^{13} m^2 + 157392470462684022000\,k^{14} m^2+\\
&723887377001928000\,k^{15} m^2 - 11963145016247853600\,k^{16} m^2 - 2634742421205642000\,k^{17} m^2 + 183099341413270800\,k^{18} m^2+\\
&87872026913313300\,k^{19} m^2 + 8053658794995300\,k^{20} m^2 - 228304777921200\,k^{21} m^2 - 217769482578600\,k^{22} m^2-\\
&16396812309600\,k^{23} m^2 - 15161203282950144000\,m^3 - 52673351115480883200000\,k\,m^3 + 291386432077362146304000\,k^2 m^3+\\
&1650077437218564076544000\,k^3 m^3 + 1525789470393168979276800\,k^4 m^3 - 1116847075700422424995200\,k^5 m^3-\\
&1937123287851580707568320\,k^6 m^3 - 154655814575565962409600\,k^7 m^3 + 718784211033585891397440\,k^8 m^3+\\
&231342854390273106395640\,k^9 m^3 - 91838945329629272215380\,k^{10} m^3 - 49166874704761526453760\,k^{11} m^3-\\
&251620166141282477520\,k^{12} m^3 + 3083062953735074679480\,k^{13} m^3 + 859247008661298794760\,k^{14} m^3+\\
&71827857172616570280\,k^{15} m^3 - 55617423166197767880\,k^{16} m^3 - 13450739164974042480\,k^{17} m^3+\\
&637038709280104500\,k^{18} m^3 + 379418324535261720\,k^{19} m^3 + 37000708852763160\,k^{20} m^3 - 548167208185440\,k^{21} m^3-\\
&831616965187560\,k^{22} m^3 - 60295099451040\,k^{23} m^3 - 69664873332874752000000\,m^4 - 832361133589762212864000\,k\,m^4+\\
&13288584593886324456960000\,k^2 m^4 + 98577429987899654811648000\,k^3 m^4 + 86352064826668900991136000\,k^4 m^4-\\
&57733033755023571347337600\,k^5 m^4 - 93271201076158133954160000\,k^6 m^4 - 7742765087488314093187200\,k^7 m^4+\\
&29601734768753519358015600\,k^8 m^4 + 9672982664236053780515400\,k^9 m^4 - 3035984146034128622518200\,k^{10} m^4-\\
&1753603279530602908155900\,k^{11} m^4 - 71574093079685223961500\,k^{12} m^4 + 82088756885566532086200\,k^{13} m^4+\\
&29058733436880797498400\,k^{14} m^4 + 3972112721445231523200\,k^{15} m^4 - 1469586816177892204800\,k^{16} m^4-\\
&414968579915800105200\,k^{17} m^4 + 7707325644787558800\,k^{18} m^4 + 9044112704819843100\,k^{19} m^4 + 990830291992598700\,k^{20} m^4+\\
&2984551785625200\,k^{21} m^4 - 16309861556715000\,k^{22} m^4 - 1097925444007200\,k^{23} m^4 - 1115729474146914281472000\,m^5-\\
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\end{aligned}
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$393852239233024 k^{15} m^{22} - 2067267117007732560120446976 m^{23} + 4527375624723633485797933056 k m^{23}+$
$8259568295760643128402395136 k^{2} m^{23} - 4222523482980798405221449728 k^{3} m^{23}-$
$1565050388156483960555995136 k^{4} m^{23} + 373584748698293799972421632 k^{5} m^{23}+$

$$
\begin{aligned}
&8557938898412486181742182 4\, k^{6} m^{23} - 1177795275276568280085299 2\, k^{7} m^{23} - 1434481826296311671062528\, k^{8} m^{23} + \\
&170801122146754769387520\, k^{9} m^{23} + 9090165203662772387840\, k^{10} m^{23} - 936922037569212186624\, k^{11} m^{23} - \\
&31272847384814632960\, k^{12} m^{23} + 865876570824736768\, k^{13} m^{23} + 20131453434019840\, k^{14} m^{23} - \\
&15560605391826303947485347 84\, m^{24} + 22130813063694343817900851 20\, k\, m^{24} + 50918292883902533552562831 36\, k^{2} m^{24} - \\
&19385639181257967989809152 00\, k^{3} m^{24} - 7365603603926418152889384 96\, k^{4} m^{24} + 1385945845739293951152291 84\, k^{5} m^{24} + \\
&305056469164406039736811 52\, k^{6} m^{24} - 331873624451099866628096 0\, k^{7} m^{24} - 385226709438610258526208\, k^{8} m^{24} + \\
&32618261544455902199808\, k^{9} m^{24} + 1683587074637153697792\, k^{10} m^{24} - 98532074696679849984\, k^{11} m^{24} - \\
&2744607650464858112\, k^{12} m^{24} + 24286892420071424\, k^{13} m^{24} - 9223293400383611154907791 36\, m^{25} + \\
&9167596265072670531388047 36\, k\, m^{25} + 2627261370499714260095926272\, k^{2} m^{25} - 7578843934029713102389248 00\, k^{3} m^{25} - \\
&29061826526910472511927091 2\, k^{4} m^{25} + 42720267965802932264173568\, k^{5} m^{25} + 892012643863346449658675 2\, k^{6} m^{25} - \\
&746117198721439336366080\, k^{7} m^{25} - 80008206433238458957824\, k^{8} m^{25} + 4630052993410275999744\, k^{9} m^{25} + \\
&212851186610198478848\, k^{10} m^{25} - 6363444942721974272\, k^{11} m^{25} - 111021447253590016\, k^{12} m^{25} - \\
&4430685706779497652462551 04\, m^{26} + 3198622320507042997602877 44\, k\, m^{26} + 1136875468263358940560883712\, k^{2} m^{26} - \\
&2506823802752017883940782 08\, k^{3} m^{26} - 9525421551223966341831065 6\, k^{4} m^{26} + 1081117161057055775155814 4\, k^{5} m^{26} + \\
&2100116097116863910903808\, k^{6} m^{26} - 130668969845615720792064\, k^{7} m^{26} - 12410034867092337328128\, k^{8} m^{26} + \\
&459081174843110195200\, k^{9} m^{26} + 16571563473426317312\, k^{10} m^{26} - 188211553812480000\, k^{11} m^{26} - \\
&1742217875809050873833717 76\, m^{27} + 9314760979137140259736780 8\, k\, m^{27} + 4115466585881354706687098 88\, k^{2} m^{27} - \\
&695152593732755270302433 28\, k^{3} m^{27} - 256159882578942438062161 92\, k^{4} m^{27} + 220766417658497300299776 0\, k^{5} m^{27} + \\
&387971016607637495611392\, k^{6} m^{27} - 17167540338440101429248\, k^{7} m^{27} - 1354380982995890995200\, k^{8} m^{27} + \\
&28286342503005683712\, k^{9} m^{27} + 601781413612093440\, k^{10} m^{27} - 5607928859454338982897254 4\, m^{28} + \\
&223511156459463397594890 24\, k\, m^{28} + 123811818881496581021368320\, k^{2} m^{28} - 159569904623775339223449 60\, k^{3} m^{28} - \\
&5554616280813285594365952\, k^{4} m^{28} + 354654282795384963072000\, k^{5} m^{28} + 54125334046489909395456\, k^{6} m^{28} - \\
&1589890215405137952768\, k^{7} m^{28} - 92834045111687970816\, k^{8} m^{28} + 811460308893696000\, k^{9} m^{28} - \\
&146708271688553595760803 84\, m^{29} + 434010115174848010282598 4\, k\, m^{29} + 30615225302328211229638656\, k^{2} m^{29} - \\
&2978445244951728944578560\, k^{3} m^{29} - 947086884055416264720384\, k^{4} m^{29} + 43132943522950004342784\, k^{5} m^{29} + \\
&5359163022969934970880\, k^{6} m^{29} - 92389595763009650688\, k^{7} m^{29} - 3008240471994531840\, k^{8} m^{29} - \\
&3075014457731019448516608\, m^{30} + 6645500799965047459676 16\, k\, m^{30} + 6118878248821068203556864\, k^{2} m^{30} - \\
&440619225231917026639872\, k^{3} m^{30} - 122235030016092159344640\, k^{4} m^{30} + 3730813500148791902208\, k^{5} m^{30} + \\
&335560290958420475904\, k^{6} m^{30} - 2528090920612528128\, k^{7} m^{30} - 504515621223056749363200\, m^{31} + \\
&77186178844898264875008\, k\, m^{31} + 964167456340528569778176\, k^{2} m^{31} - 49709856484499077988352\, k^{3} m^{31} - \\
&11223960700805940510720\, k^{4} m^{31} + 204348394531039739904\, k^{5} m^{31} + 9987576898907013120\, k^{6} m^{31} - \\
&62449229956637767237632\, m^{32} + 6387061696026441154560\, k\, m^{32} + 115312248115204963958784\, k^{2} m^{32} - \\
&4017872935777537622016\, k^{3} m^{32} - 653276292238465302528\, k^{4} m^{32} + 5323021911783899136\, k^{5} m^{32} - \\
&5485383710108911927296\, m^{33} + 335252109148972646400\, k\, m^{33} + 9837472674516534558720\, k^{2} m^{33} - \\
&207190963960846221312\, k^{3} m^{33} - 18114739899998404608\, k^{4} m^{33} - 304777274959054503936\, m^{34} + \\
&8383898537523412992\, k\, m^{34} + 533303026163461914624\, k^{2} m^{34} - 5120684935279017984\, k^{3} m^{34} - \\
&8053346106177748992\, m^{35} + 13805736182018998272\, k^{2} m^{35} \big)
\end{aligned}
$$

$$
\begin{aligned}
p_2(k,m) = {} & (2 + k - 2\, m)\, (-1 + k + 2\, m)\, (k + 2\, m)\, \big(1062475364966400000\, k + 6824384666426880000\, k^{2} + 5646425649798528000\, k^{3} - \\
&8929069272758880000\, k^{4} - 9635247752321822400\, k^{5} + 3584048710725403200\, k^{6} + 4929699282417655200\, k^{7} - \\
&354493390899790800\, k^{8} - 1011803422083734700\, k^{9} - 80536103643536100\, k^{10} + 68860903392212400\, k^{11} + \\
&19440642266123400\, k^{12} + 2528415763587000\, k^{13} - 1316426473517400\, k^{14} - 478440551912400\, k^{15} + 15019629735600\, k^{16} + \\
&15353628059700\, k^{17} + 1564852470300\, k^{18} + 14158216800\, k^{19} - 45158488200\, k^{20} - 4200789600\, k^{21} - \\
&2124950729932800000\, m + 27512626696289280000\, k\, m + 20305236148994419200 0\, k^{2} m + 16320940644683992320 0\, k^{3} m - \\
&24817487956073575200 0\, k^{4} m - 258610186249331152320\, k^{5} m + 90858331913140337760\, k^{6} m + 124865434290225234960\, k^{7} m - \\
&7482957351126399540\, k^{8} m - 23981048849405763360\, k^{9} m - 1999004844109641480\, k^{10} m + 1478485992540510720\, k^{11} m + \\
&425938510033225920\, k^{12} m + 63585689137480800\, k^{13} m - 26664921995368320\, k^{14} m - 10154738374899120\, k^{15} m + \\
&245250634494780\, k^{16} m + 300666855497760\, k^{17} m + 31342907300040\, k^{18} m + 396612750240\, k^{19} m - 833110949160\, k^{20} m - \\
&75740222880\, k^{21} m - 9082259497801728000 0\, m^{2} + 28199958990423091200 0\, k\, m^{2} + 2828082238824491942400\, k^{2} m^{2} + \\
&2188922737148451705600\, k^{3} m^{2} - 3133812661621661544480\, k^{4} m^{2} - 3113085917827500960240\, k^{5} m^{2} + \\
&1016715383306113458600\, k^{6} m^{2} + 1393056960885522112500\, k^{7} m^{2} - 63609686167572039240\, k^{8} m^{2} - \\
&245826712761423220890\, k^{9} m^{2} - 21809782802137799430\, k^{10} m^{2} + 13201390782452901420\, k^{11} m^{2} + \\
&3979871041461889320\, k^{12} m^{2} + 703050078423426480\, k^{13} m^{2} - 223888655094708240\, k^{14} m^{2} - 91352068193989440\, k^{15} m^{2} + \\
&1353892716060300\, k^{16} m^{2} + 2424643822784970\, k^{17} m^{2} + 262413574538670\, k^{18} m^{2} + 4505609041920\, k^{19} m^{2} - \\
&6216355413900\, k^{20} m^{2} - 546560410320\, k^{21} m^{2} - 17374748552301634560 00\, m^{3} + 10831541028769211904 00\, k\, m^{3} + \\
&24759629740935930677760\, k^{2} m^{3} + 18157357277649742901760\, k^{3} m^{3} - 24101718727685791831200\, k^{4} m^{3} - \\
&22278620828772269895312\, k^{5} m^{3} + 6687481462605828704496\, k^{6} m^{3} + 9028470375113342825916\, k^{7} m^{3} - \\
&263496285165384979854\, k^{8} m^{3} - 1427471888906574335076\, k^{9} m^{3} - 138148477656741437568\, k^{10} m^{3} + \\
&62148366198794228772\, k^{11} m^{3} + 20803904634986919852\, k^{12} m^{3} + 4523873191023960660\, k^{13} m^{3} - \\
&1005601836198086652\, k^{14} m^{3} - 457214972605059792\, k^{15} m^{3} + 1247823580036398\, k^{16} m^{3} + 10412060726966496\, k^{17} m^{3} + \\
&1200096344926764\, k^{18} m^{3} + 27283455042144\, k^{19} m^{3} - 24065265816396\, k^{20} m^{3} - 2009836648368\, k^{21} m^{3} -
\end{aligned}
$$

$2007834370836702796800 0 m^4 - 5399936632204430791680\, k\, m^4 + 15375960061501359729024 0 k^2 m^4+$
$1040992831701885077712 00 k^3 m^4 - 1273999184968460212802 88 k^4 m^4 - 1048160610572583150514 32 k^5 m^4+$
$289595590155973149893 44 k^6 m^4 + 371525772457934294978 94 k^7 m^4 - 389878868304215691560\, k^8 m^4-$
$5034661445157758678363\, k^9 m^4 - 555311853044009980865\, k^{10} m^4 + 145789116238537952170\, k^{11} m^4+$
$65295335959151256440\, k^{12} m^4 + 18711640297789591408\, k^{13} m^4 - 2464618138479236512\, k^{14} m^4-$
$1370372876624944912\, k^{15} m^4 - 19508356765249318\, k^{16} m^4 + 24495525834416651\, k^{17} m^4 + 3189594531981153\, k^{18} m^4+$
$95555769523808\, k^{19} m^4 - 48386593327914\, k^{20} m^4 - 3659751480024\, k^{21} m^4 - 159055534154705374817280\, m^5-$
$984660072664948622899 20\, k\, m^5 + 720304448483130891129216\, k^2 m^5 + 432536468577020658834624\, k^3 m^5-$
$492615497491543875714288\, k^4 m^5 - 330848841001791331396160\, k^5 m^5 + 87261445348108425958016\, k^6 m^5+$
$95434591266873559794016\, k^7 m^5 + 1064250707387694415805\, k^8 m^5 - 9784539530362179992507\, k^9 m^5-$
$1387837795696561521367\, k^{10} m^5 - 123168469155409370 20\, k^{11} m^5 + 110922469641587994386\, k^{12} m^5+$
$50709249845851592526\, k^{13} m^5 - 2043585698686520274\, k^{14} m^5 - 2249607654510707060\, k^{15} m^5-$
$105742918966806611\, k^{16} m^5 + 21239413713229653\, k^{17} m^5 + 4241093012038169\, k^{18} m^5 + 183786602068520\, k^{19} m^5-$
$23832561782012\, k^{20} m^5 - 684993499712\, k^{21} m^5 - 927255049639379535075840\, m^6 - 676869888891758545939968\, k\, m^6+$
$2621409047145363151357056\, k^2 m^6 + 1304673127287796854311424\, k^3 m^6 - 1430633436834732246117128\, k^4 m^6-$
$634524024050482266133068\, k^5 m^6 + 187223757774271593003990\, k^6 m^6 + 110271877585745724879461\, k^7 m^6+$
$4905540985645281078603\, k^8 m^6 - 7046012420386169046 11\, k^9 m^6 - 1510399113227826200349\, k^{10} m^6-$
$1295529411510123995683\, k^{11} m^6 - 3376091807586011295\, k^{12} m^6 + 8155710990598997657\, k^{13} m^6+$
$7137010373218083635\, k^{14} m^6 - 247996424891972786\, k^{15} m^6 - 257414596050712648\, k^{16} m^6 - 53168790146242838\, k^{17} m^6-$
$1766527186676856\, k^{18} m^6 + 88468299228060\, k^{19} m^6 + 133776076185904\, k^{20} m^6 + 12731027621824\, k^{21} m^6-$
$4164432963453328335049728\, m^7 - 2948405201386879574029824\, k\, m^7 + 7436207565053782192917120\, k^2 m^7+$
$2635670566535711825178880\, k^3 m^7 - 3060240345576518658127920\, k^4 m^7 - 2081779457291003938395 48\, k^5 m^7+$
$281301849637779861534872\, k^6 m^7 - 229021709771843056255043\, k^7 m^7 - 3827908127793656088498\, k^8 m^7+$
$6130018968369034332075 5\, k^9 m^7 + 3523221556849540967785\, k^{10} m^7 - 4768653938350786871861\, k^{11} m^7-$
$589675864007734995431\, k^{12} m^7 + 233048521037078419 67\, k^{13} m^7 + 30576316599066269923\, k^{14} m^7+$
$9167707925434694212\, k^{15} m^7 - 283864497981806875\, k^{16} m^7 - 230051727670419850\, k^{17} m^7-$
$20871415168409376\, k^{18} m^7 - 515456176743912\, k^{19} m^7 + 414740386409120\, k^{20} m^7 + 33251233684736\, k^{21} m^7-$
$14879348597707865356810752\, m^8 - 8807034349849513874504832\, k\, m^8 + 15896060671358349109507776\, k^2 m^8+$
$1965299263792810737982432\, k^3 m^8 - 4114271022827675783403896\, k^4 m^8 + 340069360324139142936 8648\, k^5 m^8+$
$244252506537178360041464\, k^6 m^8 - 1527911134646169054211153\, k^7 m^8 - 844265849427679593090 05\, k^8 m^8+$
$221375864435273383755753\, k^9 m^8 + 21890281890295021958365\, k^{10} m^8 - 971118939356351734984 9\, k^{11} m^8-$
$186356823378222202220 9\, k^{12} m^8 - 273346566205065987489\, k^{13} m^8 + 60041405784056253687\, k^{14} m^8+$
$27562236691503456126\, k^{15} m^8 + 166830501064313194\, k^{16} m^8 - 442213780782589732\, k^{17} m^8-$
$49385813827476048\, k^{18} m^8 - 1641847567767968\, k^{19} m^8 + 648539177866208\, k^{20} m^8 + 46410104770688\, k^{21} m^8-$
$43319972199399962978098944\, m^9 - 17020510036074653254113024\, k\, m^9 + 22119445439208660738703296\, k^2 m^9-$
$959567287834531729183596 8\, k^3 m^9 + 5668562169781910598984 80\, k^4 m^9 + 1446378525166250080236980 0\, k^5 m^9-$
$169156972461284717975580\, k^6 m^9 - 432885444881431167730071 7\, k^7 m^9 - 341628397372958234128973\, k^8 m^9+$
$468604462926444954562239\, k^9 m^9 + 60085590609243864162903\, k^{10} m^9 - 11736209532297470809687\, k^{11} m^9-$
$3536230390155950118831\, k^{12} m^9 - 891985102415342450907\, k^{13} m^9 + 71796774555774282281\, k^{14} m^9+$
$48716711702475640160\, k^{15} m^9 + 1211672853730065832\, k^{16} m^9 - 549544308666315400\, k^{17} m^9-$
$71655828840431616\, k^{18} m^9 - 2676671721483968\, k^{19} m^9 + 659317852002624\, k^{20} m^9 + 41904407491584\, k^{21} m^9-$
$104698629186580592850692736\, m^{10} - 11487895384235348253847104\, k\, m^{10} + 24823613769320208130910 40\, k^2 m^{10}-$
$5151975185598577643570945 6\, k^3 m^{10} + 2316462002051391799413130 4\, k^4 m^{10} + 3699357161632148475692726 8\, k^5 m^{10}-$
$1539708382009683519442038\, k^6 m^{10} - 832774650070696424033954 6\, k^7 m^{10} - 839138378536689519308310\, k^8 m^{10}+$
$682135290818509871258890\, k^9 m^{10} + 112842131538383964461134\, k^{10} m^{10} - 4774271655248121697346\, k^{11} m^{10}-$
$4756699905723786035562\, k^{12} m^{10} - 1683141307256057730422\, k^{13} m^{10} + 47448296237457023328\, k^{14} m^{10}+$
$61156755642706752004\, k^{15} m^{10} + 2383343866811206272\, k^{16} m^{10} - 477120108279390992\, k^{17} m^{10}-$
$73144538731073280\, k^{18} m^{10} - 2865868019223872\, k^{19} m^{10} + 463609986081792\, k^{20} m^{10} + 25585130452992\, k^{21} m^{10}-$
$213179534026117198336341504\, m^{11} + 56452916539792803826719936\, k\, m^{11} - 9488459110240645675526880 0\, k^2 m^{11}-$
$15213046388293661693754862 4\, k^3 m^{11} + 8342265537802206821840646 4\, k^4 m^{11} + 7049365692408777492393578 0\, k^5 m^{11}-$
$5115629567586273374519772\, k^6 m^{11} - 118895570146779856606960 16\, k^7 m^{11} - 1458869117910620337780028\, k^8 m^{11}+$
$680088243993232094237576\, k^9 m^{11} + 159135528167320721067060\, k^{10} m^{11} + 12718037035508421937544\, k^{11} m^{11}-$
$4712129431746745163588\, k^{12} m^{11} - 2277876868851651919292\, k^{13} m^{11} - 3183929249339463928\, k^{14} m^{11}+$
$57747456192005900104\, k^{15} m^{11} + 2938336984470804128\, k^{16} m^{11} - 292353360444877792\, k^{17} m^{11}-$
$54756322015283968\, k^{18} m^{11} - 2139148391574656\, k^{19} m^{11} + 228278717815808\, k^{20} m^{11} + 10525174956032\, k^{21} m^{11}-$
$369969854595824004340795776\, m^{12} + 273958101161846023856531328\, k\, m^{12} - 34432245914727845636946355 2\, k^2 m^{12}-$
$334575890397503269748012144\, k^3 m^{12} + 20148516131919274943959615 2\, k^4 m^{12} + 10683188697133936808868650 4\, k^5 m^{12}-$
$12577253645246935440376096\, k^6 m^{12} - 12774088802944868938984584\, k^7 m^{12} - 1878868127247998359934232\, k^8 m^{12}+$
$374245398539172502538024\, k^9 m^{12} + 173994226124448606636096\, k^{10} m^{12} + 33765237053460954623496\, k^{11} m^{12}-$
$3372088710348461479136\, k^{12} m^{12} - 2355018003311930637792\, k^{13} m^{12} - 48258673130457948128\, k^{14} m^{12}+$
$41903718458143322944\, k^{15} m^{12} + 2575080136692771648\, k^{16} m^{12} - 121283723563233984\, k^{17} m^{12}-$
$30353390666734080\, k^{18} m^{12} - 1125102693918720\, k^{19} m^{12} + 77458816557056\, k^{20} m^{12} + 2799584116736\, k^{21} m^{12}-$
$552127043571170657890837248\, m^{13} + 733544564186047490987029632\, k\, m^{13} - 80593191218143060669631635 2\, k^2 m^{13}-$
$592941481121519046926842736\, k^3 m^{13} + 38143924559864178280005710 4\, k^4 m^{13} + 13249459468004641470275928 0\, k^5 m^{13}-$
$24559555566321883867372688\, k^6 m^{13} - 9859955480942003755700032\, k^7 m^{13} - 1787927245009394482074992\, k^8 m^{13}-$
$1181819952114666333436 48\, k^9 m^{13} + 148506313850827318196304\, k^{10} m^{13} + 47200227420414785910176\, k^{11} m^{13}-$

$1538099058555847361392\, k^{12} m^{13} - 1908124430273842834112\, k^{13} m^{13} - 638379849564876663536\, k^{14} m^{13} +$

$23475462392376943232\, k^{15} m^{13} + 1669377191070678528\, k^{16} m^{13} - 28697554770406016\, k^{17} m^{13} -$

$12344972829412608\, k^{18} m^{13} - 409794205151232\, k^{19} m^{13} + 17275403419648\, k^{20} m^{13} + 435095339008\, k^{21} m^{13} -$

$712728598768829394102677376\, m^{14} + 1456474589251330762565325504\, k\, m^{14} - 1466422017178514198222930208\, k^2 m^{14} -$

$875359831433348650318241504\, k^3 m^{14} + 594545329058755923279167584\, k^4 m^{14} + 136316123366591997347811104\, k^5 m^{14} -$

$38782020653390706741476672\, k^6 m^{14} - 42847698889005261771087 68\, k^7 m^{14} - 1171054703462465481030112\, k^8 m^{14} -$

$546196374684305494745568\, k^9 m^{14} + 97541032848963032908800\, k^{10} m^{14} + 4677549027873035743017 6\, k^{11} m^{14} -$

$1346393425140331 63552\, k^{12} m^{14} - 1223714472544008086304\, k^{13} m^{14} - 51836718991300727808\, k^{14} m^{14} +$

$10079744766846916480\, k^{15} m^{14} + 808401596118431744\, k^{16} m^{14} - 25561599278336\, k^{17} m^{14} -$

$3583061454618624\, k^{18} m^{14} - 98474135339008\, k^{19} m^{14} + 2280499707904\, k^{20} m^{14} + 30006575104\, k^{21} m^{14} -$

$7979069032795166554136862 72\, m^{15} + 2321067520803657343515028416\, k\, m^{15} - 2199044784735735183131439168\, k^2 m^{15} -$

$1095063628515260579987490176\, k^3 m^{15} + 779519849993325101848005440\, k^4 m^{15} +$

$116902854620779245487798016\, k^5 m^{15} - 50084651768762345999799232\, k^6 m^{15} + 12780227797866052845489 28\, k^7 m^{15} -$

$352476206270460840896832\, k^8 m^{15} - 720703363931155249538240\, k^9 m^{15} + 46792002984732602482112\, k^{10} m^{15} +$

$35410805515248735933120\, k^{11} m^{15} + 465507293062149421120\, k^{12} m^{15} - 621599071972208955264\, k^{13} m^{15} -$

$30033390927561415680\, k^{14} m^{15} + 3252616049522963200\, k^{15} m^{15} + 289878604096116992\, k^{16} m^{15} +$

$2595870943891456\, k^{17} m^{15} - 702613367808000\, k^{18} m^{15} - 14044952559616\, k^{19} m^{15} + 135029587968\, k^{20} m^{15} -$

$773695804783751657854047360\, m^{16} + 3072554991347213336865524352\, k\, m^{16} - 2790262389419993640071943168\, k^2 m^{16} -$

$1171971868791346924106731136\, k^3 m^{16} + 869130607623337815171420800\, k^4 m^{16} + 83333578411176238042535296\, k^5 m^{16} -$

$53384633704041751302298240\, k^6 m^{16} + 4574960431597348624168576\, k^7 m^{16} + 2677987576411699733285 12\, k^8 m^{16} -$

$636905537858979650997376\, k^9 m^{16} + 135427792916891376264 96\, k^{10} m^{16} + 21046053829658603948672\, k^{11} m^{16} +$

$473952488392931143936\, k^{12} m^{16} - 248402705594825068288\, k^{13} m^{16} - 12895503150929662208\, k^{14} m^{16} +$

$760068870393321984\, k^{15} m^{16} + 74905108441985536\, k^{16} m^{16} + 916330988728320\, k^{17} m^{16} - 83380770570240\, k^{18} m^{16} -$

$900197253120\, k^{19} m^{16} - 645823567385721305004847104\, m^{17} + 3440979671003167347365971968\, k\, m^{17} -$

$3037369745130890301700480512\, k^2 m^{17} - 1078736447842714470836125952\, k^3 m^{17} +$

$828888637105135237011033088\, k^4 m^{17} + 48762275447918401150373120\, k^5 m^{17} - 47288737347473870414837248\, k^6 m^{17} +$

$5112174574854490369834240\, k^7 m^{17} + 505650503158117687527680\, k^8 m^{17} - 425819385177270229361408\, k^9 m^{17} -$

$609079339604449537792\, k^{10} m^{17} + 990637181989548814464 0\, k^{11} m^{17} + 277467879416717070080\, k^{12} m^{17} -$

$7696296290591639526 4\, k^{13} m^{17} - 4107987004472206592\, k^{14} m^{17} + 119995865267120128\, k^{15} m^{17} +$

$13209111970154496\, k^{16} m^{17} + 148756658372608\, k^{17} m^{17} - 4519740375040\, k^{18} m^{17} - 458035258659993287724358656\, m^{18} +$

$3295060876052684103954576384\, k\, m^{18} - 2858247440401982190905550336\, k^2 m^{18} - 856069506676856618043210752\, k^3 m^{18} +$

$678120277743710606471519232\, k^4 m^{18} + 22708691647714504809216000\, k^5 m^{18} - 34964594008783143683442176\, k^6 m^{18} +$

$3899285907761426333721088\, k^7 m^{18} + 443953362473898125879808\, k^8 m^{18} - 223872799227132272993792\, k^9 m^{18} -$

$3286751983156166058496\, k^{10} m^{18} + 3684984807856828498432\, k^{11} m^{18} + 113703149692260170240\, k^{12} m^{18} -$

$18037303059619402240\, k^{13} m^{18} - 948588366960537600\, k^{14} m^{18} + 11000547767452672\, k^{15} m^{18} +$

$1424066106867712\, k^{16} m^{18} + 9928425537536\, k^{17} m^{18} - 269061541634634256821307392\, m^{19} +$

$2714914696188626354787600384\, k\, m^{19} - 2334268097836834814856935424\, k^2 m^{19} - 585941329057602753730257920\, k^3 m^{19} +$

$476258090666158479274180608\, k^4 m^{19} + 7755825201017352083281920\, k^5 m^{19} - 21617275048669707664794624\, k^6 m^{19} +$

$2280163020019475139201024\, k^7 m^{19} + 272640725182633078547456\, k^8 m^{19} - 93715644896222827773952\, k^9 m^{19} -$

$2116688757435671557120\, k^{10} m^{19} + 1071212027969240137728\, k^{11} m^{19} + 33867033314970645504\, k^{12} m^{19} -$

$3072223792694639616\, k^{13} m^{19} - 150607575307532288\, k^{14} m^{19} + 335394859616256\, k^{15} m^{19} + 70830520532992\, k^{16} m^{19} -$

$123849155830887791836139520\, m^{20} + 1930985057385499239380865024\, k\, m^{20} - 1656696605669179352489453568\, k^2 m^{20} -$

$345400527904790673168109568\, k^3 m^{20} + 286847816112157193811195904\, k^4 m^{20} + 1371016124992174674618368\, k^5 m^{20} -$

$11165855073636103491624960\, k^6 m^{20} + 1060065160420137505757184\, k^7 m^{20} + 127137018685384483264512\, k^8 m^{20} -$

$31214333166111229544448\, k^9 m^{20} - 840553888773083533312\, k^{10} m^{20} + 238207764386560595968\, k^{11} m^{20} +$

$7255600533304172544\, k^{12} m^{20} - 355564270234804224\, k^{13} m^{20} - 14737365532606464\, k^{14} m^{20} -$

$17029669076992\, k^{15} m^{20} - 37751700416268820180967424\, m^{21} + 1186731206697376247056146432\, k\, m^{21} -$

$1021195545046089426863308800\, k^2 m^{21} - 174799507899118388709601280\, k^3 m^{21} + 147751067658788853378203648\, k^4 m^{21} -$

$4003279917680931436011 52\, k^5 m^{21} - 4800451417091381063720960\, k^6 m^{21} + 396157367791168457023488\, k^7 m^{21} +$

$46088116577417605885952\, k^8 m^{21} - 8185813633046938746880\, k^9 m^{21} - 233740052520256950272\, k^{10} m^{21} +$

$39093025980418195456\, k^{11} m^{21} + 1066389750196170752\, k^{12} m^{21} - 24645250162925568\, k^{13} m^{21} -$

$670776591355904\, k^{14} m^{21} - 4754300850636749897564 16\, m^{22} + 629555094950978895631663104\, k\, m^{22} -$

$545422417402888702823276544\, k^2 m^{22} - 75568686124927019314479104\, k^3 m^{22} + 64784018341285715457081344\, k^4 m^{22} -$

$475948484730965083701248\, k^5 m^{22} - 1705826201679518347001856\, k^6 m^{22} + 118799291730551219986432\, k^7 m^{22} +$

$12995856235901594001408\, k^8 m^{22} - 1655211703194565255168\, k^9 m^{22} - 46085908520596127744\, k^{10} m^{22} +$

$4454367209639469056\, k^{11} m^{22} + 96644281117384704\, k^{12} m^{22} - 752044945924096\, k^{13} m^{22} +$

$857203253722075728612556 8\, m^{23} + 287443952731143645103521792\, k\, m^{23} - 251406584809303064471470080\, k^2 m^{23} -$

$27711407212764666960035840\, k^3 m^{23} + 24016468292459132552544256\, k^4 m^{23} - 238766852013077063991296\, k^5 m^{23} -$

$495513622208011995136000\, k^6 m^{23} + 28265572157804876218368\, k^7 m^{23} + 2808747960261851824128\, k^8 m^{23} -$

$249092805699709616128\, k^9 m^{23} - 6198559770973487104\, k^{10} m^{23} + 313897123361964032\, k^{11} m^{23} +$

$4082168555880448\, k^{12} m^{23} + 678169840298629909118976 0\, m^{24} + 112401968631001206927261696\, k\, m^{24} -$

$$
\begin{aligned}
&9943279507184961482067148\,8\,k^2 m^{24} - 8536950076500262674759680\,k^3 m^{24} + 7456294235454414159577088\,k^4 m^{24} - \\
&8246726895982901379072\,0\,k^5 m^{24} - 1157389000420218701414\,40\,k^6 m^{24} + 52221537101093058969\,60\,k^7 m^{24} + \\
&450887990636717703168\,k^8 m^{24} - 26264108535121346560\,k^9 m^{24} - 513269809387765760\,k^{10} m^{24} + \\
&10267531718197248\,k^{11} m^{24} + 3399468189161011072598016\,m^{25} + 3737437199528890673568153\,6\,k\,m^{25} - \\
&3347975736094245969749606\,4\,k^2 m^{25} - 2180795908310801661493248\,k^3 m^{25} + 1913431288687660802506752\,k^4 m^{25} - \\
&212357259686038565683\,20\,k^5 m^{25} - 21210565895081048014848\,k^6 m^{25} + 722808747440255139840\,k^7 m^{25} + \\
&50759998899985317888\,k^8 m^{25} - 1730173876847640576\,k^9 m^{25} - 19821713375625216\,k^{10} m^{25} + \\
&1273404315084777264513024\,m^{26} + 10462248427314118546685952\,k\,m^{26} - 9497174566948263880163328\,k^2 m^{26} - \\
&4538136443875121768693\,76\,k^3 m^{26} + 39853429963828910594457\,6\,k^4 m^{26} - 41195610470039051304\,96\,k^5 m^{26} - \\
&2937337762729925541888\,k^6 m^{26} + 70538771573435793408\,k^7 m^{26} + 3582294594944237568\,k^8 m^{26} - \\
&53534775652909056\,k^9 m^{26} + 37103105029116146850201\,6\,m^{27} + 2432297805856211260145664\,k\,m^{27} - \\
&223824585347149495900569\,6\,k^2 m^{27} - 75041503800895435702272\,k^3 m^{27} + 6564534845619756859392\,0\,k^4 m^{27} - \\
&589268379220578140160\,k^5 m^{27} - 28901519145212313600\,0\,k^6 m^{27} + 4327541119934005248\,k^7 m^{27} + \\
&119415576592908288\,k^8 m^{27} + 84503658742698577231872\,m^{28} + 460908337658158660976640\,k\,m^{28} - \\
&430024753233815133487104\,k^2 m^{28} - 95094299255973945016\,32\,k^3 m^{28} + 8225897272553751183360\,k^4 m^{28} - \\
&58931437023305662464\,k^5 m^{28} - 18007162907501002752\,k^6 m^{28} + 125485156572069888\,k^7 m^{28} + \\
&1483613597289390538752\,0\,m^{29} + 6932962648788559095398\,4\,k\,m^{29} - 6558431790505907139379\,2\,k^2 m^{29} - \\
&872790448947981189120\,k^3 m^{29} + 736541088467156729856\,k^4 m^{29} - 3689739663534194688\,k^5 m^{29} - \\
&534112775917535232\,k^6 m^{29} + 1945180947224199168000\,m^{30} + 7959175277275804336128\,k\,m^{30} - \\
&763391407421934010368\,0\,k^2 m^{30} - 525954086987020369\,92\,k^3 m^{30} + 41962396448779665408\,k^4 m^{30} - \\
&109033691692400640\,k^5 m^{30} + 179688186585192333312\,m^{31} + 654706636211179487232\,k\,m^{31} - \\
&636687445408100646912\,k^2 m^{31} - 1690659020769067008\,k^3 m^{31} + 1142758643683295232\,k^4 m^{31} + \\
&10445029056788299776\,m^{32} + 34345680938018537472\,k\,m^{32} - 33867196571515355136\,k^2 m^{32} - \\
&141165332445265\,92\,k^3 m^{32} + 287619503792062464\,m^{33} + 862858511376187392\,k\,m^{33} - 862858511376187392\,k^2 m^{33} \Big)
\end{aligned}
$$

Appendix D. The certificate polynomials of $A_0(m)$

$$
\begin{aligned}
p_0(k,m) = {} & 108944415108232704000000 - 8890013394441724608000000\,k - 46903047673711680460800000\,k^2 - \\
& 154635707232024380308800 0\,k^3 + 128627993127363371107200 0\,k^4 - 548799703049891154960000\,k^5 - \\
& 42798245057299088627760 0\,k^6 + 498527413138545435073800\,k^7 + 251157706887242334803400\,k^8 - \\
& 56387104718947236284400\,k^9 - 57111445894533944180400\,k^{10} - 1764802243003052710800\,k^{11} + \\
& 5453489995304525269200\,k^{12} + 561043499497368451200\,k^{13} - 189008718749027755200\,k^{14} - 33669234551435351400\,k^{15} - \\
& 2175654522204009000\,k^{16} + 925366890732580800\,k^{17} + 25499246214355200 0\,k^{18} - 8583930975690000\,k^{19} - \\
& 3740281940833200\,k^{20} - 135388148187600\,k^{21} - 2030481658800\,k^{22} + 2033182166400\,k^{23} + 145227297600\,k^{24} + \\
& 350389909882708762752000 00\,m - 234126044145423605145600 00\,k\,m - 129139394799275879995872000\,k^2\,m - \\
& 38215651046565653575286400\,k^3\,m + 29542927184520462943221600\,k^4\,m - 19156170222278741904089040\,k^5\,m - \\
& 7894240440588132814191960\,k^6\,m + 1426802836959455315725638 0\,k^7\,m + 5517845268816203099084700\,k^8\,m - \\
& 1559886305890444056995400\,k^9\,m - 1283056454535415192430280\,k^{10}\,m - 39779881613690680294560\,k^{11}\,m + \\
& 117759232574803544674320\,k^{12}\,m + 14493477164030282822040\,k^{13}\,m - 3178019256389487266400\,k^{14}\,m - \\
& 934409961612530153100\,k^{15}\,m - 14047949020368183606 0\,k^{16}\,m + 26285582694052997280\,k^{17}\,m + 9432044926150200480\,k^{18}\,m - \\
& 70662336373646640\,k^{19}\,m - 144872661863870640\,k^{20}\,m - 12996621951374880\,k^{21}\,m - 199896062919840\,k^{22}\,m + \\
& 192586506124320\,k^{23}\,m + 13709498806080\,k^{24}\,m + 538916077696101221975040000\,m^2 - 2667010893208624198696320 00\,k\,m^2 - \\
& 169860771398055933120562560 0\,k^2\,m^2 - 49275189732890514894891264 0\,k^3\,m^2 + 32356706183268918655145760 0\,k^4\,m^2 - \\
& 2532918981325818543816556 80\,k^5\,m^2 - 5794386078079820693638152 0\,k^6\,m^2 + 1709105716384991805459398 40\,k^7\,m^2 + \\
& 528407619888572266239104 40\,k^8\,m^2 - 1671929298748890643167216 0\,k^9\,m^2 - 1224531596344121215705044 0\,k^{10}\,m^2 - \\
& 57544532828637033883536 0\,k^{11}\,m^2 + 103909412124769840921836 0\,k^{12}\,m^2 + 15463865143545768134388 0\,k^{13}\,m^2 - \\
& 1982515084505867157372 0\,k^{14}\,m^2 - 9076855778362378581120\,k^{15}\,m^2 - 1767564251088892666200\,k^{16}\,m^2 + \\
& 22774437461770173144 0\,k^{17}\,m^2 + 94375826607414786840\,k^{18}\,m^2 + 363148976538855360\,k^{19}\,m^2 - \\
& 1320410505856395240\,k^{20}\,m^2 - 135680181142083480\,k^{21}\,m^2 - 2508122290219560\,k^{22}\,m^2 + 1791113061517920\,k^{23}\,m^2 + \\
& 123530022635040\,k^{24}\,m^2 + 524709046225522315637337600 0\,m^3 - 165079339809122307169353600 0\,k\,m^3 - \\
& 1417490436286742177199339072 0\,k^2\,m^3 - 437583023303012434790113982 4\,k^3\,m^3 + 218979098928037076850125172 0\,k^4\,m^3 - \\
& 190625993092759911622076905 2\,k^5\,m^3 - 147641904166527854631443466\,k^6\,m^3 + 122172966377712509572249392 6\,k^7\,m^3 + \\
& 296494595332786108159335378\,k^8\,m^3 - 103175201532197268780158988\,k^9\,m^3 - 68770327668875486110176588\,k^{10}\,m^3 - \\
& 5074846891395951248455140\,k^{11}\,m^3 + 5244549436338994764249912\,k^{12}\,m^3 + 968326537597092391343580\,k^{13}\,m^3 - \\
& 51235172735193713577942\,k^{14}\,m^3 - 48964507460194358074554\,k^{15}\,m^3 - 1134047600388847811802 6\,k^{16}\,m^3 + \\
& 1032043572391324676652\,k^{17}\,m^3 + 497399304012100755372\,k^{18}\,m^3 + 6509593278190426104\,k^{19}\,m^3 - \\
& 6125324048331557256\,k^{20}\,m^3 - 684589335214891392\,k^{21}\,m^3 - 14804465940860256\,k^{22}\,m^3 + 7887396752334288\,k^{23}\,m^3 + \\
& 523679403605472\,k^{24}\,m^3 + 3621238458538531101867106560 0\,m^4 - 4852384576620644099371564800\,k\,m^4 - \\
& 84098862447459445513499713632\,k^2\,m^4 - 29513992989257590372476451488\,k^3\,m^4 + 9866186938879805213569549008\,k^4\,m^4 - \\
& 9461165139085252787506447632\,k^5\,m^4 + 885241760418292509291946734\,k^6\,m^4 + 6020488479943574986606986960\,k^7\,m^4 + \\
& 1069164723559276550813027628\,k^8\,m^4 - 428519279789176039436757720\,k^9\,m^4 - 257941980290934742661358264\,k^{10}\,m^4 - \\
& 29033307404094402013003620\,k^{11}\,m^4 + 17122476286485805677706464\,k^{12}\,m^4 + 4104444836404049171415660\,k^{13}\,m^4 + \\
& 44189508456071376206070\,k^{14}\,m^4 - 173575338497705591748120\,k^{15}\,m^4 - 46587114378584438654592\,k^{16}\,m^4 + \\
& 2888545218813685780536\,k^{17}\,m^4 + 1687498042263155752632\,k^{18}\,m^4 + 36543979932366115692\,k^{19}\,m^4 - \\
& 17831813214579463356\,k^{20}\,m^4 - 2150269485740844204\,k^{21}\,m^4 - 52892106799619700\,k^{22}\,m^4 + \\
& 21201192898531536\,k^{23}\,m^4 + 1346045398736208\,k^{24}\,m^4 + 188339195113544155501386800640\,m^5 + \\
& 89759042969675383184425061 76\,k\,m^5 - 376623808840479836078652985344\,k^2\,m^5 - 157720236175594521026487182736\,k^3\,m^5 + \\
& 28685030140387031346699779520\,k^4\,m^5 - 32874243807515280784027439184\,k^5\,m^5 + \\
& 11136915906320493665227428570\,k^6\,m^5 + 22081126666165949978593536183\,k^7\,m^5 + 2402136340637942195305375567\,k^8\,m^5 - \\
& 1300467317507883282894046666\,k^9\,m^5 - 682011465508321139875568830\,k^{10}\,m^5 - 116343068232304764580826392\,k^{11}\,m^5 + \\
& 37443025515458182460791144\,k^{12}\,m^5 + 12703022071527504261497766\,k^{13}\,m^5 + 830798149304617534031884\,k^{14}\,m^5 - \\
& 439382756271751694879059\,k^{15}\,m^5 - 135724569026386251046279\,k^{16}\,m^5 + 5250042486892692416072\,k^{17}\,m^5 + \\
& 4051504952481603473428\,k^{18}\,m^5 + 121098560780434549308\,k^{19}\,m^5 - 35799696954219395216\,k^{20}\,m^5 - \\
& 4678465557938349556\,k^{21}\,m^5 - 127542415305927580\,k^{22}\,m^5 + 38650146454707944\,k^{23}\,m^5 + 2328067297171072\,k^{24}\,m^5 + \\
& 766866024052478313081989021568\,m^6 + 180763178710012703868829055616\,k\,m^6 - 131789005278279721741420973884 8\,k^2\,m^6 - \\
& 68281367194553421922869017995 2\,k^3\,m^6 + 36705049239469579068366590184\,k^4\,m^6 - \\
& 79936754713983531635220166812\,k^5\,m^6 + 6300314478328192261708412814 2\,k^6\,m^6 + 6313911302308967804752236180 0\,k^7\,m^6 + \\
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$$

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$857718363001748593689395 2\, k^{10} m^{22} + 1448129522469458587320 32\, k^{11} m^{22} - 2396607414900320002867 2\, k^{12} m^{22} -$
$4661759046649105448 96\, k^{13} m^{22} + 2373220889749252505 6\, k^{14} m^{22} + 5578749133521879 04\, k^{15} m^{22} -$
$1201723011579904\, k^{16} m^{22} - 6637316018622630681608346599424\, m^{23} + 1310306331139086992863720469299 2\, k\, m^{23} +$
$1157608632036729060287373921484 8\, k^{2} m^{23} - 2481768838328844157104753704960\, k^{3} m^{23} -$
$1043570574560176018137329238016\, k^{4} m^{23} + 526486286796402436395823923 20\, k^{5} m^{23} +$
$314832798153757309505705738 24\, k^{6} m^{23} + 2003895647730090923703828 48\, k^{7} m^{23} -$
$344990755233319655234830336\, k^{8} m^{23} - 648628605326136667558707 2\, k^{9} m^{23} + 1464063508277698716467200\, k^{10} m^{23} +$
$2503898072951103938560 0\, k^{11} m^{23} - 2591960292445001646080\, k^{12} m^{23} - 46275609002445520896\, k^{13} m^{23} +$
$1171200535305551872\, k^{14} m^{23} + 16802498673934336\, k^{15} m^{23} - 2740268326011461756888701206528\, m^{24} +$
$4988200198967280966779766964224\, k\, m^{24} + 4518533779475932651292117041152\, k^{2} m^{24} -$
$806979806486336776535483678720\, k^{3} m^{24} - 3342546878015801547125490974 72\, k^{4} m^{24} +$
$138999051325910630036236533 76\, k^{5} m^{24} + 7997750396731772579644178432\, k^{6} m^{24} + 535070426589434870469099 52\, k^{7} m^{24} -$
$66718078736849477992480768\, k^{8} m^{24} - 1175706388926019724443648\, k^{9} m^{24} + 19707116172736080822272 0\, k^{10} m^{24} +$
$3181209920583953088512\, k^{11} m^{24} - 198376254154408951808\, k^{12} m^{24} - 2846986041333710848\, k^{13} m^{24} +$
$26764936664350720\, k^{14} m^{24} - 9800167395297678564699985674 24\, m^{25} + 1664835863733358529229543702528\, k\, m^{25} +$
$1539497444977094663416338186240\, k^{2} m^{25} - 227913414198726336240252354560\, k^{3} m^{25} -$
$92326728187703117752899207168\, k^{4} m^{25} + 3111179393314828890332332032\, k^{5} m^{25} +$
$1716495495066539236259659776\, k^{6} m^{25} + 11843763828070243876667392\, k^{7} m^{25} - 10507038734918652487925760\, k^{8} m^{25} -$
$16815985881358639779020 8\, k^{9} m^{25} + 20124417020450413608960\, k^{10} m^{25} + 282063626247330791424\, k^{11} m^{25} -$
$9559694667950456832\, k^{12} m^{25} - 82105367351197696\, k^{13} m^{25} - 3028054884212352397582962524 16\, m^{26} +$
$4847390636590054984294811566 08\, k\, m^{26} + 4559241501436587316463956131 84\, k^{2} m^{26} -$
$55538654876562595303400407040\, k^{3} m^{26} - 21819570684670002504385167360\, k^{4} m^{26} +$
$583570631232866579932774400\, k^{5} m^{26} + 306972156397800634259996672\, k^{6} m^{26} + 2122390293029006623113216\, k^{7} m^{26} -$
$1312610442727683249537024\, k^{8} m^{26} - 18267008948141414416384\, k^{9} m^{26} + 1464301778374959169536\, k^{10} m^{26} +$
$15648289026082865152\, k^{11} m^{26} - 217619167670829056\, k^{12} m^{26} - 8044690998100393006270827724 8\, m^{27} +$
$122336574627439766870435561472\, k\, m^{27} + 116681894302478329360973561856\, k^{2} m^{27} -$
$11577062066493205497179537408\, k^{3} m^{27} - 4366150061818885852514222080\, k^{4} m^{27} +$
$90310291192164830229037056\, k^{5} m^{27} + 44889919364237824847511552\, k^{6} m^{27} + 299382514394890271195136\, k^{7} m^{27} -$
$125098495213935036727296\, k^{8} m^{27} - 1416274555926568763392\, k^{9} m^{27} + 67590178512416800768\, k^{10} m^{27} +$
$409913326072233984\, k^{11} m^{27} - 18246860522851297037446545408\, m^{28} + 26539617987565953820372500480\, k\, m^{28} +$
$25603558453441745540805033984\, k^{2} m^{28} - 2041338128708637746179080192\, k^{3} m^{28} - 729517764258612849401659392\, k^{4} m^{28} +$
$11286654665311568667869184\, k^{5} m^{28} + 5227152763863498592616448\, k^{6} m^{28} + 31964239954867925286912\, k^{7} m^{28} -$
$8540815814456886951936\, k^{8} m^{28} - 69818318997257453568\, k^{9} m^{28} + 1486394721945255936\, k^{10} m^{28} -$
$3498681656566396379147010048\, m^{29} + 4895701469683796351681298432\, k\, m^{29} + 4767016333177204465886822400\, k^{2} m^{29} -$
$300012549151457936765091840\, k^{3} m^{29} - 99869372328810858160324608\, k^{4} m^{29} + 1105191949876227282567168\, k^{5} m^{29} +$
$465987014374368749813760\, k^{6} m^{29} + 2427073142059601756160\, k^{7} m^{29} - 371843899259168489472\, k^{8} m^{29} -$
$1644669023343869952\, k^{9} m^{29} - 559471079274383826342641664\, m^{30} + 757032665350015701114421248\, k\, m^{30} +$
$742646181750958361979912192\, k^{2} m^{30} - 360283276184638865521401344\, k^{3} m^{30} - 10907585514770831945760768\, k^{4} m^{30} +$
$81063474944326213042176\, k^{5} m^{30} + 29857465519284336721920\, k^{6} m^{30} + 116783556099028549632\, k^{7} m^{30} -$
$7753109437685956608\, k^{8} m^{30} - 73232027522407594010345472\, m^{31} + 96256799877796852515471360\, k\, m^{31} +$
$94986704876462834668535808\, k^{2} m^{31} - 3438738032100931943792640\, k^{3} m^{31} - 913740132514116484988928\, k^{4} m^{31} +$
$4143243258972107440128\, k^{5} m^{31} + 1223810372068901388288\, k^{6} m^{31} + 2677085808686530560\, k^{7} m^{31} -$
$7641857342923057617960960\, m^{32} + 9796789784736798710169600\, k\, m^{32} + 9711985387410783888998400\, k^{2} m^{32} -$
$250455981412646991691776\, k^{3} m^{32} - 55114905190335315443712\, k^{4} m^{32} + 129244885851876360192\, k^{5} m^{32} +$
$24095937072533078016\, k^{6} m^{32} - 611262224776643771105280\, m^{33} + 767115097874629178425344\, k\, m^{33} +$
$763117427617767817740288\, k^{2} m^{33} - 13045995983839651430 40\, k^{3} m^{33} - 2130390820208051748864\, k^{4} m^{33} +$
$1788949758442733568\, k^{5} m^{33} - 35186950223590777159680\, m^{34} + 43372942612316163145728\, k\, m^{34} +$
$43255589965916868182016\, k^{2} m^{34} - 431609076210745737216\, k^{3} m^{34} - 39624447712523452416\, k^{4} m^{34} -$
$1297416893044963147776\, m^{35} + 1575665507628025380864\, k\, m^{35} + 1574079789282078228480\, k^{2} m^{35} -$
$6789935825053286400\, k^{3} m^{35} - 23009560303364997120\, m^{36} + 27611472364037996544\, k\, m^{36} + 27611472364037996544\, k^{2} m^{36}$

$$
\begin{aligned}
p_1(k,m) = &-(k+2\,m)\,\big(-1618150371290931744000000 - 3196932274662397931520000\,k -\\
&1612449917359508752560000\,k^2 - 9424095513868898049600000\,k^3 - 1472576080649181368918400\,k^4 +\\
&775085514572142070920000\,k^5 + 1025932611698242081709400\,k^6 + 327720470575436691517800\,k^7 -\\
&140847170923153443936600\,k^8 - 89250868915173180041400\,k^9 + 2597451244807234359600\,k^{10} +\\
&8799561662464123639200\,k^{11} + 663314184205046413200\,k^{12} - 315266018632720948800\,k^{13} -\\
&52941408188972965200\,k^{14} - 2469231171861350400\,k^{15} + 1644673842421092000\,k^{16} + 375678709779088800\,k^{17} -\\
&16552049450391000\,k^{18} - 5743434312693000\,k^{19} - 206020374550200\,k^{20} - 1085153970600\,k^{21} +\\
&3195000547200\,k^{22} + 217840946400\,k^{23} - 3406866108017188190976000\,m - 6163494582860649681696000\,k\,m -\\
&3592549705176519862017600\,k^2\,m - 4197505885049792078431920\,k^3\,m - 455550813778085608720888\,k^4\,m +\\
&699951101808511905906708\,k^5\,m + 277510503400500494926909\,k^6\,m + 713236774669075563040386\,k^7\,m -\\
&357284079315041754607830\,k^8\,m - 201167897104817161867026\,k^9\,m + 492821991775295243649\,k^{10}\,m +\\
&1912739181300022022955\,k^{11}\,m + 1897784049631098907278\,k^{12}\,m - 556009922066413204530\,k^{13}\,m -\\
&154613399043854573928\,k^{14}\,m - 190630228077651683160\,k^{15}\,m + 48546729680648769360\,k^{16}\,m +\\
&14156082836168938560\,k^{17}\,m - 240724731541525560\,k^{18}\,m - 229928922954816120\,k^{19}\,m - 19782198274224960\,k^{20}\,m -\\
&116435974420080\,k^{21}\,m + 302444030694960\,k^{22}\,m + 20564248209120\,k^{23}\,m - 3311957325992345401392000000\,m^2 -\\
&5292767922920904769991424000\,k\,m^2 - 3774762582337677539400624000\,k^2\,m^2 - 665668444687956343218221280\,k^3\,m^2 -\\
&594155590533566034810182640\,k^4\,m^2 + 1213473150861533472363843600\,k^5\,m^2 + 321037165208086389877094280\,k^6\,m^2 +\\
&701385096933666807924326400\,k^7\,m^2 - 364453137330381670226317200\,k^8\,m^2 - 194471826639017277742597800\,k^9\,m^2 +\\
&4959896902205998309452\,k^{10}\,m^2 + 16968473252879997447282600\,k^{11}\,m^2 + 2165468000387803705175400\,k^{12}\,m^2 -\\
&362506051938383421125400\,k^{13}\,m^2 - 152791306216347948458400\,k^{14}\,m^2 - 25149753808836110823600\,k^{15}\,m^2 +\\
&4250288792768755388400\,k^{16}\,m^2 + 1427782817451722293800\,k^{17}\,m^2 - 543615258174710400\,k^{18}\,m^2 -\\
&2099645803947329820\,k^{19}\,m^2 - 206942952758254020\,k^{20}\,m^2 - 2252259979675020\,k^{21}\,m^2 + 2796490116105840\,k^{22}\,m^2 +\\
&185295033952560\,k^{23}\,m^2 - 1951640073149476224561235200\,m^3 - 2503969924093511369838935040\,k\,m^3 -\\
&2473964966853099585629784720\,k^2\,m^3 - 608664445810574784262207063\,k^3\,m^3 - 465558162170237476961545746\,k^4\,m^3 +\\
&112773518718490475191785630\,k^5\,m^3 + 223856537461969895236763249\,k^6\,m^3 + 415048870854286978579787160\,k^7\,m^3 -\\
&216534496873877219298323484\,k^8\,m^3 - 111063334276748417311838682\,k^9\,m^3 - 320541742495833586269287\,k^{10}\,m^3 +\\
&8596832219979724377623664\,k^{11}\,m^3 + 1415689208299583854104654\,k^{12}\,m^3 - 104389700169981507701706\,k^{13}\,m^3 -\\
&82902444279609889376280\,k^{14}\,m^3 - 1660308969246542935837\,k^{15}\,m^3 + 193655653654993689767\,k^{16}\,m^3 +\\
&75616625804331690023\,k^{17}\,m^3 + 5472267950056774716\,k^{18}\,m^3 - 971083576845684622\,k^{19}\,m^3 -\\
&104482397719078898\,k^{20}\,m^3 - 16564476972322512\,k^{21}\,m^3 + 12231244509471864\,k^{22}\,m^3 + 785519105408208\,k^{23}\,m^3 -\\
&762799463211476220137857152\,m^4 - 543971698861318716188541888\,k\,m^4 - 111215185457487341923790546\,k^2\,m^4 -\\
&382313241356896640989055078\,k^3\,m^4 - 253968279770081350755152529\,k^4\,m^4 +\\
&708669146128820132066947072\,k^5\,m^4 + 108570893689446029911822675\,k^6\,m^4 + 164103296658368508490957290\,k^7\,m^4 -\\
&872171118983953054630147806\,k^8\,m^4 - 425256694695349593083000028\,k^9\,m^4 - 291872262626621548440607\,k^{10}\,m^4 +\\
&2812765304915348957391173\,k^{11}\,m^4 + 614976805563431197804930\,k^{12}\,m^4 - 87993569246680915561\,k^{13}\,m^4 -\\
&293280657665915095747344\,k^{14}\,m^4 - 69451986800363215068126\,k^{15}\,m^4 + 5447625523250486820648\,k^{16}\,m^4 +\\
&2571385416779608787988\,k^{17}\,m^4 + 447691990499027606\,k^{18}\,m^4 - 28104612478233731562\,k^{19}\,m^4 -\\
&3278003866062272802\,k^{20}\,m^4 - 67250625469271190\,k^{21}\,m^4 + 32624155342928040\,k^{22}\,m^4 + 2019068098104312\,k^{23}\,m^4 -\\
&198803745497647177093149552\,m^5 + 119105131108108860978800915\,k\,m^5 - 345635906970814474169681853\,k^2\,m^5 -\\
&180183600794499560841833500680\,k^3\,m^5 - 104664730525026990476285149380\,k^4\,m^5 +\\
&33288747997659062098507952622\,k^5\,m^5 + 39641227193413342546900924521\,k^6\,m^5 +\\
&44683641594528496564544197\,k^7\,m^5 - 2592448271595261833784625011\,k^8\,m^5 - 11573828981253421070102014\,k^9\,m^5 -\\
&13906845845883996423674857\,k^{10}\,m^5 + 615710748862767783747650\,k^{11}\,m^5 + 19210969622445636417473889\,k^{12}\,m^5 +\\
&1160629220374145017033501\,k^{13}\,m^5 - 735276225753434720692740\,k^{14}\,m^5 - 20445468393985941375295\,k^{15}\,m^5 +\\
&999510574491419752867\,k^{16}\,m^5 + 61720382492084726160\,k^{17}\,m^5 + 16676628441713456515\,k^{18}\,m^5 -\\
&559365096082193390\,k^{19}\,m^5 - 7109427325544375466\,k^{20}\,m^5 - 175626076234351026\,k^{21}\,m^5 + 58957241580496780\,k^{22}\,m^5 +\\
&3492100945756608\,k^{23}\,m^5 - 30044260345288243106309444928\,m^6 + 157691679858303619512955978080\,k\,m^6 -\\
&643065291536977546237443623\,k^2\,m^6 - 670974810701808962355373335312\,k^3\,m^6 -\\
&343475913426985112539830047592\,k^4\,m^6 + 123195237608031302212669784384\,k^5\,m^6 +\\
&114623533445913895594784505438\,k^6\,m^6 + 7922825162869615577202899542\,k^7\,m^6 -\\
&6030494300312815574710082378\,k^8\,m^6 - 2274998175497384691620309350\,k^9\,m^6 - 443945563715852226213548102\,k^{10}\,m^6 +\\
&8419446914863263395926238\,k^{11}\,m^6 + 45264403602445669089479586\,k^{12}\,m^6 + 5102424062712183717326986\,k^{13}\,m^6 -\\
&1365901053087606870061316\,k^{14}\,m^6 - 449034568188159841247872\,k^{15}\,m^6 + 11416382345342863787380\,k^{16}\,m^6 +\\
&11028522581274082481228\,k^{17}\,m^6 + 398844595042476889092\,k^{18}\,m^6 - 80399258324999304896\,k^{19}\,m^6 -\\
&11280921629958165020\,k^{20}\,m^6 - 318995499760363720\,k^{21}\,m^6 + 76233496205131360\,k^{22}\,m^6 + 4285030175890752\,k^{23}\,m^6 +\\
&64370466923603944866823238\,m^7 + 7643601559087039496850309043\,k\,m^7 + 9492954335944948897160192496\,k^2\,m^7 -\\
&20401535236746965791096100966\,k^3\,m^7 - 93132645150610405986354933609\,k^4\,m^7 +\\
&37036944843724263850314258195\,k^5\,m^7 + 27197458222378313787755382783\,k^6\,m^7 +\\
&5708124813654516108241223431\,k^7\,m^7 - 1153160546396465651221565785\,k^8\,m^7 - 309826631854670944215512173\,k^9\,m^7 -\\
&10394411983969121846736609\,k^{10}\,m^7 + 3600959268772701657838666\,k^{11}\,m^7 + 829322520754656247705086\,k^{12}\,m^7 +\\
&133118479242139018534125\,k^{13}\,m^7 - 19203341645424561068225\,k^{14}\,m^7 - 762922601555236287712996\,k^{15}\,m^7 +\\
&498613449349053426243\,k^{16}\,m^7 + 151545101526633401140\,k^{17}\,m^7 + 680500647438524914408\,k^{18}\,m^7 -\\
&85491597472898111994\,k^{19}\,m^7 - 1353664004231314902\,k^{20}\,m^7 - 421191674974995976\,k^{21}\,m^7 + 72686560553890848\,k^{22}\,m^7 +\\
&3835950787248384\,k^{23}\,m^7 + 13654923858200965256319887827\,m^8 + 2542959236559146784741956958624\,k\,m^8 +\\
&5930667618890047966218296138\,k^2\,m^8 - 51778253435036328597787801555\,k^3\,m^8 -\\
&214305148871141724972744909482\,k^4\,m^8 + 92236344858259881120247589716\,k^5\,m^8 +\\
&543637006151963133529810950594\,k^6\,m^8 - 160230154940693282315563703\,k^7\,m^8 -
\end{aligned}
$$

$189862083707985170291892127778\,k^{8}\,m^{8} - 225742357665747641412268159\,0\,k^{9}\,m^{8} -$
$186871781995632701163838124 2\,k^{10}\,m^{8} - 1446686472652619001287594 22\,k^{11}\,m^{8} + 120479836212560633619345378\,k^{12}\,m^{8} +$
$24966450819726787111299654\,k^{13}\,m^{8} - 2043054562488282562929100\,k^{14}\,m^{8} - 102692911022980300 0914284\,k^{15}\,m^{8} -$
$9032878106546438295088\,k^{16}\,m^{8} + 1634159160259796186945 6\,k^{17}\,m^{8} + 8711441228585078012 28\,k^{18}\,m^{8} -$
$67586363133133872100\,k^{19}\,m^{8} - 1252534769995122282 4\,k^{20}\,m^{8} - 414675485098780320\,k^{21}\,m^{8} + 51902671075771520\,k^{22}\,m^{8} +$
$2536240435771776\,k^{23}\,m^{8} + 381583868626777498743963656448\,m^{9} + 655961834224334752558818013 9008\,k\,m^{9} +$
$2642965832764567543526867981040\,k^{2}\,m^{9} - 1113828168543836804934548657159 2\,k^{3}\,m^{9} -$
$4265172158579302921152872351324\,k^{4}\,m^{9} + 1928783657237737373412782039274\,k^{5}\,m^{9} +$
$933543053712361591286618918060\,k^{6}\,m^{9} - 735558551575649784894703678 42\,k^{7}\,m^{9} -$
$280484929218263311196612491 82\,k^{8}\,m^{9} + 147380671149622886875840260 4\,k^{9}\,m^{9} -$
$2645752612305350638335355320\,k^{10}\,m^{9} - 453842868626287423097692046\,k^{11}\,m^{9} + 140349159086973033671563716\,k^{12}\,m^{9} +$
$36041242832013477642621616\,k^{13}\,m^{9} - 1589319578935673329038392\,k^{14}\,m^{9} - 1112667509967351211356376\,k^{15}\,m^{9} -$
$23701337465486221425210\,k^{16}\,m^{9} + 13994849921330157589826\,k^{17}\,m^{9} + 860564232631400540692\,k^{18}\,m^{9} -$
$38983557658047866328\,k^{19}\,m^{9} - 9029073729351348944\,k^{20}\,m^{9} - 308420947664255072\,k^{21}\,m^{9} +$
$27888112072671680\,k^{22}\,m^{9} + 1238581163787264\,k^{23}\,m^{9} + 505218696082367932722742970112\,m^{10} +$
$1379480258696397406946779072320 0\,k\,m^{10} + 7713534445526705701342004915664\,k^{2}\,m^{10} -$
$2053349135964249580478139962120 8\,k^{3}\,m^{10} - 7434680859021724849645308512188\,k^{4}\,m^{10} +$
$3421121508640169146480701223006\,k^{5}\,m^{10} + 1396920912984462366087691415454\,k^{6}\,m^{10} -$
$171753911276049550933230797522\,k^{7}\,m^{10} - 381667895967890616142500641 42\,k^{8}\,m^{10} +$
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$$229903600495053225590784\, k\, m^{31} + 45517997544459172388536 32\, k^{2}\, m^{31} - 14517530303136008660582 4\, k^{3}\, m^{31} -$$
$$2406752704342117475942 4\, k^{4}\, m^{31} + 289516745119582126080\, k^{5}\, m^{31} + 9987576898907013120\, k^{6}\, m^{31} -$$
$$20229171776955887910912 0\, m^{32} + 13094563266872148492288\, k\, m^{32} + 3637211145780676106649 60\, k^{2}\, m^{32} -$$
$$815469990330438254592 0\, k^{3}\, m^{32} - 952169500588438978560\, k^{4}\, m^{32} + 5323021911783899136\, k^{5}\, m^{32} -$$
$$1186453261770677865676 8\, m^{33} + 477778384286870667264\, k\, m^{33} + 209572273763707131002 88\, k^{2}\, m^{33} -$$
$$294242607860589527040\, k^{3}\, m^{33} - 1811473989999840460 8\, k^{4}\, m^{33} - 445710831817165111296\, m^{34} +$$
$$8383898537523412992\, k\, m^{34} + 774903409348794384384\, k^{2}\, m^{34} - 5120684935279017984\, k^{3}\, m^{34} -$$
$$8053346106177748992\, m^{35} + 1380573618201899827 2\, k^{2}\, m^{35} \Big)$$

$$p_2(k,m) = (1 + k - 2\, m)\, (k + 2\, m)\, (1 + k + 2\, m)\, \Big(- 6472601485163726976 00000 + 3215378374476915225 60000\, k -$$
$$2259206489959932864960 00\, k^{2} - 3435968966901065400960 00\, k^{3} + 154117686824094018412800\, k^{4} +$$
$$135837221219064175708800\, k^{5} - 39062305025433465804 00\, k^{6} - 25994425539749159959200\, k^{7} -$$
$$3275034055865506481400\, k^{8} + 2380363475501834285400\, k^{9} + 360984269380792314600\, k^{10} - 77848975657439443800\, k^{11} -$$
$$15411942318402844200\, k^{12} - 1485761766687815400\, k^{13} + 331830618073038000\, k^{14} + 130811834187997200\, k^{15} -$$
$$2459087921631600\, k^{16} - 1802418991077600\, k^{17} - 66116077486200\, k^{18} - 1995525088200\, k^{19} +$$
$$943977434400\, k^{20} + 72613648800\, k^{21} - 1398014127322201428480 0000\, m + 10447091895288923392896000\, k\, m -$$
$$9469124380096635304915200\, k^{2}\, m - 9489479421499534208001600\, k^{3}\, m + 5172019515439979087380320\, k^{4}\, m +$$
$$3326731487088351480739440\, k^{5}\, m - 270981910116174952923480\, k^{6}\, m - 593720302262642132016600\, k^{7}\, m -$$
$$67636867852120351348620\, k^{8}\, m + 50360275604814290391780\, k^{9}\, m + 8363494403163203585400\, k^{10}\, m -$$
$$113687854954505892396 0\, k^{11}\, m - 38624776476441857730 0\, k^{12}\, m - 8124294110450520114 0\, k^{13}\, m +$$
$$8367039916017519240\, k^{14}\, m + 4714394889987445320\, k^{15}\, m + 33635701890501480\, k^{16}\, m - 65928072709374120\, k^{17}\, m -$$
$$6346479119520960\, k^{18}\, m - 192959137118160\, k^{19}\, m + 89438503659120\, k^{20}\, m + 6854749403040\, k^{21}\, m -$$
$$14463796589468472738240 0000\, m^{2} + 15048678057005866182336 0000\, k\, m^{2} - 146253858149545333728266880\, k^{2}\, m^{2} -$$
$$117353975995329436879084800\, k^{3}\, m^{2} + 70206429936778706416950720\, k^{4}\, m^{2} + 35627570307714908050012800\, k^{5}\, m^{2} -$$
$$4194347859956162402870040\, k^{6}\, m^{2} - 5693877418788131312048400\, k^{7}\, m^{2} - 649028009499262230846960\, k^{8}\, m^{2} +$$
$$430152459948693732024180\, k^{9}\, m^{2} + 81228343865440072855680\, k^{10}\, m^{2} - 5384296365746215478400\, k^{11}\, m^{2} -$$
$$3538566381861868216860\, k^{12}\, m^{2} - 982123683517950501780\, k^{13}\, m^{2} + 66040140935138929020\, k^{14}\, m^{2} +$$
$$46719304271573926860\, k^{15}\, m^{2} + 814366548089817660\, k^{16}\, m^{2} - 591653709231444720\, k^{17}\, m^{2} -$$
$$66219409942689780\, k^{18}\, m^{2} - 2132008987123260\, k^{19}\, m^{2} + 833791519441440\, k^{20}\, m^{2} + 61765011317520\, k^{21}\, m^{2} -$$
$$955134107749212800247360000\, m^{3} + 1319090393288249287101319680\, k\, m^{3} - 1299212224946104581844799904\, k^{2}\, m^{3} -$$
$$890806197347489856358740528\, k^{3}\, m^{3} + 563849507946244200106072968\, k^{4}\, m^{3} + 230483417693552635583912004\, k^{5}\, m^{3} -$$
$$34819043676426201460509288\, k^{6}\, m^{3} - 31924244637485590406397210\, k^{7}\, m^{3} - 3815822168158190627259612\, k^{8}\, m^{3} +$$
$$2058839260589082372636960\, k^{9}\, m^{3} + 464293772042933147220990\, k^{10}\, m^{3} - 12608570400920504355 6\, k^{11}\, m^{3} -$$
$$18072157196229893649756\, k^{12}\, m^{3} - 6123679369819833237912\, k^{13}\, m^{3} + 263227478241063419406\, k^{14}\, m^{3} +$$
$$244113661898476778358\, k^{15}\, m^{3} + 6198211022881683132\, k^{16}\, m^{3} - 2714316842607456708\, k^{17}\, m^{3} -$$
$$333440290604213064\, k^{18}\, m^{3} - 11297413021912704\, k^{19}\, m^{3} + 3681858674364408\, k^{20}\, m^{3} + 261839701802736\, k^{21}\, m^{3} -$$
$$4525302333674638965308889600\, m^{4} + 8033744666076186644985392640\, k\, m^{4} - 7847942413370662302207217056\, k^{2}\, m^{4} -$$
$$4732907621431170549693045600\, k^{3}\, m^{4} + 3118452715963078062958103760\, k^{4}\, m^{4} + 1029918584456344362818296080\, k^{5}\, m^{4} -$$

$$
\begin{aligned}
&193215961003916212878287898\, k^{6} m^{4} - 119209937835492152877918906\, k^{7} m^{4} - 153416549044689761148 47232\, k^{8} m^{4} + \\
&612328609930126207345 1418\, k^{9} m^{4} + 179387697837306407781 3084\, k^{10} m^{4} + 116412233349734325272772\, k^{11} m^{4} - \\
&60255786876072420538188\, k^{12} m^{4} - 24515598540307345779006\, k^{13} m^{4} + 5855529703390973307 48\, k^{14} m^{4} + \\
&8203457374137735173 40\, k^{15} m^{4} + 2673677720591340184 8\, k^{16} m^{4} - 7818486815535150348\, k^{17} m^{4} - \\
&1043466440632209306\, k^{18} m^{4} - 36863227835091918\, k^{19} m^{4} + 9927573749897664\, k^{20} m^{4} + 673022699368104\, k^{21} m^{4} - \\
&1638963470782056051 0914442624\, m^{5} + 3657012233495637021 1715231424\, k\, m^{5} - 35136190596695293129267929888\, k^{2} m^{5} - \\
&18910600958853225312870784464\, k^{3} m^{5} + 12866150375946255695339461824\, k^{4} m^{5} + \\
&3418594867201007439885829260\, k^{5} m^{5} - 7907667393545894030251674 06\, k^{6} m^{5} - 314755424402685457544554640\, k^{7} m^{5} - \\
&44581382276347476664722775\, k^{8} m^{5} + 10983222403531238841494045\, k^{9} m^{5} + 5040724342731493362747474\, k^{10} m^{5} + \\
&662429030341639990830492\, k^{11} m^{5} - 141544373730349554678483\, k^{12} m^{5} - 69657830692885793021705\, k^{13} m^{5} + \\
&5711357508427919600 72\, k^{14} m^{5} + 1948255165712535028640\, k^{15} m^{5} + 768395896432172643 48\, k^{16} m^{5} - \\
&15521956383624485802\, k^{17} m^{5} - 2258038725787895634\, k^{18} m^{5} - 82289271243944786\, k^{19} m^{5} + 18161039578768436\, k^{20} m^{5} + \\
&1164033648585536\, k^{21} m^{5} - 47195119673881056589864622976\, m^{6} + 130269149140928017076236840320\, k\, m^{6} - \\
&122789749318038255185665949952\, k^{2} m^{6} - 59425367184205558631199871344\, k^{3} m^{6} + \\
&41567172164384172772436452848\, k^{4} m^{6} + 8814844457066078189800273292\, k^{5} m^{6} - \\
&2518393029568879050768825542\, k^{6} m^{6} - 596958400116579673608137266\, k^{7} m^{6} - 96081573670133764041526202\, k^{8} m^{6} + \\
&68535841077359003195612 62\, k^{9} m^{6} + 10755978480779171035031826\, k^{10} m^{6} + 2172209798353678536960062\, k^{11} m^{6} - \\
&243009504912840147450194\, k^{12} m^{6} - 148607186267091531745222\, k^{13} m^{6} - 7196240264667732394 52\, k^{14} m^{6} + \\
&3449170038854173280428\, k^{15} m^{6} + 1588897726771669538 04\, k^{16} m^{6} - 22309689231297801868\, k^{17} m^{6} - \\
&3576775406462856992\, k^{18} m^{6} - 132849043219954096\, k^{19} m^{6} + 23806615514161504\, k^{20} m^{6} + \\
&1428343391963584\, k^{21} m^{6} - 110912764994952527956726328832\, m^{7} + 374677337074554800178774100992\, k\, m^{7} - \\
&346579140319969156345937639808\, k^{2} m^{7} - 151379892857724349633999162896\, k^{3} m^{7} + \\
&108575568573926742201736926256\, k^{4} m^{7} + 18185403535076647964868621188\, k^{5} m^{7} - \\
&6448454469899775078343626522\, k^{6} m^{7} - 777190983472525890538276844\, k^{7} m^{7} - 153342996947159280323497155\, k^{8} m^{7} - \\
&24556929294868161598118155\, k^{9} m^{7} + 17890395194290164137375052\, k^{10} m^{7} + 5018367553318608424374142\, k^{11} m^{7} - \\
&306896550623525886610843\, k^{12} m^{7} - 246543927573743554451837\, k^{13} m^{7} - 38707929353968266873 52\, k^{14} m^{7} + \\
&4704272473039744083648\, k^{15} m^{7} + 247044676494329638004\, k^{16} m^{7} - 23814600586886354166\, k^{17} m^{7} - \\
&4287761464281447920\, k^{18} m^{7} - 160128242164362296\, k^{19} m^{7} + 23049998675245792\, k^{20} m^{7} + \\
&1278650262416128\, k^{21} m^{7} - 216569565796164611186263590144\, m^{8} + 889900936961583443863892628672\, k\, m^{8} - \\
&809202852438993337993977963552\, k^{2} m^{8} - 319398175843923940209669622992\, k^{3} m^{8} + \\
&234462665547170568129832460208\, k^{4} m^{8} + 30625372148930578587939959028\, k^{5} m^{8} - \\
&13563741231414678720759555606\, k^{6} m^{8} - 519717081859578763318523174\, k^{7} m^{8} - 171994394908470444165248170\, k^{8} m^{8} - \\
&97467189738742647003920074\, k^{9} m^{8} + 23529507211337985645736706\, k^{10} m^{8} + 8797844725559262876639410\, k^{11} m^{8} - \\
&274214656049500503103702\, k^{12} m^{8} - 325511494557087089708822\, k^{13} m^{8} - 8041306838098387882060\, k^{14} m^{8} + \\
&5047263649776776477396\, k^{15} m^{8} + 296803275239163015256\, k^{16} m^{8} - 19036089058040939588\, k^{17} m^{8} - \\
&3967472833907941736\, k^{18} m^{8} - 146709031512626912\, k^{19} m^{8} + 16744301402550272\, k^{20} m^{8} + \\
&845413478590592\, k^{21} m^{8} - 355632265417016622827888515584\, m^{9} + 1774878946281351538567046539776\, k\, m^{9} - \\
&1590374822725846906238603088384\, k^{2} m^{9} - 567056611363353215164377086992\, k^{3} m^{9} + \\
&425384207477471213127241151184\, k^{4} m^{9} + 42671299427806026946464884644\, k^{5} m^{9} - \\
&23790786069873091903785883160\, k^{6} m^{9} + 425072296976400186898800618\, k^{7} m^{9} - 103604112881736359408099054\, k^{8} m^{9} - \\
&202626240981448247494905198\, k^{9} m^{9} + 24555962971568058695959438\, k^{10} m^{9} + 12163600735503573516732064\, k^{11} m^{9} - \\
&142358644849315570693190\, k^{12} m^{9} - 347360865739235454248578\, k^{13} m^{9} - 11153910240460401048358\, k^{14} m^{9} + \\
&4315318325077746658042\, k^{15} m^{9} + 280281174479198044100\, k^{16} m^{9} - 11274355944164665736\, k^{17} m^{9} - \\
&2863655365292438960\, k^{18} m^{9} - 103011907229498336\, k^{19} m^{9} + 9171545696735808\, k^{20} m^{9} + \\
&412860387929088\, k^{21} m^{9} - 494802259849464620118614065152\, m^{10} + 3010844115572071129891854112512\, k\, m^{10} - \\
&2665649917726479635892831881472\, k^{2} m^{10} - 857231913324334401251413530880\, k^{3} m^{10} + \\
&656224802417236878500205879048\, k^{4} m^{10} + 49597439632352230883984939296\, k^{5} m^{10} - \\
&35179218741151821354678766242\, k^{6} m^{10} + 1924856192659604937061379970\, k^{7} m^{10} + \\
&58666965869618100122676692\, k^{8} m^{10} - 302339307767107473063308518\, k^{9} m^{10} + 20119812270722107895716198\, k^{10} m^{10} + \\
&13571328179156836626354438\, k^{11} m^{10} + 224770536851057320940000\, k^{12} m^{10} - 302657417996252586058242\, k^{13} m^{10} - \\
&11539817179978504649912\, k^{14} m^{10} + 2960551869826293379184\, k^{15} m^{10} + 210127830198311021288\, k^{16} m^{10} - \\
&4731220853805014512\, k^{17} m^{10} - 1617688926928391744\, k^{18} m^{10} - 55436977746909696\, k^{19} m^{10} + 3770154525384192\, k^{20} m^{10} + \\
&147098675361792\, k^{21} m^{10} - 585097680888963695425277881728\, m^{11} + 4387295211245360933964265516032\, k\, m^{11} - \\
&3848486796501875695114152094848\, k^{2} m^{11} - 1113395448855398801847260945216\, k^{3} m^{11} + \\
&868552655566578978962226133432\, k^{4} m^{11} + 48248125989816137110618510872\, k^{5} m^{11} - \\
&44213148703631696817765665032\, k^{6} m^{11} + 3425243444444871296323153132\, k^{7} m^{11} + \\
&257394926434721670500886464\, k^{8} m^{11} - 352045754197832084465133452\, k^{9} m^{11} + 12443403053724716112935816\, k^{10} m^{11} + \\
&12399110299335466201858780\, k^{11} m^{11} + 1380503267163716868974 16\, k^{12} m^{11} - 216632237245483195496076\, k^{13} m^{11} - \\
&9298363307292057892224\, k^{14} m^{11} + 1632840525834385822248\, k^{15} m^{11} + 125608043989047850992\, k^{16} m^{11} - \\
&1203047379140134816\, k^{17} m^{11} - 712830115293362816\, k^{18} m^{11} - 22682411653753984\, k^{19} m^{11} + 1145184432274432\, k^{20} m^{11} + \\
&37172337934336\, k^{21} m^{11} - 586852575151148128652417361792\, m^{12} + 5534029190735328373970840168256\, k\, m^{12} - \\
&4822507252516141920653680699104\, k^{2} m^{12} - 1250929380517011535943147455040\, k^{3} m^{12} + \\
&993018784045701137970184425896\, k^{4} m^{12} + 39182688101613905582350579072\, k^{5} m^{12} - \\
&47518079142276095403939078872\, k^{6} m^{12} + 4282635976264015876525047224\, k^{7} m^{12} + \\
&403566107567807386590405264\, k^{8} m^{12} - 331716734789881452727940936\, k^{9} m^{12} + 5074394029804576570777168\, k^{10} m^{12} + \\
&9361581646203809616712968\, k^{11} m^{12} + 1681695713476529380979 76\, k^{12} m^{12} - 127720230224235847302072\, k^{13} m^{12} -
\end{aligned}
$$

$$5953959939172620496520\, k^{14} m^{12} + 7217774865991671317l2\, k^{15} m^{12} + 5981973296303951990 4\, k^{16} m^{12} -$$
$$2488086330749 7664\, k^{17} m^{12} - 242430064891957888\, k^{18} m^{12} - 6930255720734720\, k^{19} m^{12} + 249310114955264\, k^{20} m^{12} +$$
$$6310353403904\, k^{21} m^{12} - 4945481002192285425292994 06208\, m^{13} + 60788434916088005220823287623 04\, k\, m^{13} -$$
$$527570668335379821039494518 4448\, k^{2} m^{13} - 1221940373087884169326074562272\, k^{3} m^{13} +$$
$$985664712790671415016142370432\, k^{4} m^{13} + 262598370040233535928052185 44\, k^{5} m^{13} -$$
$$43869102603995574822561825808\, k^{6} m^{13} + 4196332617063844536879036048\, k^{7} m^{13} +$$
$$439189312625000454185043040\, k^{8} m^{13} - 25778715462550787013 2206800\, k^{9} m^{13} + 3923999952872887535 45024\, k^{10} m^{13} +$$
$$5871770202184069531820432\, k^{11} m^{13} + 1343308179553745806090 88\, k^{12} m^{13} - 6199510959579002616 7808\, k^{13} m^{13} -$$
$$305611105721919885057 6\, k^{14} m^{13} + 253546322475999066688\, k^{15} m^{13} + 22557036131834525760\, k^{16} m^{13} +$$
$$1281941843933 79456\, k^{17} m^{13} - 62378969237406976\, k^{18} m^{13} - 1530806964932608\, k^{19} m^{13} +$$
$$36783428059136\, k^{20} m^{13} + 645141364736\, k^{21} m^{13} - 34184043684841009758632069 4144\, m^{14} +$$
$$584148278467523653729716072 6144\, k\, m^{14} - 5060688514994955694781086348896\, k^{2} m^{14} -$$
$$1041556246140842061797571915744\, k^{3} m^{14} + 85248775476130854493807837 3248\, k^{4} m^{14} +$$
$$140947727967199205121421552 64\, k^{5} m^{14} - 34898028553291584972047092768\, k^{6} m^{14} +$$
$$3358521547502795924219323328\, k^{7} m^{14} + 371740975262987419791164224\, k^{8} m^{14} - 16698265477103895005678553 6\, k^{9} m^{14} -$$
$$138463462484309723822864 0\, k^{10} m^{14} + 3065647539924254468419520\, k^{11} m^{14} + 807023258453484091840 64\, k^{12} m^{14} -$$
$$2468431071912870626588 8\, k^{13} m^{14} - 1258810369325010801920\, k^{14} m^{14} + 69691495072077638144\, k^{15} m^{14} +$$
$$6650509580899133312\, k^{16} m^{14} + 61476527487482624\, k^{17} m^{14} - 11738262762164224\, k^{18} m^{14} -$$
$$230817197129728\, k^{19} m^{14} + 3293221617664\, k^{20} m^{14} + 30006575104\, k^{21} m^{14} - 18205903899462293593245516 7872\, m^{15} +$$
$$4927398192313093759161867804288\, k\, m^{15} - 4269987250075339637584988603520\, k^{2} m^{15} -$$
$$776567885846955416851665433024\, k^{3} m^{15} + 643996472813392777640367172992\, k^{4} m^{15} +$$
$$55732886077320903031818624 00\, k^{5} m^{15} - 23965708333599194609406837568\, k^{6} m^{15} +$$
$$2239632016941584879359230400\, k^{7} m^{15} + 255009750931856201624230464\, k^{8} m^{15} - 90650581912844276105639488\, k^{9} m^{15} -$$
$$1393671486825348115927744\, k^{10} m^{15} + 1331085415080678531414592\, k^{11} m^{15} + 38049133814209739754176\, k^{12} m^{15} -$$
$$8003781567688901195904\, k^{13} m^{15} - 413818661138221988352\, k^{14} m^{15} + 14608080834106031360\, k^{15} m^{15} +$$
$$1500642715116878080\, k^{16} m^{15} + 15996944643215360\, k^{17} m^{15} - 1523330705121280\, k^{18} m^{15} - 21246530584576\, k^{19} m^{15} +$$
$$135029587968\, k^{20} m^{15} - 5923417098131028742834824768 0\, m^{16} + 3656951737850960795782174853376\, k\, m^{16} -$$
$$3175690768189351630395686855040\, k^{2} m^{16} - 507119164637533871361710892160\, k^{3} m^{16} +$$
$$425491153453892173748904302592\, k^{4} m^{16} + 11070055447515417555032401 92\, k^{5} m^{16} -$$
$$14217035625005970379786502144\, k^{6} m^{16} + 1257198525654911475369508736\, k^{7} m^{16} +$$
$$144424806309782901476208256\, k^{8} m^{16} - 41317726217840765077698432\, k^{9} m^{16} - 844057533356486378548096\, k^{10} m^{16} +$$
$$478766956528732793102720\, k^{11} m^{16} + 14279418946850063400192\, k^{12} m^{16} - 2088855968556811743488\, k^{13} m^{16} -$$
$$107228449811448140032\, k^{14} m^{16} + 2236272582972550144\, k^{15} m^{16} + 250232435575547392\, k^{16} m^{16} +$$
$$2560524861706240\, k^{17} m^{16} - 121798563758080\, k^{18} m^{16} - 900197253120\, k^{19} m^{16} + 9722025099398335735797889536\, m^{17} +$$
$$2391260777030475351648824931072\, k\, m^{17} - 2084182112583620167335134801664\, k^{2} m^{17} -$$
$$290131579743515872600011408128\, k^{3} m^{17} + 245944813633359772991239433728\, k^{4} m^{17} -$$
$$4813243109567643248378429 44\, k^{5} m^{17} - 7281337711556841210625603840\, k^{6} m^{17} + 596918822961389208391399936\, k^{7} m^{17} +$$
$$68094964097781673590824960\, k^{8} m^{17} - 15790748897757471802187264\, k^{9} m^{17} - 377064994702352446460416\, k^{10} m^{17} +$$
$$141621928273555295580672\, k^{11} m^{17} + 4267402387589104675328\, k^{12} m^{17} - 431141915965402216960\, k^{13} m^{17} -$$
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$$238112488210432\, k^{17} m^{17} - 4519740375040\, k^{18} m^{17} + 3319918090930357145274936 7296\, m^{18} +$$
$$1378306277305681236861559472640\, k\, m^{18} - 1207312168948607273065984720896\, k^{2} m^{18} -$$
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$$9928425537536\, k^{17} m^{18} + 30828244950886228775844323328\, m^{19} + 699989178318300157070682694656\, k\, m^{19} -$$
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$14116533244526592\, k^{3} m^{32} + 287619503792062464\, m^{33} + 862858511376187392\, k\, m^{33} - 862858511376187392\, k^{2} m^{33})$

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