Contributions to Symbolic q-Hypergeometric Summation

Dissertation

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Eidesstattliche Erklärung

Ich versichere, daß ich die Dissertation selbständig verfaßt habe, andere als die angegebenen Quellen und Hilfsmittel nicht verwendet und mich auch sonst keiner unerlaubten Hilfe bedient habe.

Axel Riese
Linz, im November 1997

Zusammenfassung

Diese Dissertation besteht aus fünf großteils unabhängigen Kapiteln, die sich — mit Ausnahme des vierten — mit der (symbolischen) Summation von q-hypergeometrischen Folgen befassen. Einen besonderen Schwerpunkt bildet das automatische Beweisen und Finden von q-Identitäten.

Im ersten Kapitel stellen wir eine neu entwickelte Mathematica Implementierung eines bibasischen Analogons des Gosperschen Algorithmus zur indefiniten hypergeometrischen Summation vor, die auf einer Verallgemeinerung des von Paule entwickelten Konzepts der „greatest factorial factorization" von Polynomen beruht.

Im zweiten Kapitel wird zunächst die von Wilf und Zeilberger präsentierte Theorie der WZ-Paare in die q-hypergeometrische Welt übertragen und danach meine Mathematica Implementierung `qZeil` des q-Zeilbergerschen Algorithmus zur systematischen Erzeugung von „companion" und dualen Identitäten verwendet, wodurch eine große Anzahl bekannter sowie neuer Identitäten algorithmisch gefunden werden kann.

Das dritte Kapitel ist dem Konzept der Bailey-Paare und dem zugrunde liegenden Iterationsmechanismus gewidmet. Dieser kann auf einfache Weise zum Verifizieren und Finden von q-Identitäten einer bestimmten Klasse herangezogen werden. Insbesondere beschreiben wir das Mathematica Paket `Bailey`, das einen halbautomatischen „Spaziergang" entlang sogenannter Bailey-Ketten ermöglicht. Mit einer erweiterten Version des q-Zeilbergerschen Algorithmus werden anschließend neue Bailey-Paare hergeleitet.

In Anhang A wird kurz eine allgemeine Definition des q-Binomialkoeffizienten, basierend auf der q-Gamma Funktion, für komplexe Parameter vorgestellt.

In Anhang B schließlich findet sich eine detaillierte Anleitung für die Benutzung des `qZeil` Pakets.

Abstract

This thesis consists of five mostly self-contained parts which all — except the fourth — deal with (symbolic) summation of q-hypergeometric sequences. The main emphasis has been put on automatically proving and finding q-identities.

In Chapter 1 we introduce a newly developed Mathematica implementation of a bibasic analogue of Gosper's algorithm for indefinite hypergeometric summation together with its theoretical background based on a bibasic variant of Paule's concept of greatest factorial factorization of polynomials.

In Chapter 2 the theory of WZ-pairs presented by Wilf and Zeilberger is generalized to the q-case. The author's Mathematica implementation `qZeil` of the q-Zeilberger algorithm is then used to systematically generate companion and dual identities. Proceeding this way, a large number of known as well as new identities can be found algorithmically.

Chapter 3 is devoted to the concept of Bailey pairs and its underlying iteration mechanism that can be used to easily prove and find q-identities of certain type. In particular, the author's package `Bailey`, which allows to walk along Bailey chains semi-automatically, is described. With the help of an extended version of the q-Zeilberger algorithm some new Bailey pairs are derived.

In Appendix A we shortly present a general definition of the q-binomial coefficient for complex parameters in terms of the q-gamma function.

Finally, Appendix B serves as a detailed manual for using the `qZeil` package.

Contents

Introduction

Thanks to Zeilberger's [**46**] algorithm, proving most definite hypergeometric summation and transformation formulas has become routine work that can be performed by a computer. It was also Zeilberger who first observed that his algorithm can be carried over to the q-hypergeometric case. Based on Paule's [**32**] algebraic concept of greatest factorial factorization (GFF), which provides an explanation of hypergeometric telescoping and extends beautifully to the q- (and even multibasic) hypergeometric case, I implemented a Mathematica q-analogue of Zeilberger's algorithm in the frame of my diploma thesis [**37**] trying to overcome shortcomings of the already existing Maple implementations of Zeilberger (see Petkovšek, Wilf, and Zeilberger [**36**]) and Koornwinder [**25**]. Since I came up with a first prototype of `qZeil` in late 1993, the program has constantly undergone substantial improvements. Special attention has been directed to advancing the capabilities for *finding* q-identities automatically. As an example, `qZeil` now discovers polynomial multipliers or suggests powers of q which make a given input summable. This *Extended q-Zeilberger Algorithm*[†] builds the algorithmic backbone of this thesis.

The thesis consists of five self-contained parts, which are all devoted to the field of q-series but can mostly be read independently from each other. To this end some of the basic definitions appear more than once. Readers being familiar with the subject may safely skip these repetitions.

In Chapter 1 a newly developed Mathematica implementation of a *bibasic analogue of Gosper's algorithm* for indefinite hypergeometric summation is introduced together with its theoretical background based on a bibasic variant of Paule's [**32**] concept of *greatest factorial factorization* of polynomials. This chapter has been already published in the Electronic Journal of Combinatorics [**38**].

In Chapter 2 the theory of *WZ-pairs* developed by Wilf and Zeilberger [**44**] is generalized to the q-case. The author's Mathematica implementation `qZeil` of the q-Zeilberger algorithm (cf. Paule and Riese [**33**]) is then used to systematically generate *companion* and *dual* identities from "standard" qWZ-pairs employing a recently established *shadowing* strategy. Proceeding this way, a large number of known as well as new identities can be found automatically.

In Chapter 3 it is shown how the concept of *Bailey pairs* and its underlying iteration mechanism (cf. Andrews [**9**, **10**] or Paule [**30**]) can be used to easily prove and find q-identities of certain type. In particular, the author's package `Bailey`, which allows to walk along Bailey chains semi-automatically, is described. With the help of `qZeil` some new Bailey pairs are derived.

In Appendix A we shortly present a general definition of the *q-binomial coefficient* for complex parameters in terms of the q-gamma function, which has been stimulated by several

[†]Latest information on the package can be retrieved via the World Wide Web from the `qZeil` homepage at `http://www.risc.uni-linz.ac.at/research/combinat/risc/software/qZeil`

inaccurate definitions found in literature.

Appendix B serves as a *manual* for the `qZeil` package including hints on installation, usage, and new features.

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Chapter 1

A Generalization of Gosper's Algorithm to Bibasic Hypergeometric Summation

Recently, Paule and Strehl [**34**] from a normal form point of view described how the algorithm presented by Gosper [**24**] for indefinite hypergeometric summation extends quite naturally to the q-hypergeometric case by introducing a q-analogue of the canonical Gosper-Petkovšek (GP) representation for rational functions. Based on the new algebraic concept of greatest factorial factorization (GFF), Paule [**32**] developed a general approach to hypergeometric telescoping. For instance, it was shown by Paule (cf. Paule and Riese [**33**]) that the problem of q-hypergeometric telescoping can be treated along the same lines as the $q = 1$ case by making use of a q-version of GFF. Built on these concepts, a Mathematica implementation of q-analogues of Gosper's as well as of Zeilberger's [**46**] fast algorithm for definite q-hypergeometric summation has been carried out by the author (cf. Paule and Riese [**33**], and Riese [**37**]). The original approach to definite q-hypergeometric summation is due to Wilf and Zeilberger [**45**].

The object of this chapter is to describe how the algorithm qTelescope presented in [**33**], a q-analogue of Gosper's algorithm, generalizes to the bibasic hypergeometric case. In Section 1.1 the underlying theoretical background based on a bibasic extension of GFF is discussed, which leads to the bibasic counterpart of the algorithm qTelescope. In Section 1.2 the degree setting for solving the bibasic key equation is established. Applications are given in Section 1.3 to illustrate the usage of the newly developed Mathematica implementation which is available by email request to the author†.

1.1 Theoretical Background

In this section, q-greatest factorial factorization (qGFF) of polynomials, which has been introduced by Paule (cf. Paule and Riese [**33**]) providing an algebraic explanation of q-hypergeometric telescoping, is extended to the bibasic hypergeometric case. It turns out that to this end the argumentation can be carried over almost word by word.

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1.1.1 Bibasic Greatest Factorial Factorization

Let $\mathbb{Z}$ denote the set of all integers, and $\mathbb{N}$ the set of all non-negative integers. Let p, q, x, and y be fixed indeterminates. Assume $K = L(\kappa_1, \ldots, \kappa_n)$ to be the field of rational functions in a fixed number of indeterminates $\kappa_1, \ldots, \kappa_n$, $n \in \mathbb{N}$, where $p \neq \kappa_i \neq y$ and $q \neq \kappa_i \neq x$, $1 \leq i \leq n$, over some computable field L of characteristic 0 and not containing p, q, x, and y. (For the sake of simplicity with regard to the implementation we will restrict ourselves to the case where L is the rational number field $\mathbb{Q}$.) The transcendental extension of K by the indeterminates p and q is denoted by F, i.e., $F = K(p, q)$.

For $P \in F[x, y]$, let the *bibasic shift operator* ϵ be given by $(\epsilon P)(x, y) = P(qx, py)$. The extension of this shift operator to the rational function field $F(x, y)$, the quotient field of the polynomial ring $F[x, y]$, will be also denoted by ϵ.

Definition 1.1. A polynomial $P \in F[x]$ (resp. $P \in F[y]$) is called *q-monic* (resp. *p-monic*) if $P(0) = 1$. A polynomial $P \in F[x, y]$ is called *bibasic monic* if $P(x, 0) \neq 0 \neq P(0, y)$ and either $P(0, 0) = 1$, or $P(0, 0) = 0$ and the coefficients of P are relatively prime polynomials in F.†

Example 1.1. (i) The following polynomials are bibasic monic:

$$P_1(x, y) = 1, \qquad P_2(x, y) = 1 - apqx^2y^3, \qquad P_3(x, y) = (1 - q)^2x^2 + py.$$

(ii) The following polynomials are not bibasic monic:

$$P_4(x, y) = q, \qquad P_5(x, y) = xy - apqx^2y^3, \qquad P_6(x, y) = (1 - q)^{-1}px^2 + py. \qquad \square$$

The properties of being q-monic, p-monic, and bibasic monic are clearly invariant with respect to the bibasic shift operator ϵ, i.e., if P is q-monic, p-monic, or bibasic monic, then the same holds true for ϵP. Furthermore, the product of two bibasic monic polynomials is again bibasic monic. Also note that a bibasic monic polynomial P satisfies $\gcd(x, P) = 1 = \gcd(y, P)$.

Evidently, any non-zero polynomial $P \in F[x, y]$ has a unique factorization, the *bibasic monic decomposition*, in the form

$$P = z \cdot x^\alpha \cdot y^\beta \cdot P^*,$$

where $z \in F$, $\alpha, \beta \in \mathbb{N}$, and $P^* \in F[x, y]$ is bibasic monic.

The bibasic monic decomposition of a polynomial $P \neq 0$ can be computed easily as follows. Define $\alpha := \max\{i \in \mathbb{N} : x^i | P\}$, $\beta := \max\{j \in \mathbb{N} : y^j | P\}$, and put $\bar{P} := x^{-\alpha} \cdot y^{-\beta} \cdot P$. If $\bar{P}(0, 0) \neq 0$ define $z := \bar{P}(0, 0)$, otherwise let l denote the least common multiple of all coefficient-denominators of $\bar{P}$, let g denote the greatest common divisor of all coefficients of $l \cdot \bar{P}$, and define $z := g/l$. Then, for $P^* := z^{-1} \cdot \bar{P}$, the bibasic monic decomposition of P is given by $P = z \cdot x^\alpha \cdot y^\beta \cdot P^*$.

Example 1.2. The bibasic monic decompositions of the polynomials P_4, P_5, and P_6 from the example above are given by

$$P_4 = q \cdot x^0 \cdot y^0 \cdot 1, \quad P_5 = 1 \cdot x \cdot y \cdot (1 - apqxy^2), \quad P_6 = \frac{p}{1 - q} \cdot x^0 \cdot y^0 \cdot (x^2 + (1 - q)y). \quad \square$$

†In other words, P is assumed to be primitive over $L[\kappa_1, \ldots, \kappa_n, p, q]$ in this case, which will guarantee the uniqueness of the so-called bibasic monic decomposition of a polynomial as shown below.

Moreover, we assume the result of any gcd computation over $F[x,y]$ as being normalized in the following sense. If $P_1 = z_1 \cdot x^{\alpha_1} \cdot y^{\beta_1} \cdot P_1^*$ and $P_2 = z_2 \cdot x^{\alpha_2} \cdot y^{\beta_2} \cdot P_2^*$ are the bibasic monic decompositions of $P_1, P_2 \in F[x,y]$, we define

$$\gcd_{p,q}(P_1, P_2) := \gcd(x^{\alpha_1}, x^{\alpha_2}) \cdot \gcd(y^{\beta_1}, y^{\beta_2}) \cdot \gcd_{p,q}(P_1^*, P_2^*),$$

where the $\gcd_{p,q}$ of two bibasic monic polynomials is understood to be bibasic monic.

The polynomial degree in x and y of any $P \in F[x,y]$ is denoted by $\deg_x(P)$ and $\deg_y(P)$, respectively.

Definition 1.2. For any bibasic monic polynomial $P \in F[x,y]$ and $k \in \mathbb{N}$, the *k-th falling bibasic factorial* $[P]_{p,q}^{\underline{k}}$ of P is defined as

$$[P]_{p,q}^{\underline{k}} := \prod_{i=0}^{k-1} \epsilon^{-i} P.$$

Note that by the null convention $\prod_{i\in\emptyset} P_i := 1$ we have $[P]_{p,q}^{\underline{0}} = 1$. In general, polynomials arising in bibasic hypergeometric summation have several different representations in terms of falling bibasic factorials. From all possibilities, we shall consider only the one taking care of maximal chains, which informally can be obtained as follows. One selects irreducible factors of P in such a way that their product, say

$$P_{k,1}(x,y) \cdot P_{k,1}(q^{-1}x, p^{-1}y) \cdots P_{k,1}(q^{-k+1}x, p^{-k+1}y),$$

forms a falling bibasic factorial $[P_{k,1}]_{p,q}^{\underline{k}}$ of maximal length k. For the remaining irreducible factors of P this procedure is applied again in order to find all k-th falling factorial divisors $[P_{k,1}]_{p,q}^{\underline{k}}, \ldots, [P_{k,l}]_{p,q}^{\underline{k}}$ of that type. Then $[P_k]_{p,q}^{\underline{k}} := [P_{k,1} \cdots P_{k,l}]_{p,q}^{\underline{k}}$ forms the bibasic factorial factor of P of maximal length k. Iterating this procedure one gets a factorization of P in terms of "greatest" factorial factors.

Definition 1.3. We say that $\langle P_1, \ldots, P_k \rangle$, $P_i \in F[x,y]$, is a *bibasic GFF-form* of a bibasic monic polynomial $P \in F[x,y]$, written as $\mathrm{GFF}_{p,q}(P) = \langle P_1, \ldots, P_k \rangle$, if the following conditions hold:

($\mathrm{GFF}_{p,q}$ 1) $P = [P_1]_{p,q}^{\underline{1}} \cdots [P_k]_{p,q}^{\underline{k}}$,

($\mathrm{GFF}_{p,q}$ 2) each P_i is bibasic monic, and $k > 0$ implies $P_k \neq 1$,

($\mathrm{GFF}_{p,q}$ 3) for $i \leq j$ we have $\gcd_{p,q}([P_i]_{p,q}^{\underline{i}}, \epsilon P_j) = 1 = \gcd_{p,q}([P_i]_{p,q}^{\underline{i}}, \epsilon^{-j} P_j)$.

Note that $\mathrm{GFF}_{p,q}(1) = \langle\rangle$. Condition ($\mathrm{GFF}_{p,q}$ 3) intuitively can be understood as prohibiting "overlaps" of bibasic factorials that violate length maximality. The following theorem states that, as in the q-hypergeometric case, the bibasic GFF-form is unique and thus provides a canonical form.

Theorem 1.1. *If $\langle P_1, \ldots, P_k \rangle$ and $\langle P_1', \ldots, P_l' \rangle$ are bibasic GFF-forms of a bibasic monic polynomial $P \in F[x,y]$, then $k = l$ and $P_i = P_i'$ for all $1 \leq i \leq k$.*

Proof. The corresponding result for the ordinary hypergeometric case ($p = q = 1$) has been proved by Paule [**32**, Thm. 2.1]. The arguments used there extend immediately to the bibasic hypergeometric case proceeding by induction on $d := \deg_x(P) + \deg_y(P)$. □

From algorithmic point of view it is important to note that the bibasic GFF-form can be computed in an iterative manner essentially involving only gcd computations.

In q-hypergeometric summation, the normalized gcd of a polynomial P and its q-shift ϵP plays a fundamental role, as the gcd of P and its shift EP does in ordinary hypergeometric summation, where $(EP)(x) = P(x+1)$. The same is true for bibasic hypergeometric summation with respect to the bibasic shift operator ϵ. The mathematical and algorithmic essence lies in the following lemma.

Lemma 1.2 (Fundamental $\mathbf{GFF_{p,q}}$ Lemma). *Let $P \in F[x, y]$ be a bibasic monic polynomial with* $\mathrm{GFF}_{p,q}(P) = \langle P_1, \ldots, P_k \rangle$*. Then*

$$\mathrm{gcd}_{p,q}(P, \epsilon P) = [P_1]_{p,q}^{0} \cdots [P_k]_{p,q}^{k-1}.$$

Proof. Due to the choice of the bibasic shift operator ϵ, the proof of the so-called Fundamental qGFF Lemma (cf. Paule and Riese [**33**, Lemma 1]) can be carried over to the bibasic hypergeometric case completely unchanged. □

Thus, if $\mathrm{GFF}_{p,q}(P) = \langle P_1, \ldots, P_k \rangle$, then $\mathrm{GFF}_{p,q}(\mathrm{gcd}_{p,q}(P, \epsilon P)) = \langle P_2, \ldots, P_k \rangle$. Consequently, dividing P with $\mathrm{GFF}_{p,q}(P) = \langle P_1, \ldots, P_k \rangle$ by $\epsilon^{-1}\,\mathrm{gcd}_{p,q}(P, \epsilon P)$ or $\mathrm{gcd}_{p,q}(P, \epsilon P)$ results in separating the product of the first, respectively last, falling bibasic factorial entries, or in other words

$$\frac{P}{\epsilon^{-1}\,\mathrm{gcd}_{p,q}(P, \epsilon P)} = P_1 \cdot P_2 \cdots P_k \quad \text{and} \quad \frac{P}{\mathrm{gcd}_{p,q}(P, \epsilon P)} = P_1 \cdot (\epsilon^{-1} P_2) \cdots (\epsilon^{-k+1} P_k).$$

1.1.2 Bibasic Hypergeometric Telescoping

A sequence $(f_k)_{k \in \mathbb{Z}}$ is said to be *bibasic hypergeometric* (see, e.g., Petkovšek, Wilf, and Zeilberger [**36**]) in p and q over F, if there exists a rational function $\rho \in F(x, y)$ such that $f_{k+1}/f_k = \rho(q^k, p^k)$ for all k where the quotient is well-defined.

Assume we are given a bibasic hypergeometric sequence $(f_k)_{k \in \mathbb{Z}}$. Then the problem of bibasic hypergeometric telescoping is to decide whether there exists a bibasic hypergeometric sequence $(g_k)_{k \in \mathbb{Z}}$ such that

$$g_{k+1} - g_k = f_k, \tag{1.1}$$

and if so, to determine $(g_k)_{k \in \mathbb{Z}}$ with the motive that for $a, b \in \mathbb{Z}$, $a \leq b$,

$$\sum_{k=a}^{b} f_k = g_{b+1} - g_a,$$

which solves the indefinite summation problem.

For the rational function ρ, related to f_{k+1}/f_k as above, there exists a representation $\rho(x, y) = z \cdot x^{\alpha} \cdot y^{\beta} \cdot A^*(x, y)/B^*(x, y)$ with bibasic monic $A^*, B^* \in F[x, y]$, $z \in F$, and $\alpha, \beta \in \mathbb{Z}$, which we call a *rational representation* of the bibasic hypergeometric sequence $(f_k)_{k \in \mathbb{Z}}$. If additionally A^* and B^* are relatively prime, then $\rho(x, y)$ is called the *reduced rational representation* of $(f_k)_{k \in \mathbb{Z}}$. For $\alpha \in \mathbb{Z}$, let $\alpha_+ := \max(\alpha, 0)$ and $\alpha_- := \max(-\alpha, 0)$.

It will be shown below that bibasic hypergeometric telescoping can be decided constructively as follows.

Algorithm Telescope$_{p,q}$. INPUT: *a bibasic hypergeometric sequence* $(f_k)_{k\in\mathbb{Z}}$ *specified by its reduced rational representation* $\rho = z \cdot x^\alpha \cdot y^\beta \cdot A^*/B^*$*;*
OUTPUT: *a bibasic hypergeometric solution* $(g_k)_{k\in\mathbb{Z}}$ *of* (1.1)*; in case such a solution does not exist, the algorithm stops.*

(i) Compute the bibasic GP form of $(f_k)_{k\in\mathbb{Z}}$*, i.e.,*

(a) determine unique bibasic monic polynomials P^*, Q^*, $R^* \in F[x,y]$ *such that*

$$\frac{A^*}{B^*} = \frac{\epsilon P^*}{P^*} \cdot \frac{Q^*}{\epsilon R^*}, \tag{1.2}$$

where $\gcd_{p,q}(P^*, Q^*) = 1 = \gcd_{p,q}(P^*, R^*)$ *and* $\gcd_{p,q}(Q^*, \epsilon^j R^*) = 1$ *for all* $j \geq 1$*, and*

(b) let a_x*,* b_x*,* a_y*, and* b_y *denote the coefficients of the lowest occurring powers of* x *and* y *in* $A^*(x,0)$*,* $B^*(x,0)$*,* $A^*(0,y)$*, and* $B^*(0,y)$*, respectively. Define*

$$(\gamma,\delta) := \begin{cases} (\varphi,\psi) & \text{if } \alpha = 0 = \beta \text{ and } q^\varphi \cdot p^\mu \cdot b_y/a_y = z = p^\psi \cdot q^\nu \cdot b_x/a_x \\ & \text{for } \varphi,\psi \in \mathbb{N} \text{ and } \mu,\nu \in \mathbb{Z}, \\ (\varphi,0) & \text{if } \alpha = 0 \neq \beta \text{ and } z = q^\varphi \cdot p^\mu \cdot b_y/a_y \text{ for } \varphi \in \mathbb{N},\ \mu \in \mathbb{Z}, \\ (0,\psi) & \text{if } \alpha \neq 0 = \beta \text{ and } z = p^\psi \cdot q^\nu \cdot b_x/a_x \text{ for } \psi \in \mathbb{N},\ \nu \in \mathbb{Z}, \\ (0,0) & \text{otherwise,} \end{cases}$$

and put

$$\begin{aligned} P &:= x^\gamma \cdot y^\delta \cdot P^*, \\ Q &:= z \cdot q^{-\gamma} \cdot p^{-\delta} \cdot x^{\alpha_+} \cdot y^{\beta_+} \cdot Q^*, \\ \epsilon R &:= x^{\alpha_-} \cdot y^{\beta_-} \cdot \epsilon R^*, \end{aligned} \tag{1.3}$$

with the motive that then

$$\rho = \frac{\epsilon P}{P} \cdot \frac{Q}{\epsilon R}.$$

(ii) Try to solve the bibasic key equation

$$P = Q \cdot \epsilon Y - R \cdot Y \tag{1.4}$$

for a polynomial $Y \in F[x,y]$.

(iii) If such a polynomial solution Y *exists, then*

$$g_k = \frac{R(q^k,p^k) \cdot Y(q^k,p^k)}{P(q^k,p^k)} \cdot f_k \tag{1.5}$$

is a bibasic hypergeometric solution of (1.1)*, otherwise no bibasic hypergeometric solution* $(g_k)_{k\in\mathbb{Z}}$ *exists.* □

The steps of Algorithm Telescope$_{p,q}$ are derived as follows. First, assume that a bibasic hypergeometric solution $(g_k)_{k\in\mathbb{Z}}$ with rational representation $g_{k+1}/g_k = \sigma(q^k,p^k)$ of (1.1) exists. Then evidently we have

$$g_k = \tau(q^k,p^k) \cdot f_k, \tag{1.6}$$

where $\tau(x, y) = 1/(\sigma(x, y) - 1) \in F(x, y)$.

By relation (1.6), equation (1.1) is equivalent to

$$z \cdot x^{\alpha_+} \cdot y^{\beta_+} \cdot A^* \cdot \epsilon\tau - x^{\alpha_-} \cdot y^{\beta_-} \cdot B^* \cdot \tau = x^{\alpha_-} \cdot y^{\beta_-} \cdot B^*, \tag{1.7}$$

where the reduced rational representation of $(f_k)_{k\in\mathbb{Z}}$ is given by $\rho = z \cdot x^\alpha \cdot y^\beta \cdot A^*/B^*$.

Vice versa, any rational solution $\tau \in F(x, y)$ of (1.7) gives rise to a bibasic hypergeometric solution $g_k := \tau(q^k, p^k) \cdot f_k$ of (1.1). This means, bibasic hypergeometric telescoping is equivalent to finding a *rational* solution τ of (1.7).

Any $\tau \in F(x, y)$ can be represented as the quotient of relatively prime polynomials in the form $\tau = \mathcal{U}/\mathcal{V}$ where $\mathcal{U}, \mathcal{V} \in F[x, y]$ with $\mathcal{V} = x^\varphi \cdot y^\psi \cdot \mathcal{V}^*$ the bibasic monic decomposition of $\mathcal{V}$. In case such a solution τ of (1.7) exists, assume we know $\mathcal{V}$ or a multiple $V \in F[x, y]$ of $\mathcal{V}$. Then by clearing denominators in

$$z \cdot x^{\alpha_+} \cdot y^{\beta_+} \cdot A^* \cdot \frac{\epsilon U}{\epsilon V} - x^{\alpha_-} \cdot y^{\beta_-} \cdot B^* \cdot \frac{U}{V} = x^{\alpha_-} \cdot y^{\beta_-} \cdot B^*,$$

the problem reduces further to finding a polynomial solution $U \in F[x, y]$ of the resulting difference equation with polynomial coefficients,

$$z \cdot x^{\alpha_+} \cdot y^{\beta_+} \cdot A^* \cdot V \cdot \epsilon U - x^{\alpha_-} \cdot y^{\beta_-} \cdot B^* \cdot (\epsilon V) \cdot U = x^{\alpha_-} \cdot y^{\beta_-} \cdot B^* \cdot V \cdot \epsilon V. \tag{1.8}$$

Note that at least one polynomial solution, namely $U = \mathcal{U} \cdot V/\mathcal{V}$, exists. Furthermore, equations of that type simplify by canceling $\gcd_{p,q}$'s. For instance, in order to get more information about the denominator $\mathcal{V}$, let $\mathcal{V}_i := \epsilon^i\mathcal{V}/\gcd_{p,q}(\mathcal{V}, \epsilon\mathcal{V})$, $i \in \{0, 1\}$. Then (1.7) is equivalent to

$$z \cdot x^{\alpha_+} \cdot y^{\beta_+} \cdot A^* \cdot \mathcal{V}_0 \cdot \epsilon\mathcal{U} - x^{\alpha_-} \cdot y^{\beta_-} \cdot B^* \cdot \mathcal{V}_1 \cdot \mathcal{U} = x^{\alpha_-} \cdot y^{\beta_-} \cdot B^* \cdot \mathcal{V}_0 \cdot \mathcal{V}_1 \cdot \gcd_{p,q}(\mathcal{V}, \epsilon\mathcal{V}). \tag{1.9}$$

Now, if $\langle \mathcal{P}_1, \ldots, \mathcal{P}_m\rangle$, $m \in \mathbb{N}$, is the bibasic GFF-form of $\mathcal{V}^*$, it follows from $\gcd_{p,q}(\mathcal{U}, \mathcal{V}) = 1 = \gcd_{p,q}(\mathcal{V}_0, \mathcal{V}_1)$ and the Fundamental $\mathrm{GFF}_{p,q}$ Lemma that

$$\mathcal{V}_0 = (\epsilon^0\mathcal{P}_1) \cdots (\epsilon^{-m+1}\mathcal{P}_m) \mid B^* \quad \text{and} \quad \mathcal{V}_1 = q^\varphi \cdot p^\psi \cdot (\epsilon\mathcal{P}_1) \cdots (\epsilon\mathcal{P}_m) \mid A^*.$$

This observation gives rise to a simple and straightforward algorithm for computing a multiple $V^* := [P_1]^{\underline{1}}_{p,q} \cdots [P_n]^{\underline{n}}_{p,q}$ of $\mathcal{V}^*$. For instance, if $P_1 := \gcd_{p,q}(\epsilon^{-1}A^*, B^*)$ then obviously $\mathcal{P}_1|P_1$. Actually, one can iteratively extract bibasic monic $\mathcal{P}_i$-multiples P_i such that $\epsilon P_i|A^*$ and $\epsilon^{-i+1}P_i|B^*$ by the following algorithm.

Algorithm V*MULT. INPUT: *relatively prime and bibasic monic polynomials* $A^*, B^* \in F[x, y]$ *that constitute the bibasic monic quotient of* $\rho = z \cdot x^\alpha \cdot y^\beta \cdot A^*/B^* \in F(x, y)$;
OUTPUT: *bibasic monic polynomials* $P_1, \ldots, P_n$ *such that* $V^* := [P_1]^{\underline{1}}_{p,q} \cdots [P_n]^{\underline{n}}_{p,q}$ *is a multiple of* $\mathcal{V}^*$, *the bibasic monic part of the denominator* $\mathcal{V} = x^\varphi \cdot y^\psi \cdot \mathcal{V}^*$ *of* $\tau \in F(x, y)$.

(i) Compute $n = \min\{j \in \mathbb{N} \mid \gcd_{p,q}(\epsilon^{-1}A^*, \epsilon^{k-1}B^*) = 1$ *for all integers* $k > j\}$.

(ii) Set $A_0 = A^*$, $B_0 = B^*$, *and compute for* i *from* 1 *to* n:

$$P_i = \gcd_{p,q}(\epsilon^{-1}A_{i-1}, \epsilon^{i-1}B_{i-1}),$$
$$A_i = A_{i-1}/\epsilon P_i,$$
$$B_i = B_{i-1}/\epsilon^{-i+1}P_i.$$

□

A proof for the fact that the P_i are indeed multiples of the $\mathcal{P}_i$ has been worked out for the ordinary hypergeometric case by Paule [**32**, Lemma 5.1]. It can be carried over to the bibasic hypergeometric world almost word by word. Hence we leave the steps of the verification to the reader.

Note that in general step (i) of Algorithm V*MULT would be a rather time-consuming task involving resultant computations which could be solved by generalizing the univariate case (cf. Abramov, Paule, and Petkovšek [**1**]) in a straightforward way, for instance, as follows. Define $R_1(v,w) := \mathrm{Res}_x(A^*(x,y), B^*(vx,wy))$ and $R_2(v,w) := \mathrm{Res}_y(A^*(x,y), B^*(vx,wy))$, viewed as polynomials of v and w over $F[y]$, respectively $F[x]$. Then n is the maximal positive integer such that $R_1(q^n,p^n)\cdot R_2(q^n,p^n) = 0$ if such an integer exists, and $n = 0$ otherwise. However, in our implementation we make use of the fact that A^* and B^* already come in nicely factored form so that the computation of n boils down to a comparison of those factors.

Moreover, Algorithm V*MULT also delivers the constituents of the bibasic monic part of the GP representation (1.2) as stated in the following lemma.

Lemma 1.3. *Let n, A_n, B_n, and the tuple $\langle P_1,\ldots,P_n\rangle$ be computed as in Algorithm* V*MULT*. Then for $P^* = V^*$, $Q^* = A_n$, and $R^* = \epsilon^{-1}B_n$ we have*

$$\frac{A^*}{B^*} = \frac{\epsilon P^*}{P^*}\cdot\frac{Q^*}{\epsilon R^*},$$

where $\gcd_{p,q}(P^*,Q^*) = 1 = \gcd_{p,q}(P^*,R^*)$ *and* $\gcd_{p,q}(Q^*,\epsilon^j R^*) = 1$ *for all* $j \geq 1$.

For more details on GP representations in the q-hypergeometric case, see Abramov, Paule, and Petkovšek [**1**], or Paule and Strehl [**34**]. The results obtained there also apply to the bibasic hypergeometric case, in particular we have the following.

Lemma 1.4. *The polynomials P^*, Q^*, and R^* of the bibasic monic part of the GP representation* (1.2) *are unique.*

Proof. The corresponding result for the case $p = q = 1$ has been proved by Petkovšek [**35**]. The argumentation extends directly to the q- and bibasic hypergeometric case. □

With the multiple V^* of $\mathcal{V}^*$ in hands, all what is left for solving (1.7), and thus the bibasic hypergeometric telescoping problem (1.1), is to determine appropriate multiplicities γ and δ such that

$$V = x^\gamma \cdot y^\delta \cdot V^* \text{ is a multiple of } \mathcal{V} = x^\varphi \cdot y^\psi \cdot \mathcal{V}^*.$$

For that we consider equation (1.9) again in the equivalent version

$$z\cdot x^{\alpha_+}\cdot y^{\beta_+}\cdot A^*\cdot \mathcal{V}^*\cdot\epsilon\mathcal{U} - x^{\alpha_-}\cdot y^{\beta_-}\cdot B^*\cdot q^\varphi\cdot p^\psi\cdot(\epsilon\mathcal{V}^*)\cdot\mathcal{U} = x^{\alpha_-}\cdot y^{\beta_-}\cdot B^*\cdot\mathcal{V}^*\cdot\epsilon\mathcal{V}, \tag{1.10}$$

and distinguish the following cases corresponding to step (ib) of Algorithm $\text{Telescope}_{p,q}$.

(i) Assume that either $\alpha_- \neq 0$ or $\alpha_+ \neq 0$. In the first case we have $\alpha_+ = 0$ and $x^{\alpha_-} \mid \mathcal{U}$, hence φ must be 0 because of $\gcd_{p,q}(\mathcal{U},\mathcal{V}) = 1$. This means, we can choose $\gamma := 0$. In the second case we have $\alpha_- = 0$ and $x^{\min(\alpha_+,\varphi)} \mid \mathcal{U}$, because of $\epsilon\mathcal{V} = x^\varphi\cdot y^\psi\cdot q^\varphi\cdot p^\psi\cdot\epsilon\mathcal{V}^*$. Again φ must be 0, and again we can choose $\gamma := 0$. Analogously, if $\beta \neq 0$ we can choose $\delta := 0$.

(ii) Assume that $\alpha = 0$ and $\beta \neq 0$, hence $\psi = 0$ by (i). For $\varphi > 0$, evaluating equation (1.10) at $x = 0$ results in

$$z \cdot y^{\beta_+} \cdot A^*(0,y) \cdot \mathcal{V}^*(0,y) \cdot \mathcal{U}(0,py) - y^{\beta_-} \cdot B^*(0,y) \cdot q^{\varphi} \cdot \mathcal{V}^*(0,py) \cdot \mathcal{U}(0,y) = 0. \tag{1.11}$$

In order to evaluate (1.11) at $y = 0$, note that $P \in F[x,y]$ being bibasic monic does not necessarily imply that $P(0,y) \in F[y]$ is p-monic. To overcome this problem, let us consider the p-monic decompositions of $\mathcal{U}(0,y)$ and $\mathcal{V}^*(0,y)$, say $\mathcal{U}(0,y) = u \cdot y^{\beta_u} \cdot \bar{U}(y)$ and $\mathcal{V}^*(0,y) = v \cdot y^{\beta_v} \cdot \bar{V}(y)$, respectively. Now, dividing equation (1.11) by $\mathcal{U}(0,y) \cdot \mathcal{V}^*(0,y) \neq 0$ leads to

$$z \cdot y^{\beta_+} \cdot A^*(0,y) \cdot p^{\beta_u} \cdot \frac{\bar{U}(py)}{\bar{U}(y)} - y^{\beta_-} \cdot B^*(0,y) \cdot q^{\varphi} \cdot p^{\beta_v} \cdot \frac{\bar{V}(py)}{\bar{V}(y)} = 0. \tag{1.12}$$

Additionally, let the p-monic decompositions of $A^*(0,y)$ and $B^*(0,y)$ be given by $A^*(0,y) = a_y \cdot y^{\beta_a} \cdot \bar{A}(y)$ and $B^*(0,y) = b_y \cdot y^{\beta_b} \cdot \bar{B}(y)$, respectively. Then the powers $y^{\beta_a+\beta_+}$ and $y^{\beta_b+\beta_-}$ must be equal, and after cancellation eq. (1.12) at $y = 0$ turns into

$$z \cdot a_y \cdot p^{\beta_u} - b_y \cdot q^{\varphi} \cdot p^{\beta_v} = 0.$$

This means, we obtain as a condition for $\varphi > 0$ that $z = q^{\varphi} \cdot p^{\mu} \cdot b_y/a_y$ with $\mu \in \mathbb{Z}$. Hence, in this case we choose $\gamma := \varphi$, i.e., we set γ to this q-power if z has this particular form, and $\gamma := 0$ otherwise. Analogously, if $\alpha \neq 0$ and $\beta = 0$ we define $\delta := \psi > 0$, if $z = p^{\psi} \cdot q^{\nu} \cdot b_x/a_x$ with $\nu \in \mathbb{Z}$, and $\delta := 0$ otherwise.

(iii) Finally, for the case $\alpha = 0 = \beta$ similar reasoning as in case (ii) leads to the conditions

$$q^{\varphi} \cdot p^{\mu} \cdot b_y/a_y = z = p^{\psi} \cdot q^{\nu} \cdot b_x/a_x, \tag{1.13}$$

for $\varphi > 0$ or $\psi > 0$, and $\mu, \nu \in \mathbb{Z}$. Thus, if both conditions (1.13) are satisfied we choose $\gamma := \varphi$ and $\delta := \psi$, and otherwise $\gamma = \delta := 0$.

The remaining steps of Algorithm Telescope$_{p,q}$ now are explained as follows. Once again, employing the GP representation for the bibasic monic quotient of ρ,

$$\frac{A^*}{B^*} = \frac{\epsilon P^*}{P^*} \cdot \frac{Q^*}{\epsilon R^*},$$

it is easily seen that equation (1.8) can be written as

$$z \cdot q^{-\gamma} \cdot p^{-\delta} \cdot x^{\alpha_+} \cdot y^{\beta_+} \cdot \frac{Q^*}{\epsilon R^*} \cdot \epsilon U - x^{\alpha_-} \cdot y^{\beta_-} \cdot U = x^{\gamma+\alpha_-} \cdot y^{\delta+\beta_-} \cdot P^*. \tag{1.14}$$

Because of relative primeness of certain polynomials, we observe that $x^{\alpha_-} \,|\, U$, $y^{\beta_-} \,|\, U$, and $\epsilon R^* \mid \epsilon U$. Hence by defining Y by the relation

$$U = x^{\alpha_-} \cdot y^{\beta_-} \cdot q^{-\alpha_-} \cdot p^{-\beta_-} \cdot R^* \cdot Y,$$

the task to solve equation (1.8) for U reduces to solve

$$z \cdot q^{-\gamma} \cdot p^{-\delta} \cdot x^{\alpha_+} \cdot y^{\beta_+} \cdot Q^* \cdot \epsilon Y - x^{\alpha_-} \cdot y^{\beta_-} \cdot q^{-\alpha_-} \cdot p^{-\beta_-} \cdot R^* \cdot Y = x^{\gamma} \cdot y^{\delta} \cdot P^* \tag{1.15}$$

for $Y \in F[x,y]$. By definition (1.3) of P, Q, and R, equation (1.15) immediately turns into the bibasic key equation (1.4),

$$Q \cdot \epsilon Y - R \cdot Y = P.$$

Finally, from $U/V = R \cdot Y/P$, again by definition (1.3), it follows directly that

$$g_k = \frac{R(q^k, p^k) \cdot Y(q^k, p^k)}{P(q^k, p^k)} \cdot f_k$$

as in (1.5) actually is a solution of the bibasic hypergeometric telescoping problem (1.1). This completes the proof of the correctness of Algorithm Telescope$_{p,q}$.

1.2 Degree Setting for Solving the Bibasic Key Equation

To solve the bibasic key equation

$$P = Q \cdot \epsilon Y - R \cdot Y \tag{1.16}$$

we first have to determine degree bounds d_1 and d_2, say, for the solution polynomial $Y \in F[x, y]$ with respect to x and y, respectively, as shown in Theorem 1.5 below. Then we put

$$Y(x, y) := \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} y_{i,j} \cdot x^i \cdot y^j$$

with undetermined $y_{i,j}$ and solve (1.16) for the $y_{i,j}$ by equating to zero all coefficients of $x^i y^j$ in the equation

$$P - Q \cdot \epsilon Y + R \cdot Y = 0,$$

which corresponds to solving a system of linear equations.

Theorem 1.5. *Let $l_Q^x(y)$ and $l_R^x(y)$ denote the leading coefficient polynomials of Q and R with respect to x. Let $QR^+ := Q + R$ and $QR^- := Q - R$. Then a bound for $\deg_x(Y)$ is given by:*

(A_x) If $\deg_x(QR^+) \neq \deg_x(QR^-)$, then

$$\deg_x(Y) \leq \max\{\deg_x(P) - \max\{\deg_x(QR^+), \deg_x(QR^-)\}, 0\}.$$

(B_x) If $\deg_x(QR^+) = \deg_x(QR^-)$, then

($B1_x$) if $\deg_x(Q) \neq \deg_x(R)$, then

$$\deg_x(Y) = \deg_x(P) - \deg_x(QR^+),$$

($B2_x$) if $\deg_x(Q) = \deg_x(R)$, then

($B2a_x$) if $l_R^x(y)/l_Q^x(y)$ is of the form $p^\mu \cdot q^\nu \cdot r(y)$ with $\mu, \nu \in \mathbb{N}$, and $r(y)$ a rational function with $r(0) = 1$, then

$$\deg_x(Y) \leq \max\{\deg_x(P) - \deg_x(QR^+), \nu\},$$

($B2b_x$) otherwise

$$\deg_x(Y) = \deg_x(P) - \deg_x(QR^+).$$

A bound for $\deg_y(Y)$ is given by interchanging x with y and p with q in both (A_x) and (B_x).

Proof. We rewrite the key equation to obtain

$$2\,P = QR^{+} \cdot (\epsilon Y - Y) + QR^{-} \cdot (\epsilon Y + Y). \tag{1.17}$$

Cases (A_x) and (A_y) follow immediately. Note that it might happen that

$$\deg_x(QR^{+}) > \deg_x(P) \text{ and } \deg_x(QR^{-}) = \deg_x(P),$$

and simultaneously

$$\deg_y(QR^{+}) > \deg_y(P) \text{ and } \deg_y(QR^{-}) = \deg_y(P).$$

In this case, setting $\deg_x(Y) = \deg_y(Y) = 0$ could yield a solution, since $\epsilon Y - Y = 0$ then.

For Case ($\mathrm{B1}_x$) let $a := \deg_x(Q)$, $c := \deg_x(Y)$, and let $l_Y^x(y)$ denote the leading coefficient polynomial of Y with respect to x. Assume that $\deg_x(Q) > \deg_x(R)$. Then (1.17) gives

$$\begin{aligned} 2\,P(x,y) &= (l_Q^x(y)\,x^a + \dots) \cdot [(l_Y^x(py)\,q^c - l_Y^x(y))\,x^c + \dots] \\ &+ (l_Q^x(y)\,x^a + \dots) \cdot [(l_Y^x(py)\,q^c + l_Y^x(y))\,x^c + \dots] \\ &= 2\,l_Q^x(y)\,l_Y^x(py)\,q^c\,x^{a+c} + \dots. \end{aligned} \tag{1.18}$$

Clearly, the coefficient of x^{a+c} in (1.18) will never vanish. Therefore we have

$$\deg_x(Y) = \deg_x(P) - \deg_x(Q).$$

Including the case $\deg_x(Q) < \deg_x(R)$, we obtain

$$\deg_x(Y) = \deg_x(P) - \max\{\deg_x(Q), \deg_x(R)\} = \deg_x(P) - \deg_x(QR^{+}).$$

Analogous reasoning leads to Case ($\mathrm{B1}_y$).

For Case ($\mathrm{B2}_x$) we similarly observe that

$$\begin{aligned} 2\,P(x,y) &= [(l_Q^x(y) + l_R^x(y))\,x^a + \dots] \cdot [(l_Y^x(py)\,q^c - l_Y^x(y))\,x^c + \dots] \\ &+ [(l_Q^x(y) - l_R^x(y))\,x^a + \dots] \cdot [(l_Y^x(py)\,q^c + l_Y^x(y))\,x^c + \dots] \\ &= 2\,[l_Q^x(y)\,l_Y^x(py)\,q^c - l_R^x(y)\,l_Y^x(y)]\,x^{a+c} + \dots. \end{aligned} \tag{1.19}$$

Now we no longer have the guarantee that the coefficient of x^{a+c} in (1.19) does not vanish, but it is easily seen that this happens only for

$$q^c = \frac{l_R^x(y)}{l_Q^x(y)} \cdot \frac{l_Y^x(y)}{l_Y^x(py)}. \tag{1.20}$$

Note that $l_Y^x(y)$ is actually *not* known. However, for any non-zero polynomial $h(y) = h_0 + h_1\,y + \dots + h_d\,y^d$, the quotient $h(y)/h(py)$ is of the form $p^{-m} \cdot s(y)$, where $s(y)$ is a rational function with $s(0) = 1$ and m is the zero-root multiplicity of $h(y)$. Hence, the rightmost fraction in (1.20) may eliminate only positive integer powers of p and a rational function of y but never introduce a power of q. This proves Cases ($\mathrm{B2a}_x$) and ($\mathrm{B2a}_y$).

On the other hand, if the coefficient of x^{a+c} in (1.19) does not vanish, we obtain Case ($\mathrm{B2b}_x$) and analogously Case ($\mathrm{B2b}_y$). □

1.3 Applications

In this section we shall illustrate the method of bibasic hypergeometric telescoping using the author's Mathematica implementation qTelescope, which is a bibasic extension of a q-analogue of Gosper's algorithm originally described in Paule and Riese [**33**].

Let the *q-shifted factorial* of $a \in F$ be defined as usual (see, e.g., Gasper and Rahman [**20**]) by

$$(a;q)_k := \begin{cases} (1-a)(1-aq)\cdots(1-aq^{k-1}), & \text{if } k>0, \\ 1, & \text{if } k=0, \\ \left[(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{k})\right]^{-1}, & \text{if } k<0, \end{cases}$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$

where products of q-shifted factorials will be abbreviated by

$$(a_1,a_2,\ldots,a_n;q)_k := (a_1;q)_k\,(a_2;q)_k\cdots(a_n;q)_k.$$

In the present implementation we allow as summand any bibasic hypergeometric sequence $(f_k)_{k\in\mathbb{Z}}$ of the form

$$f_k = \frac{\prod_r (C_r\, q^{(c_r i_r)k+d_r}; q^{i_r})_{a_r k+b_r}}{\prod_s (D_s\, q^{(v_s j_s)k+w_s}; q^{j_s})_{t_s k+u_s}} \cdot \frac{\prod_r (C'_r\, p^{(c'_r i'_r)k+d'_r}; p^{i'_r})_{a'_r k+b'_r}}{\prod_s (D'_s\, p^{(v'_s j'_s)k+w'_s}; p^{j'_s})_{t'_s k+u'_s}}$$
$$\times R(q^k,p^k)\cdot q^{\alpha\binom{k}{2}}\cdot p^{\beta\binom{k}{2}}\cdot z^k,$$

with

C_r, D_s	power products in $K(p)$,
C'_r, D'_s	power products in $K(q)$,
a_r, t_s, a'_r, t'_s	specific integers (i.e., integers free of any parameters),
b_r, u_s, b'_r, u'_s	integer parameters free of k, or $\pm\infty$ if a_r (resp. t_s, a'_r, t'_s) $= 0$,
c_r, v_s, c'_r, v'_s	specific integers,
d_r, w_s, d'_r, w'_s	integer parameters free of k,
i_r, j_s, i'_r, j'_s	specific non-zero integers,
R	a rational function in $F(q^k,p^k)$ s.t. the denominator factors completely into a product of terms of the form $(1-D\,q^{vk+w})$ and $(1-D'p^{v'k+w'})$,
α, β	specific integers, and
z	a rational function in F.

For the actual computation of the GP representation let $\rho(x,y)$ denote the possibly non-reduced rational representation of the summand f_k. It is obvious from the input specification that ρ can always be converted into the form

$$\rho(x,y) = \frac{(\epsilon\bar{P})(x,y)}{\bar{P}(x,y)} \cdot \frac{\prod_i (1-\Gamma_i\, x^{\gamma_i})}{\prod_j (1-\Delta_j\, x^{\delta_j})} \cdot \frac{\prod_i (1-\Gamma'_i\, y^{\gamma'_i})}{\prod_j (1-\Delta'_j\, y^{\delta'_j})} \cdot x^{\bar{\alpha}} \cdot y^{\bar{\beta}} \cdot \bar{z}$$
$$= \frac{(\epsilon\bar{P})(x,y)}{\bar{P}(x,y)} \cdot \frac{\bar{A}(x,y)}{\bar{B}(x,y)} \cdot x^{\bar{\alpha}} \cdot y^{\bar{\beta}} \cdot \bar{z},$$

where $\bar{P}$ is bibasic monic and satisfies $\gcd_{p,q}(\bar{P}, \bar{A}) = 1 = \gcd_{p,q}(\epsilon\bar{P}, \bar{B})$; the $\Gamma_i, \Delta_j, \Gamma'_i, \Delta'_j$ are power products in F, the $\gamma_i, \delta_j, \gamma'_i, \delta'_j$ are positive integers, $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}$, and $\bar{z} \in F$.

Concerning Algorithm V*MULT, it is clear from above that any $\bar{P} \neq 1$ will actually contribute to $[P_1]^{\frac{1}{p,q}}$ and thus can be treated separately. Due to our input restrictions — this is the reason for admitting only power products instead of arbitrary rational functions — it is possible to find n in step (i) of Algorithm V*MULT simply by comparing all factors in $\bar{A}$ and $\bar{B}$ as already mentioned.

Furthermore, since $\bar{A}$ and $\bar{B}$ are both products of a q-monic and a p-monic polynomial, they will never contribute to b_x/a_x and b_y/a_y. Thus, b_x/a_x and b_y/a_y are in any case integer powers of q and p, respectively, coming from $\epsilon\bar{P}/\bar{P}$. Therefore, they do not take influence on the computation of γ and δ at all.

1.3.1 Bibasic Summation Formulas

In 1989, Gasper [**19**] derived the indefinite bibasic summation formula

$$\sum_{k=0}^{n} f_k = \sum_{k=0}^{n} \frac{(1-ap^kq^k)\,(1-bp^kq^{-k})}{(1-a)\,(1-b)} \; \frac{(a,b;p)_k\,(c,a/bc;q)_k}{(q,aq/b;q)_k\,(ap/c,bcp;p)_k} \; q^k$$
$$= \frac{(ap,bp;p)_n\,(cq,aq/bc;q)_n}{(q,aq/b;q)_n\,(ap/c,bcp;p)_n} = g_n \tag{1.21}$$

by showing that g_k is a bibasic hypergeometric solution of the equation $f_k = g_k - g_{k-1}$, however, without revealing how to come up with g_k. With our implementation the job of finding g_k is left to the computer.

```
In[1]:= (* first of all load the package *)
        <<qTelescope.m

Out[1]= Axel Riese's qTelescope implementation version 2.1 loaded

In[2]:= qTelescope[(1-a p^k q^k) (1-b p^k/q^k) qfac[a,p,k] qfac[b,p,k] qfac[c,q,k] *
                   qfac[a/b/c,q,k] q^k / ((1-a) (1-b) qfac[q,q,k] qfac[a q/b,q,k] *
                   qfac[a p/c,p,k] qfac[b c p,p,k]), {k, 0, n}]

                                                 a q
        qfac[a p, p, n] qfac[b p, p, n] qfac[---, q, n] qfac[c q, q, n]
                                                 b c
Out[2]= ----------------------------------------------------------------
            a p                                               a q
        qfac[---, p, n] qfac[b c p, p, n] qfac[q, q, n] qfac[---, q, n]
             c                                                 b
```

Applying the same argumentation, Gasper and Rahman [**21**] generalized (1.21) to

$$\sum_{k=-m}^{n} (1-adp^kq^k)\,(1-bp^k/dq^k)\; \frac{(a,b;p)_k\,(c,ad^2/bc;q)_k}{(dq,adq/b;q)_k\,(adp/c,bcp/d;p)_k}\; q^k$$
$$= \frac{(1-a)\,(1-b)\,(1-c)\,(1-ad^2/bc)}{d\,(1-c/d)\,(1-ad/bc)}$$
$$\times \left\{ \frac{(ap,bp;p)_n\,(cq,ad^2q/bc;q)_n}{(dq,adq/b;q)_n\,(adp/c,bcp/d;p)_n} - \frac{(c/ad,d/bc;p)_{m+1}\,(1/d,b/ad;q)_{m+1}}{(1/c,bc/ad^2;q)_{m+1}\,(1/a,1/b;p)_{m+1}} \right\}. \tag{1.22}$$

Obviously, (1.21) is the case $d = 1$, $m = 0$ of (1.22). Since the output of `qTelescope` for identity (1.22) is quite lengthy, here we shall consider only the case $m = -1$ after dividing the summand by the constant fraction on the right hand side. Of course, the algorithm works for symbolic m as well.

```
In[3]:= qTelescope[(1-a d p^k q^k) (1-b/d p^k/q^k) qfac[a,p,k] qfac[b,p,k] *
                 qfac[c,q,k] qfac[a d^2/b/c,q,k] q^k d (1-c/d) (1-a d/b/c) /
                (qfac[d q,q,k] qfac[a d q/b,q,k] qfac[a d p/c,p,k] *
                 qfac[b c p/d,p,k] (1-a) (1-b) (1-c) (1-a d^2/b/c)), {k, 1, n}]

                                                                   2
                                                                a d  q
Out[3]= -1 + (qfac[a p, p, n] qfac[b p, p, n] qfac[c q, q, n] qfac[------, q, n]) /
                                                                 b c

         b c p            a d p                          a d q
   (qfac[-----, p, n] qfac[-----, p, n] qfac[d q, q, n] qfac[-----, q, n])
           d                c                              b
```

1.3.2 Bibasic Matrix Inverses

Al-Salam and Verma [**4**] showed that the triangular matrices $H = (h_{n,k})$ and $G = (g_{k,n})$, where

$$h_{n,k} = \frac{(-1)^{n+k}\,(hqp^n;q)_{n-1}\,(1-hq^kp^k)}{(p;p)_{n-k}\,(hqp^n;q)_k}$$

and

$$g_{k,n} = \frac{(hp^nq^n;q)_{k-n}}{(p;p)_{k-n}}\;p^{\binom{k-n}{2}}$$

are inverse to each other. This result is equivalent to the fact that

$$\sum_{k=m}^{n} h_{n,k}\cdot g_{k,m} = \delta_{n,m}, \tag{1.23}$$

where $\delta_{n,m}$ denotes the Kronecker symbol. Running the algorithm we obtain:

```
In[4]:= qTelescope[(-1)^(n+k) qfac[h q p^n,q,n-1] (1-h q^k p^k) *
                 qfac[h p^m q^m,q,k-m] p^Binomial[k-m,2] /
                (qfac[p,p,n-k] qfac[h q p^n,q,k] qfac[p,p,k-m]), {k, m, n}]

Out[4]= {0, {-m + n != 0}}
```

This means, we algorithmically proved identity (1.23) for $m \neq n$, but evaluation failed for $m = n$. However, it is easily seen that $h_{n,n}\cdot g_{n,n} = 1$, which completes the proof.

These matrices were used in a slightly modified form also by Gessel and Stanton [**23**] in the derivation of a family of q-Lagrange inversion formulas.

Al-Salam and Verma [**4**] employed the fact that the n-th q-difference of a polynomial of degree less than n is equal to zero, to show that

$$\Big(1-\frac{a}{q}\Big)\sum_{k=0}^{n}\frac{(-1)^k\,(ap^k;q)_{n-1}}{(p;p)_k\,(p;p)_{n-k}}\;p^{\binom{k}{2}} = \delta_{n,0}. \tag{1.24}$$

Unfortunately, for $d_k := (ap^k; q)_{n-1}$, we find that

$$\frac{d_{k+1}}{d_k} = \frac{(1 - ap^{k+1})\,(1 - ap^{k+1}q) \cdots (1 - ap^{k+1}q^{n-2})}{(1 - ap^k)\,(1 - ap^k q) \cdots (1 - ap^k q^{n-2})}$$

is a rational function of q^k and p^k only for fixed n. Therefore d_k is *not* a valid input for the algorithm. To overcome the problem, we replace k, n, and a in (1.24) by $k - m$, $n - m$, and $a^{-1}p^m q^{1-n}$, respectively, such that (1.24) turns into the orthogonality relation

$$c_{n,m} \sum_{k=m}^{n} a_{n,k} \cdot b_{k,m} = \delta_{n,m} \tag{1.25}$$

with

$$\begin{aligned} c_{n,m} &= (1 - a^{-1}p^m q^{-n})\, a^{1+m-n}\, q^{\binom{m+1}{2} - \binom{n}{2}}, \\ a_{n,k} &= \frac{(ap^{-k}; q)_n}{(p;p)_{n-k}}\, (-1)^{1+k+n}\, p^{\binom{n-k}{2}}, \\ b_{k,m} &= \frac{p^{-k(m+1)}}{(p;p)_{k-m}\, (ap^{-k}; q)_{m+1}}. \end{aligned}$$

Note that $a_{n,k}$ and $b_{k,m}$ still do not fit into the input specification of the algorithm. For $A = (a_{n,k})$, $B = (b_{k,m})$, and $C = (c_{n,m})$, relation (1.25) could be rewritten as $A \cdot B = \operatorname{diag}(C)^{-1}$, showing that the matrix $\operatorname{diag}(C) \cdot A = (c_{n,n} \cdot a_{n,k})$ is inverse to the matrix B. Since inverse matrices commute, we exchange $\operatorname{diag}(C) \cdot A$ with B and find that (1.25) is equivalent to

$$\sum_{k=m}^{n} b_{n,k} \cdot c_{k,k} \cdot a_{k,m} = \delta_{n,m},$$

or, in other words

$$\sum_{k=m}^{n} \frac{(-1)^{k+m}\,(1 - ap^{-k}q^k)\,(ap^{-m}; q)_k}{(p;p)_{n-k}\,(p;p)_{k-m}\,(ap^{-n}; q)_{k+1}}\, p^{\binom{k-m}{2} - n(k+1) + k(m+1)} = \delta_{n,m}. \tag{1.26}$$

Now, in this form we are faced with an admissible input and compute:

```
In[5]:= qTelescope[(-1)^(k+m) (1-a q^k/p^k) qfac[a/p^m,q,k] *
                 p^(Binomial[k-m,2]-n(k+1)+k(m+1)) /
                 (qfac[p,p,n-k] qfac[p,p,k-m] qfac[a/p^n,q,k+1]), {k, m, n}]

Out[5]= {0, {-m + n != 0}}
```

For $m = 0$, (1.26) reduces to

$$\sum_{k=0}^{n} \frac{(-1)^k\,(1 - ap^{-k}q^k)\,(a;q)_k}{(p;p)_{n-k}\,(p;p)_k\,(ap^{-n}; q)_{k+1}}\, p^{\binom{n-k}{2}} = \delta_{n,0}.$$

```
In[6]:= qTelescope[(-1)^k (1-a q^k/p^k) qfac[a,q,k] p^Binomial[n-k,2] /
                 (qfac[p,p,n-k] qfac[p,p,k] qfac[a/p^n,q,k+1]), {k, 0, n}]

Out[6]= {0, {n != 0}}
```

Proceeding in the same way, for instance, we also can prove the bibasic identity (cf. Gasper [**19**])

$$\Big(1-\frac{a}{q}\Big)\Big(1-\frac{b}{q}\Big)\sum_{k=0}^{n}\frac{(-1)^k\,(ap^k,bp^{-k};q)_{n-1}\,(1-ap^{2k}/b)}{(p;p)_k\,(p;p)_{n-k}\,(ap^k/b;p)_{n+1}}\,p^{k(n-1)+\binom{n-k}{2}}=\delta_{n,0},$$

by transforming it into the equivalent version

$$\Big(1-\frac{b}{a}\Big)\sum_{k=0}^{n}(1-ap^{-k}q^k)(1-bp^kq^k)\,\frac{(-1)^k\,(a,b;q)_k\,(bp^{k+1}/a;p)_{n-1}}{(p;p)_k\,(p;p)_{n-k}\,(ap^{-n},bp^n;q)_{k+1}}\,p^{\binom{n-k}{2}}=\delta_{n,0}.$$

```
In[7]:= qTelescope[(1-b/a) (1-a q^k/p^k) (1-b p^k q^k) (-1)^k qfac[a,q,k] *
                    qfac[b,q,k] qfac[b/a p^(k+1),p,n-1] p^Binomial[n-k,2] /
                   (qfac[p,p,k] qfac[p,p,n-k] qfac[a/p^n,q,k+1] qfac[b p^n,q,k+1]),
                    {k, 0, n}]

Out[7]= {0, {n != 0}}
```

1.3.3 Extensions and Open Problems

With the input specification described above we actually have not taken into account that a bibasic hypergeometric summand f_k could involve q-shifted factorials with mixed bases such as $(a;p^iq^j)_k$ for $i,j\in\mathbb{Z}$ as well. However, since to our knowledge applications of this type have not arisen in practice up to now, this feature has not been implemented yet.

For the sake of simplicity we restricted ourselves to discuss in detail the bibasic case. Nevertheless, the presented algorithm should easily extend to the *multibasic* case, i.e., to sequences being hypergeometric in independent bases $q_1,\ldots,q_m$. Recently, Bauer and Petkovšek [**16**] developed a different approach to multibasic hypergeometric telescoping which also covers the "mixed" (ordinary and multibasic) hypergeometric case.

A generalization of bibasic hypergeometric telescoping to definite summation can only be regarded as a useful extension, if one can prove that sums like $\sum_k \begin{bmatrix}n\\k\end{bmatrix}_p\begin{bmatrix}n\\k\end{bmatrix}_q$ satisfy a linear recurrence with polynomial coefficients in p^n and q^n. However, this seems to be untrue in general.

Finally, we would like to remark that so far we found only one single bibasic example in the literature which we could not handle with our machinery, namely Gasper's [**19**] transformation formulas

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{1-ap^kq^k}{1-a}\,\frac{(a;p)_k\,(c/b;q)_k}{(q;q)_k\,(abp;p)_k}\,b^k\\
&\quad=\frac{1-c}{1-b}\,\sum_{k=0}^{\infty}\frac{(ap;p)_k\,(c/b;q)_k}{(q;q)_k\,(abp;p)_k}\,(bq)^k\\
&\quad=\frac{1-c}{1-abp}\,\sum_{k=0}^{\infty}\frac{(ap;p)_k\,(cq/b;q)_k}{(q;q)_k\,(abp^2;p)_k}\,b^k\\
&\quad=\frac{(1-c)\,(ap;p)_\infty}{(1-b)\,(abp;p)_\infty}\,\sum_{k=0}^{\infty}\frac{(b;p)_k\,(cqp^k;q)_\infty}{(p;p)_k\,(bqp^k;q)_\infty}\,(ap)^k,
\end{aligned}$$

when $\max(|p|,|q|,|ap|,|b|)<1$.

Chapter 2

Automatic Generation of q-Identities

Using the author's Mathematica implementation `qZeil` of a q-analogue of Zeilberger's algorithm (cf. Paule and Riese [**33**]) for definite q-hypergeometric summation we will show in this chapter how the concept of WZ-pairs introduced by Wilf and Zeilberger [**44**] generalizes to the q-case giving new identities from existing ones "for free", i.e., without too much additional effort. In particular, we shall focus our attention on generating companion and dual identities from qWZ-pairs. Similar to Gessel's [**22**] systematic investigation of dual identity production for the $q = 1$ case, we shall apply this method to several "standard" terminating q-identities leading to a large number of new identities as well as identities appearing in the context of Bailey chains (see, e.g., Andrews [**10**], Paule [**30**], or Chapter 3).

2.1 q-Hypergeometric Telescoping and qWZ-Certification

Analogous to Zeilberger's [**46**] algorithm its q-analogue takes terminating q-hypergeometric sums as input. The output is a linear recurrence that is satisfied by the input sum, together with a rational function which serves as the proof certificate. It is important to note that the proof certificate enables an independent verification of the output recurrence merely by checking a rational function identity. This means, the algorithm itself supplies complete information for a correctness check which works independently of the steps in which the output recurrence was manufactured.

The backbone of the author's q-Zeilberger implementation is Algorithm qTelescope, a q-analogue of Gosper's [**24**] algorithm for indefinite hypergeometric summation based on a q-version of Paule's [**32**] concept of greatest factorial factorization. A detailed description of Algorithm qTelescope is given in Paule and Riese [**33**].

Let $\mathbb{Z}$ denote the set of all integers, $\mathbb{N}$ the set of all non-negative integers, and $\mathbb{N}^+ := \mathbb{N}\backslash\{0\}$ the set of all positive integers. Assume $K = L(\kappa_1, \ldots, \kappa_m)$ to be the field of rational functions in a fixed number of indeterminates $\kappa_1, \ldots, \kappa_m$, all different from q, over some computable field L of characteristic 0. (For the sake of simplicity with regard to the implementation we will restrict ourselves to the case where L is the rational number field $\mathbb{Q}$.) The transcendental extension of K by the indeterminate q is denoted by F, i.e., $F = K(q)$.

A sequence (f_k) with values in F, where k runs through all integers, is said to be

q-hypergeometric in k, if the quotient f_{k+1}/f_k is a rational function of q^k over F for all k where the quotient is well-defined. Given a q-hypergeometric sequence (f_k), the problem of *q-hypergeometric telescoping* then consists in constructively deciding whether there exists a q-hypergeometric sequence (g_k) such that

$$f_k = g_k - g_{k-1},$$

with the motive that for $a, b \in \mathbb{Z}$, $a \leq b$,

$$\sum_{k=a}^{b} f_k = g_b - g_{a-1}.$$

It is well known that Algorithm qTelescope in general fails as soon as we turn to *definite* q-hypergeometric summation. However, it can be used in a non-obvious way also for this purpose thanks to an observation by Zeilberger [**46**, **47**, **48**]. For this, let $f := (f_{n,k})$ be a double-indexed sequence with values in F. We shall consider only sequences where n runs through $\mathbb{N}$, whereas the second parameter k might run through all integers.

The sequence f is called q-hypergeometric in n and k, if both quotients

$$\frac{f_{n+1,k}}{f_{n,k}} \quad \text{and} \quad \frac{f_{n,k+1}}{f_{n,k}}$$

are rational functions of q^n and q^k over F for all n and k where the quotients are well-defined.

Let the *q-shifted factorial* of $a \in F$ be defined as usual (see, e.g., Gasper and Rahman [**20**]) by

$$(a;q)_k := \begin{cases} (1-a)\,(1-aq)\cdots(1-aq^{k-1}), & \text{if } k > 0, \\ 1, & \text{if } k = 0, \\ \left[(1-aq^{-1})\,(1-aq^{-2})\cdots(1-aq^{k})\right]^{-1}, & \text{if } k < 0, \end{cases}$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty} (1-aq^k),$$

where products of q-shifted factorials will be abbreviated by

$$(a_1, a_2, \ldots, a_m; q)_k := (a_1;q)_k\,(a_2;q)_k \cdots (a_m;q)_k.$$

Example 2.1. The sequence of *Gaussian polynomials* (also called *q-binomial coefficients*)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise,} \end{cases}$$

is q-hypergeometric in n and k. □

We say that the sequence f has *finite support* with respect to k, if for all n there exists a finite integer interval I_n such that $f_{n,k} \neq 0$ for $k \in I_n$, and $f_{n,k} = 0$ for $k \notin I_n$. As an example, consider $f_{n,k} := \begin{bmatrix} n \\ k \end{bmatrix}_q$ with $I_n = \{0, 1, \ldots, n\}$.

Given f being q-hypergeometric in n and k, one can prove under mild side-conditions, as demonstrated in Wilf and Zeilberger [**45**], that for a certain integer $d \geq 0$ and $n \geq d$ there exists a linear recurrence

$$\sigma_0(n)\, f_{n,k} + \sigma_1(n)\, f_{n-1,k} + \cdots + \sigma_d(n)\, f_{n-d,k} = g_{n,k} - g_{n,k-1}, \tag{2.1}$$

where the coefficients are polynomials in q^n not depending on k and not all zero, and where $g_{n,k}$ is a rational function multiple of $f_{n,k}$ and thus q-hypergeometric in n and k, too. Given the order d, which in general is not known a priori, $g_{n,k}$ and also the coefficient polynomials $\sigma_i(n)$ are determined by q-hypergeometric telescoping, i.e., by Algorithm qTelescope.

Assume that f has finite support with respect to k. Then summing both sides of (2.1) over all k results in

$$\sigma_0(n)\, S_n + \sigma_1(n)\, S_{n-1} + \cdots + \sigma_d(n)\, S_{n-d} = 0, \tag{2.2}$$

a recurrence for the sum sequence $S_n := \sum_k f_{n,k}$, a finite sum due to the finite support property. We use the convention that the summation parameter k runs through all the integers, in case the summation range is not specified explicitly.

Now the *qWZ-certificate* (for short: *certificate*) of recurrence (2.1) or (2.2), respectively, by definition is the rational function $cert(n,k)$, rational in q^n and q^k, such that

$$g_{n,k} = cert(n,k) \cdot f_{n,k}.$$

Evidently, with the certificate in hands the verification of (2.1), and therefore (2.2), reduces to checking the *rational function identity*

$$r(n,k) = cert(n,k) - cert(n,k-1) \cdot \frac{f_{n,k-1}}{f_{n,k}},$$

where $r(n,k)$, rational in q^n and q^k, comes from rewriting the left hand side of (2.1) as $r(n,k) \cdot f_{n,k}$. The computation of $r(n,k)$ is straightforward, because any $f_{n-i,k}$ can be written as a rational function multiple of $f_{n,k}$, for instance, $f_{n-1,k} = (f_{n-1,k}/f_{n,k}) \cdot f_{n,k}$.

In the inhomogeneous case, i.e., if f does not have finite support, or, if one is interested in summation with bounds not naturally induced by the finite support, we have to introduce the corresponding correction terms in (2.2). For more details, see Paule and Riese [**33**].

It is well known that Zeilberger's algorithm and especially its q-analogue do not always deliver the recurrence with minimal order. However, several approaches have been developed to decrease the order to the expected one, such as Paule's [**31**] method of *creative symmetrizing* (see also Paule and Riese [**33**], or Petkovšek, Wilf, and Zeilberger [**36**]).

Finally, suppose that we want to prove a closed form summation identity

$$\sum\nolimits_k a_{n,k} = b_n,$$

where $(a_{n,k})$ has finite support and $b_n \neq 0$ for all n. By putting $f_{n,k} := a_{n,k}/b_n$, the identity to be proved may be rewritten as

$$\sum\nolimits_k f_{n,k} = 1. \tag{2.3}$$

In this situation, it turns out that in many instances $f_k := f_{n,k} - f_{n-1,k}$ is Gosper-summable, i.e., q-hypergeometric telescoping applied to f_k, a rational function multiple of $f_{n,k}$, leads to a so-called qWZ-pair defined as follows.

Definition 2.1. Let $f = (f_{n,k})$ and $g = (g_{n,k})$ denote q-hypergeometric sequences, where $g_{n,k}$ is a rational function multiple of $f_{n,k}$. We say that (f, g) forms a *qWZ-pair* if

$$f_{n,k} - f_{n-1,k} = g_{n,k} - g_{n,k-1}, \tag{2.4}$$

for all n and k where both sides are well-defined.

Now, under the additional assumption that f has finite support, the same holds true for g. Summing both sides of the qWZ-equation (2.4) over all k then gives $S_n = S_{n-1}$. Thus, S_n is free of n, and therefore we have $S_n = S_0$ for all n. Checking that $S_0 = 1$ completes the proof of (2.3). The proof strategy based on these observations is called the qWZ method, originally introduced for the $q = 1$ case by Wilf and Zeilberger [**44**] (cf. also Gessel [**22**], or Petkovšek, Wilf, and Zeilberger [**36**]). Note that with the q-Zeilberger algorithm in hands, computing a qWZ-pair just consists in applying the algorithm to $f_{n,k}$ with order 1 and checking that $\sigma_0 + \sigma_1 = 0$ in case a solution exists (w.l.o.g. we may assume that σ_0 has been normalized to 1). This will be true in almost all instances of closed form summation formulas, eventually with the help of creative symmetrizing.

However, the remarkable fact about qWZ-pairs is that they can be used to produce new identities from existing ones easily — as shown in the following sections.

2.2 Companion Identities

As in the $q = 1$ case (cf. Wilf and Zeilberger [**44**]), a certain type of identities we get "for free" from a qWZ-pair is called the *companion identity.* It is based on the symmetry of f and g in the qWZ-equation (2.4).

Theorem 2.1. *Let (f, g) form a qWZ-pair satisfying the following conditions:*

(F) For each integer k, the limit $f_k := \lim_{n\to\infty} f_{n,k}$ exists and is finite.

(G) We have $\lim_{k\to-\infty} \sum_{n\geq 0} g_{n+1,k} = 0$.

Then the companion identity is given by

$$\sum_{n\geq 0} g_{n+1,k} = \sum_{j\leq k} (f_j - f_{0,j}),$$

provided that both series either converge absolutely or are treated as formal power (Laurent) series.

Proof. Since f and g form a qWZ-pair we have

$$f_{n+1,k} - f_{n,k} = g_{n+1,k} - g_{n+1,k-1}.$$

Summing both sides for n from 0 to N gives

$$f_{N+1,k} - f_{0,k} = \sum_{n=0}^{N} g_{n+1,k} - \sum_{n=0}^{N} g_{n+1,k-1}.$$

Now we let $N \to \infty$ and use (F) to get

$$f_k - f_{0,k} = \sum_{n\geq 0} g_{n+1,k} - \sum_{n\geq 0} g_{n+1,k-1}.$$

If we first replace k by j and then sum over both sides for j from $-l$ to k, we obtain

$$\sum_{j=-l}^{k} (f_j - f_{0,j}) = \sum_{n\geq 0} g_{n+1,k} - \sum_{n\geq 0} g_{n+1,-l-1}.$$

Letting $l \to \infty$ and using (G) gives the companion identity

$$\sum_{j\leq k} (f_j - f_{0,j}) = \sum_{n\geq 0} g_{n+1,k}. \qquad \square$$

Note that condition (G) is satisfied automatically if f (and therefore g) has finite support with respect to k.

The actual computation of f_k depends on whether we treat the companion identity analytically or in the sense of formal power (Laurent) series. In the first case one usually needs at least to make the assumption $|q| < 1$. In general, most of the factors of $f_{n,k}$ have the same limit for $n \to \infty$ whatever our point of view is. For those factors, the computation is carried out fully automatically by our implementation. However, for the remaining factors, the "critical" ones such as b^n or b^{-n}, etc., which analytically need further assumptions ($|b| < 1$ or $|b| > 1$, respectively), or whose limits are not defined in the sense of formal Laurent series (e.g., $\lim_{n\to\infty} b^{-n}$), the limit is kept in an unevaluated form.

As an example, let us consider the q-Chu-Vandermonde identity in the form

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} b \\ k \end{bmatrix}_q q^{k^2} = \begin{bmatrix} b+n \\ n \end{bmatrix}_q.$$

Setting $f_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} b \\ k \end{bmatrix}_q q^{k^2} / \begin{bmatrix} b+n \\ n \end{bmatrix}_q$ we have

$$f_k = \lim_{n\to\infty} f_{n,k} = \frac{(q;q)_b}{(q;q)_k} \begin{bmatrix} b \\ k \end{bmatrix}_q q^{k^2} \qquad \text{and} \qquad f_{0,k} = \frac{\begin{bmatrix} 0 \\ k \end{bmatrix}_q \begin{bmatrix} b \\ k \end{bmatrix}_q}{\begin{bmatrix} b \\ 0 \end{bmatrix}_q} q^{k^2} = \delta_{k,0},$$

where $\delta_{k,0}$ denotes the Kronecker symbol. Running `qZeil` as follows (cf. Riese [**37**] or Appendix B for more details on how to use the `qZeil` package) we obtain:

```
In[1]:= (* first of all load the package *)
        <<qZeil.m

Out[1]= Axel Riese's qZeilberger implementation version 1.8 loaded

In[2]:= (* tell qZeil to compute the companion identity, too *)
        Companion = True;
        qZeil[qBinomial[n,k,q] qBinomial[b,k,q] q^(k^2) / qBinomial[b+n,n,q],
              {k, 0, n}, n, 1, {b}]

Out[3]= SUM[n] == 1
```

```
In[4]:= (* the companion identity is assigned to CompId *)
        CompId

                      2
              1 + k + k  + n                                2
             q               qBinomial[n, k, q] qfac[q, q, b]  qfac[q, q, n]
Out[4]= Sum[-(-------------------------------------------------------------),
               qfac[q, q, -1 + b - k] qfac[q, q, k] qfac[q, q, 1 + b + n]

           {n, 0, ∞}] == -If[k >= 0, 1, 0] +

                 2
               jj
              q    qBinomial[b, b - jj, q] qfac[q, q, b]
           Sum[----------------------------------------, {jj, 0, k}]
                             qfac[q, q, jj]
```

Thus, we automatically find that for $b \geq 0$ and $k \geq 0$ the companion identity reads as

$$\frac{q^{k^2+k+1}\,(1-q^{k+1})}{(1-q^{b+1})} \begin{bmatrix} b \\ k+1 \end{bmatrix}_q \sum_{n=k}^{\infty} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\begin{bmatrix} b+n+1 \\ n \end{bmatrix}_q}\, q^n = 1 - (q;q)_b \sum_{j=0}^{k} \begin{bmatrix} b \\ j \end{bmatrix}_q \frac{q^{j^2}}{(q;q)_j}.$$

For $k = 0$ this identity becomes

$$\frac{q\,(1-q^b)}{1-q^{b+1}} \sum_{n=0}^{\infty} \frac{q^n}{\begin{bmatrix} b+n+1 \\ n \end{bmatrix}_q} = 1 - (q;q)_b. \tag{2.5}$$

Surprisingly, in many instances the $k = 0$ case of the companion identity turns out to be the limiting case of a *Gosper-summable* identity. The only counterexample we have found so far is the companion identity of Ramanujan's bilateral sum (2.22) in Section 2.4.11 below. For instance, we obtain by q-hypergeometric telescoping

$$\frac{q\,(1-q^b)}{1-q^{b+1}} \sum_{n=0}^{m-1} \frac{q^n}{\begin{bmatrix} b+n+1 \\ n \end{bmatrix}_q} = 1 - \begin{bmatrix} b+m \\ b \end{bmatrix}_q^{-1},$$

which for $m \to \infty$ reduces to identity (2.5).

For the special case $b = n$ of the q-Chu-Vandermonde identity we get the result spelled out in Wilf and Zeilberger [**45**] with its $k = 0$ case

$$q \sum_{n=0}^{\infty} \frac{(2-q^n-q^{2n+1})\,(q;q)_n^2}{(1+q^{n+1})\,(q;q)_{2n+1}}\, q^n = 1 - (q;q)_\infty,$$

the $m \to \infty$ case of

$$q \sum_{n=0}^{m-1} \frac{(2-q^n-q^{2n+1})\,(q;q)_n^2}{(1+q^{n+1})\,(q;q)_{2n+1}}\, q^n = 1 - \begin{bmatrix} 2m \\ m \end{bmatrix}_q^{-1}.$$

This shows that the companion identity of some special case of an identity is not the same as the specialization of the companion identity in general.

For further applications of companion identities see Section 2.4.

2.3 Dual Identities

Another method for discovering new identities is based on the fact that to any qWZ-pair one can associate a dual pair that may produce new identities. Once we have found a qWZ-pair, we can easily construct other ones in a way as listed in the following theorem due to Gessel [**22**] and Wilf and Zeilberger [**44**]. Since during the process of dualization the domain of n is temporarily transformed to the negative integers, we shall assume that now *both* parameters of double-indexed sequences run through all integers. However, this is done only for technical reasons concerning intermediate steps. For the final result n will be non-negative again.

Theorem 2.2. *Let (f, g) form a qWZ-pair.*

(i) For integers a and b, $(f^*_{n,k}, g^*_{n,k}) := (f_{n+a,k+b}, g_{n+a,k+b})$ is a qWZ-pair.

(ii) For any $c \in F$, $(f^*_{n,k}, g^*_{n,k}) := (c\cdot f_{n,k}, c\cdot g_{n,k})$ is a qWZ-pair.

(iii) $(f^*_{n,k}, g^*_{n,k}) := (f_{-n,k}, -g_{-n+1,k})$ is a qWZ-pair.

(iv) $(f^*_{n,k}, g^*_{n,k}) := (f_{n,-k}, -g_{n,-k-1})$ is a qWZ-pair.

(v) $(f^*_{n,k}, g^*_{n,k}) := (g_{k,n}, f_{k,n})$ is a qWZ-pair.

(vi) Let $R_1(n,k) := f_{n,k}/f_{n-1,k}$, $R_2(n,k) := f_{n,k}/f_{n,k-1}$, and $R_3(n,k) := g_{n,k}/f_{n,k}$ for all n and k where all quotients are well-defined. Any pair of sequences $(f^*_{n,k}, g^*_{n,k})$ which produces the same R_1, R_2, R_3 over some suitable domain for n and k is a qWZ-pair over this domain.

Proof. (i) – (v) Straightforward by plugging in f^* and g^* into the qWZ-equation (2.4).
(vi) Dividing the qWZ-equation (2.4) by $f_{n,k}$ we get

$$1 - \frac{f_{n-1,k}}{f_{n,k}} = \frac{g_{n,k}}{f_{n,k}} - \frac{g_{n,k-1}}{f_{n,k}} = \frac{g_{n,k}}{f_{n,k}} - \frac{g_{n,k-1}}{f_{n,k-1}} \cdot \frac{f_{n,k-1}}{f_{n,k}}.$$

By our assumptions we may replace f and g by f^* and g^*, respectively. Multiplying through by $f^*_{n,k}$ proves that (f^*, g^*) forms a qWZ-pair. ☐

As in the $q = 1$ case one introduces the operation of *shadowing* (see the work of Wilf and Zeilberger, e.g., [**43, 44, 45, 49**]). Let us consider a sequence defined on $\mathbb{N}$, for instance, $a_n = (q;q)_n$. Then the defining property of a_n is that it satisfies the first order recurrence equation $a_n = (1-q^n)\, a_{n-1}$ together with the initial condition $a_0 = 1$. Trying to extend this sequence to the "opposite side", one could ask for a sequence $\bar{a}_n$ defined on the negative integers such that $\bar{a}_n = (1-q^n)\, \bar{a}_{n-1}$. A sequence that satisfies this condition is

$$\bar{a}_n = \frac{(-1)^n\, q^{\binom{n+1}{2}}}{(q;q)_{-n-1}} \quad \text{for } n < 0.$$

We call $\bar{a}_n$ the *shadow* of a_n. More generally, for $a_{n,k} = (\alpha;q)_{an+bk+c}$, where α is free of n and k, the shadow is defined by

$$\bar{a}_{n,k} = \frac{(-1)^{an+bk+c}\, \alpha^{an+bk+c}\, q^{\binom{an+bk+c}{2}}}{(q^2/\alpha;q)_{-an-bk-c-1}}, \tag{2.6}$$

with the property that $a_{n,k}$ $(n > 0)$ and $\bar{a}_{n,k}$ $(n < 0)$ produce the same R_1 and R_2, defined in Theorem 2.2 (vi). The reason for choosing the denominator of $\bar{a}_{n,k}$ as shown above instead of simply taking $(q/\alpha;q)_{-an-bk-c}$ is that we want to include the case $\alpha = q$ directly.

The shadow $\bar{f}_{n,k}$ of a summand term $f_{n,k}$ is then defined to be the result of formally replacing each factor of the form $(\alpha;q)_{an+bk+c}$ in f according to the shadowing rule (2.6). Since $f_{n,k}$ $(n>0)$ and $\bar{f}_{n,k}$ $(n<0)$ also produce the same R_1 and R_2, the sequences f and $\bar{f}$ are in a certain sense equivalent. Thus, it follows from Theorem 2.2 (vi) — the assumption on R_3 is trivially satisfied, because the certificate (a rational function) is invariant under taking the shadow — that, if f and g form a qWZ-pair for $n>0$, then so do $\bar{f}$ and $\bar{g}$ for $n<0$.

Evidently, one is free to shadow only some of the factors of $f_{n,k}$ and fixing the others, this way getting different shadow pairs. A strategy that gives fruitful results, i.e., non-trivial, well-defined dual qWZ-pairs with finite support in the end, is the following:

Algorithm *q*Shadow. Input: *a qWZ-pair* (f,g)*;* Output: *the shadow pair* $(\bar{f},\bar{g})$*.*

(S1) Let $c_1,\ldots,c_m$ *denote all non-negative integer parameters that* f *and* g *depend on. Define* f^1 *and* g^1 *to be the result of replacing each* $c_i \neq n$ *by* $-c_i-1$ *in* f *and* g*, respectively, to preserve non-negativity under shadowing.*

(S2) Let f^2 *denote the result of rewriting all terms of the form* $(\alpha';q)_{a'n+b'k+c'}$ *in* f^1*, for which* α' *depends on* n *or* k*, using the rule*†

$$(\alpha q^d;q)_e = \frac{(\alpha;q)_{d+e}}{(\alpha;q)_d} \quad \text{for all } d,e\in\mathbb{Z},$$

because the shadowing rule is then also applicable to those factors.

(S3) Since we are dealing with terminating *identities, the upper summation bound is typically induced by factors of* f *like* $(q;q)^{-1}_{ln-mk+d}$*,* $(q^{-ln+d};q)_{mk+e}$*, or* $\begin{bmatrix} ln+d \\ mk+e \end{bmatrix}_q$*, for some* $l,m\in\mathbb{N}^+$ *and* $d,e\in\mathbb{Z}$*. Let* $\bar{f}$ *denote the result of applying the shadowing rule* (2.6) *to all terms of* f^2 *except to those of the form* $(q;q)_{an+bk+c}$*, for which* $a+(l/m)\cdot b=0$ *(but not* $a=0=b$*).*

(S4) Put $\bar{g}_{n,k} := cert^1(n,k)\cdot \bar{f}_{n,k}$*, where* $cert^1(n,k)$ *denotes the certificate after performing step (S1), i.e.,* $cert^1(n,k) = g^1_{n,k}/f^1_{n,k}$*.* □

This is a powerful generalization of the shadowing strategy described by Wilf and Zeilberger [**44**], who only considered parts of the $l=m$ case. Beginning with version 1.6 of the author's package `qZeil`, also step (S1) is performed automatically. Furthermore, the shadowing strategy can now be changed manually by calling `qZeil` with the option `Shadow->`s, where $s=l/m\in\mathbb{Q}$ as in (S3) above. For example, if in the original identity the upper summation bound is induced by the factor $\begin{bmatrix} n \\ k \end{bmatrix}_q$ or $(q^{-n};q)_k$, then for computing the dual identity `qZeil` has to be called with `qZeil[..., Shadow->1]`, since $l=m=1$ in this case. Because the default value for `Shadow` is 1, the option could also be omitted here. On the other hand, if the factor in consideration is $\begin{bmatrix} 2n \\ k \end{bmatrix}_q$ or $(q^{-2n};q)_k$, then the corresponding call should be `qZeil[..., Shadow->2]`, since $l=2$ and $m=1$. This will prevent shadowing of factors of the form $(q;q)_{an-ak+c}$ for $a\neq 0$ in the first case and of factors of the form $(q;q)_{2an-ak+c}$ for $a\neq 0$ in the second case. We will take a closer look at the main idea behind step (S3) below.

The final step in dualization is to pass from the shadow pair $(\bar{f},\bar{g})$ to the *dual pair* (f',g') by a flip of variables and sequences, transforming the domain of n back to the non-negative integers. The dual pair is defined as follows.

†cf. Gasper and Rahman [**20**, (I.17)]

Definition 2.2. With the notation introduced above, the *dual pair* of a qWZ-pair (f, g) is given by

$$(f'_{n,k}, g'_{n,k}) := (\bar{g}_{-k,-n-1}, \bar{f}_{-k-1,-n}).$$

It is easily seen from Theorem 2.2 (iii), (iv), (v), and (i) that (f', g') again forms a qWZ-pair. Note that its certificate is altered via the same change of variables.

Now we shall investigate step (S3) of our shadowing strategy more closely. Suppose that the summand $f_{n,k}$ contains the factor

$$h_{n,k} = \frac{(q;q)_{in+jk+d_1}}{(q;q)_{ln-mk+d_2}\,(q;q)_{on+pk+d_3}},$$

where $i, l, m, p \in \mathbb{N}^+$, $j, o \in \mathbb{N}$ with $o = 0$ if $j \neq 0$, and $d_1, d_2, d_3 \in \mathbb{Z}$. This is usually true for terminating q-hypergeometric identities, however, $h_{n,k}$ might appear in an equivalent form such as $(q^{-n}; q)_k/(q; q)_k$, $\begin{bmatrix} n \\ k \end{bmatrix}_q$, $\begin{bmatrix} 2n \\ n-k \end{bmatrix}_q$, $\begin{bmatrix} n+k \\ 2k \end{bmatrix}_q$, etc. Since $g_{n,k}$ is a rational function multiple of $f_{n,k}$, we may conclude that $g_{n,k} = h_{n,k} \cdot a_{n,k}$ for some q-hypergeometric sequence $a_{n,k}$. Thus, by fixing the term $(q; q)_{ln-mk+d_2}$ in $h_{n,k}$, according to (S3), the shadow of $g_{n,k}$ is then given by

$$\bar{g}_{n,k} = \frac{(q;q)_{-on-pk-d_3-1}}{(q;q)_{ln-mk+d_2}\,(q;q)_{-in-jk-d_1-1}} \cdot b_{n,k},$$

where $b_{n,k}$ equals $\bar{a}_{n,k}$ multiplied with a power of -1 and a power of q. This means, we end up with the dual summand

$$f'_{n,k} = \frac{(q;q)_{pn+ok+p-d_3-1}}{(q;q)_{mn-lk+m+d_2}\,(q;q)_{jn+ik+j-d_1-1}} \cdot b_{-k,-n-1},$$

which again has finite support. Note that $f'_{n,k}$ is well-defined for all n and k satisfying $p(n+1) + ok - d_3 > 0$, a condition that cannot be guaranteed to hold for all n and k in general. However, for the case $j \neq 0$ and $o = 0$ it is immediately clear that $p(n+1) - d_3 > 0$ holds for sufficiently large n and all k. On the other hand we find that for $j = 0$ the lower summation bound $a := \lceil (d_1 + 1)/i \rceil$ in the dual identity does not depend on n. Thus, it is easily seen that $p(n+1) + ok - d_3 > 0$ is satisfied for sufficiently large n and all $k \geq a$. In other words, after substituting $n + n_0$ for n in the dual pair, with $n_0 \in \mathbb{N}$ sufficiently large, the dual summand is well-defined for all $n \in \mathbb{N}$ over the whole summation range. However, in practice we did not find an application up to now which actually needs this transformation.

On the other hand, suppose our shadowing strategy is to regardlessly shadow all factors in $f_{n,k}$. Then the shadow of $g_{n,k}$ becomes

$$\bar{g}_{n,k} = \frac{(q;q)_{-ln+mk-d_2-1}\,(q;q)_{-on-pk-d_3-1}}{(q;q)_{-in-jk-d_1-1}} \cdot c_{n,k}$$

for some q-hypergeometric sequence $c_{n,k}$. Consequently, we are lead to

$$f'_{n,k} = \frac{(q;q)_{-mn+lk-m-d_2-1}\,(q;q)_{pn+ok+p-d_3-1}}{(q;q)_{jn+ik+j-d_1-1}} \cdot c_{-k,-n-1},$$

which neither has finite support nor is well-defined over reasonable domains for n and k.

As with companion identities, note that dualization does not commute with specialization in general, i.e., the dual identity of some special case of an identity is not the same as the specialization of the dual identity. Hence, we get substantially different results by specializing parameters, for instance, to non-zero integer multiples of n, respectively powers of q^n, or, by replacing n by $2n$, etc., as shown below. However, the dualization operation itself is an involution up to constant factors.

For the q-Chu-Vandermonde identity above one gets the following result (cf. Riese [**37**]):

```
In[5]:= (* tell qZeil to compute the dual identity, too *)
        Dual = True;
        qZeil[qBinomial[n,k,q] qBinomial[b,k,q] q^(k^2) / qBinomial[b+n,n,q],
              {k, 0, n}, n, 1, {b}, Shadow->1]

Out[6]= SUM[n] == 1

In[7]:= (* the dual pair is assigned to DualPair *)
        DualPair
                                         2                  2
              k + n  -k/2 - b k + k /2 + n/2 + b n - n /2
Out[7]= {((-1)      q                                      qBinomial[n, k, q]

             qfac[q, q, b + k] qfac[q, q, b - n] qfac[q, q, n]) / qfac[q, q, k],

                                                  2                     2
               k + n  -1 - b - k/2 - b k + k /2 + (3 n)/2 + b n - n /2
          ((-1)      q

             qBinomial[-1 + n, k, q] qfac[q, q, 1 + b + k] qfac[q, q, b - n]

             qfac[q, q, -1 + n]) / qfac[q, q, k]}

In[8]:= (* compute the dual identity from the dual pair *)
        DualId[{k, 0, n}, n]
                                       2
                 k  -k/2 - b k + k /2
             (-1)  q                   qBinomial[n, k, q] qfac[q, q, b + k]
Out[8]= Sum[------------------------------------------------------------, {k, 0, n}]
                                     qfac[q, q, k]

                                         2
                   n  -n/2 - b n + n /2
          == (-1)  q                   qBinomial[b, b - n, q] qfac[q, q, b]
```

Therefore the dual identity reads as

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} b+k \\ k \end{bmatrix}_q q^{\binom{k}{2}-bk} = (-1)^n \begin{bmatrix} b \\ n \end{bmatrix}_q q^{\binom{n}{2}-bn},$$

which is the same as the original identity modulo a renaming of the parameters. An identity satisfying this property is called *self-dual*.

As mentioned above, for the special case $b = n$ we do not obtain just the dual identity with b replaced by n, but

$$\sum_{k=0}^{n} \frac{q^{n-k} + q^n - 2q^k}{1+q^k} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2k \\ k \end{bmatrix}_q = 0,$$

presented by Wilf and Zeilberger [**45**].

Next, let us consider the q-Saalschütz identity in the form

$$\sum_{k=0}^{n} \begin{bmatrix} r-s+m \\ k \end{bmatrix}_q \begin{bmatrix} s-r+n \\ n-k \end{bmatrix}_q \begin{bmatrix} s+k \\ m+n \end{bmatrix}_q q^{(n-k)(r-s+m-k)} = \begin{bmatrix} r \\ n \end{bmatrix}_q \begin{bmatrix} s \\ m \end{bmatrix}_q .$$

The program computes the following dual identity (cf. Riese [**37**]):

$$\sum_{k=0}^{n} \begin{bmatrix} m+k \\ k \end{bmatrix}_q \begin{bmatrix} s \\ r-k \end{bmatrix}_q \begin{bmatrix} m-s \\ n-k \end{bmatrix}_q q^{(n-k)(r-k)} = \begin{bmatrix} m+r-s \\ n \end{bmatrix}_q \begin{bmatrix} n+s \\ r \end{bmatrix}_q .$$

Renaming the parameters we find the q-Saalschütz identity also to be self-dual.

For the special case $m = n$ and $r = s$, the process of dualization leads to the following result (cf. Riese [**37**]):

$$\sum_{k=0}^{n} \frac{(2q^n - q^k - q^{k+n-s} - q^{2k} - q^{2k+n-s} + 2q^{3k-s})\,(q;q)_{n+s-2k-1}}{(1+q^k)\,(q;q)_{s-k}^2} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2k \\ k \end{bmatrix}_q q^{-2k} = 0,$$

where $s \geq n+1$.

Further applications of dual identities are given in the following section.

2.4 Applications

In the following we shall present several dual and companion identities of "standard" terminating q-identities taken from Appendix II of Gasper and Rahman [**20**]. Unfortunately, it turned out that the full power of Gessel's [**22**] method for systematically producing dual identities in the $q = 1$ case cannot be carried over to the q-case completely, since factorization of q-polynomials (where the variables occur as exponents of q) is much harder to handle algorithmically than the $q = 1$ case.

Recall the usual definitions of an ${}_r\phi_s$ *basic hypergeometric series*

$${}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; q, z) \equiv {}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, z \right]$$
$$:= \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(q, b_1, \ldots, b_s; q)_k} \left((-1)^k \, q^{\binom{k}{2}} \right)^{1+s-r} z^k,$$

and an ${}_r\psi_s$ *basic bilateral hypergeometric series*

$${}_r\psi_s(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z) \equiv {}_r\psi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; q, z \right]$$
$$:= \sum_{k=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_s; q)_k} \left((-1)^k \, q^{\binom{k}{2}} \right)^{s-r} z^k .$$

All finite versions of companion identities for $k = 0$ have been found algorithmically by q-hypergeometric telescoping. Krattenthaler [**26**] kindly pointed out many connections between several identities to me.

2.4.1 The q-Binomial Theorem

From the q-binomial theorem [**20**, (II.4)],

$$ {}_1\phi_0(q^{-n}, —; q, z) = (zq^{-n}; q)_n, \tag{2.7} $$

we obtain the dual identity

$$ {}_2\phi_1(q^{-n}, z; 0; q, q) = z^n, $$

which is a special case of the q-Chu-Vandermonde identity [**20**, (II.6)]. The companion identity after replacing z by $-q/z$ is a special case of the ${}_1\phi_1$ summation formula (2.8) below,

$$ \frac{(-q/z)^k}{(q;q)_k} \sum_{n=k}^{\infty} \frac{z^n\,(q^{-n};q)_k}{(-z;q)_{n+1}}\, q^{\binom{n}{2}} = 1 \qquad (k \geq 0), $$

which for $k = 0$ reduces to the $m \to \infty$ case of

$$ \sum_{n=0}^{m-1} \frac{z^n\, q^{\binom{n}{2}}}{(-z;q)_{n+1}} = 1 - \frac{z^m\, q^{\binom{m}{2}}}{(-z;q)_m}. $$

For $z = -q$ and $z = q$ we immediately get

$$ \sum_{n=0}^{m} \frac{(-1)^n\, q^{\binom{n}{2}}}{(q;q)_n} = \frac{(-1)^m\, q^{\binom{m+1}{2}}}{(q;q)_m} \qquad \text{and} \qquad \sum_{n=0}^{m} \frac{q^{\binom{n}{2}}}{(-q;q)_n} = 2 - \frac{q^{\binom{m+1}{2}}}{(-q;q)_m}, $$

respectively, with the limiting cases

$$ \sum_{n=0}^{\infty} \frac{(-1)^n\, q^{\binom{n}{2}}}{(q;q)_n} = 0 \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(-q;q)_n} = 2, $$

respectively. The first identity is a special case of Euler's q-analogue of the exponential function (cf. Andrews [**7**]), whereas the second one was derived by Andrews [**5**] in the context of mock-theta-functions. The dual identity of (2.7) for $z = q^{-n}$ reads as

$$ 2\sum_{k=0}^{n} \frac{(-1)^k\,(q^k + q^{2k} - q^n)}{(1+q^k)\,(q;q)_{n-k}} \begin{bmatrix} 2k \\ k \end{bmatrix}_q q^{\binom{k}{2} - 2nk} = \frac{q^n}{(q;q)_n}. $$

The companion identity of (2.7) for $z = q^{-n}$ is

$$ \frac{q^{-k}}{(q;q)_k} \sum_{n=k}^{\infty} \frac{(-1)^n\,(q^k + q^{n+k+1} - q^{2n+1})\,(q^{-n};q)_k}{(q^{n+2};q)_{n+1}}\, q^{n(3n-2k+1)/2} = 1 \qquad (k \geq 0), $$

which for $k = 0$ reduces to the $m \to \infty$ case of

$$ \sum_{n=0}^{m-1} \frac{(-1)^n\,(1 + q^{n+1} - q^{2n+1})}{(q^{n+2};q)_{n+1}}\, q^{n(3n+1)/2} = 1 - \frac{(-1)^m\, q^{m(3m+1)/2}}{(q^{m+1};q)_m}. $$

2.4.2 The Sum of a ${}_1\phi_1$ Series

The sum of a ${}_1\phi_1$ series [**20**, (II.5)],

$$ {}_1\phi_1(a;c;q,c/a) = \frac{(c/a;q)_\infty}{(c;q)_\infty}, \tag{2.8}$$

turns out to be self-dual for $a = q^{-n}$. The companion identity in this case is

$$\frac{(-1)^k\, c^{k+1}\, q^{\binom{k+1}{2}}}{(c,q;q)_k} \sum_{n=k}^{\infty} (q^{-n};q)_k\,(c;q)_n\, q^{n(k+1)} = 1-(c;q)_\infty \sum_{j=0}^{k} \frac{c^j\, q^{j(j-1)}}{(c,q;q)_j} \qquad (k \ge 0), \tag{2.9}$$

which for $k=0$ reduces to the $m \to \infty$ case of

$$c \sum_{n=0}^{m-1} (c;q)_n\, q^n = 1-(c;q)_m. \tag{2.10}$$

Apparently, identity (2.10) — despite its simplicity — has not been included into the q-hypergeometric database in this form up to now. However, this result can also be derived from the original ${}_1\phi_1$ summation formula. Letting $k \to \infty$ in (2.9) we obtain

$$\sum_{j=0}^{\infty} \frac{c^j\, q^{j(j-1)}}{(c,q;q)_j} = \frac{1}{(c;q)_\infty},$$

which is known as Cauchy's formula (cf., for instance, Andrews [**7**]). Note that letting $k \to \infty$ in the companion identity, in general corresponds to letting $n \to \infty$ in the original identity. The dual identity of (2.8) for $a = q^{-n}$ and $c = q^n$ reads as

$$\sum_{k=0}^{n} \frac{(1-q^{2k+2}-q^{2k+n+2}+q^{3k+2})\,(q;q)_{k+n+1}}{(q;q)_{2k+2}} \begin{bmatrix} n\\ k\end{bmatrix}_q q^{k(2k+1)} = 1,$$

which for $n \to \infty$ becomes

$$\sum_{k=0}^{\infty} \frac{q^{k(2k+1)}}{(q;q)_{2k+1}\,(q;q)_k} = \frac{1}{(q;q)_\infty} - q^2 \sum_{k=0}^{\infty} \frac{q^{2k(k+2)}}{(q;q)_{2k+2}\,(q;q)_k}.$$

The companion identity does not exist in this case because the limit involved is not finite.

2.4.3 The q-Chu-Vandermonde Identity

From the q-Chu-Vandermonde identity [**20**, (II.7)],

$${}_2\phi_1(a,q^{-n};c;q,cq^n/a) = \frac{(c/a;q)_n}{(c;q)_n}, \tag{2.11}$$

we obtain the dual identity

$${}_2\phi_1(aq/c,q^{-n};q^2/c;q,q^{1+n}/a) = \frac{(q/a;q)_n}{(q^2/c;q)_n},$$

which is again q-Chu-Vandermonde. The companion identity is

$$\frac{a^{k+1}\,(c/a;q)_{k+1}}{(c,q;q)_k}\sum_{n=k}^{\infty}\frac{(q^{-n};q)_k\,(c;q)_n}{(a;q)_{n+1}}\,q^{n(k+1)} = \frac{(c;q)_\infty}{(a;q)_\infty}\sum_{j=0}^{k}\frac{(-1)^j\,a^j\,(c/a;q)_j}{(c,q;q)_j}\,q^{\binom{j}{2}}-1 \qquad (k\geq 0),$$

which for $k\to\infty$ turns into the ${}_1\phi_1$ summation formula (2.8) above and for $k=0$ reduces to the $m\to\infty$ case of

$$a\,(1-c/a)\sum_{n=0}^{m-1}\frac{(c;q)_n\,q^n}{(a;q)_{n+1}}=\frac{(c;q)_m}{(a;q)_m}-1.$$

For $a=0$ this identity again gives equation (2.9), whereas the case $c=0$ leads to the likewise simple result

$$a\sum_{n=0}^{m-1}\frac{q^n}{(a;q)_{n+1}}=\frac{1}{(a;q)_m}-1, \tag{2.12}$$

an extended version of the well-known identity

$$\sum_{n=0}^{m}\frac{q^n}{(q;q)_n}=\frac{1}{(q;q)_m}.$$

The dual identity of (2.11) for $a=q^n$ reads as

$$\sum_{k=-n-1}^{n}\frac{c^k\,(q^2/c;q)_k}{(c;q)_k}\begin{bmatrix}2n+1\\ n-k\end{bmatrix}_q q^{k(k-1)}=\frac{(q^{n+1};q)_{n+1}}{(c;q)_n}, \tag{2.13}$$

a special case of Carlitz' [**18**] summation formula

$${}_3\phi_2\left[\begin{matrix}q^{-2n-1},b,c\\ q^{-2n}/b,q^{-2n}/c\end{matrix};q,\frac{q^{2-n}}{bc}\right]=\frac{(bq,cq;q)_n\,(q^2,bcq;q)_{2n}}{(q^2,bcq;q)_n\,(bq,cq;q)_{2n}}. \tag{2.14}$$

Identity (2.13) turns out to be of special interest in the frame of Bailey chains (see, e.g., Andrews [**10**], or Paule [**30**]). For instance, for $c=0$ we obtain

$$\sum_{k=-n-1}^{n}(-1)^k\begin{bmatrix}2n+1\\ n-k\end{bmatrix}_q q^{k(3k+1)/2}=(q^{n+1};q)_{n+1},$$

the unsymmetric counterpart of the Bailey pair identity (cf. Paule [**30**])

$$\sum_{k=-n}^{n}(-1)^k\begin{bmatrix}2n\\ n-k\end{bmatrix}_q q^{k(3k+1)/2}=(q^{n+1};q)_n.$$

In the limit $n\to\infty$ both identities turn into Euler's pentagonal number theorem (see, e.g., Andrews [**7**])

$$\sum_{k=-\infty}^{\infty}(-1)^k\,q^{k(3k+1)/2}=(q;q)_\infty.$$

For $c = q$ identity (2.13) turns into

$$\sum_{k=0}^{n}(1-q^{2k+1})\begin{bmatrix}2n+1\\n-k\end{bmatrix}_q q^{k^2} = \frac{(q^{n+1};q)_{n+1}}{(q;q)_n},$$

the $(2n+1)$ counterpart of the trivial Bailey pair identity (see Chapter 3)

$$\sum_{k=0}^{n}\begin{bmatrix}2n\\n-k\end{bmatrix}_q \delta_{k,0} = \begin{bmatrix}2n\\n\end{bmatrix}_q.$$

Similarly, by putting $c = -q$ in equation (2.13) we obtain

$$\sum_{k=-n-1}^{n}(-1)^k\begin{bmatrix}2n+1\\n-k\end{bmatrix}_q q^{k^2} = (q;q^2)_{n+1},$$

corresponding to the Bailey pair identity (cf. Paule [**30**])

$$\sum_{k=-n}^{n}(-1)^k\begin{bmatrix}2n\\n-k\end{bmatrix}_q q^{k^2} = (q;q^2)_n.$$

For $n \to \infty$ both identities become

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{k^2} = \frac{(q;q)_\infty}{(-q;q)_\infty},$$

a special case of Jacobi's triple product identity (see, e.g., Andrews [**6**, **7**]). The companion identity of (2.11) for $a = q^n$ is a limiting case of the ${}_6\phi_5$ summation formula (2.21) below,

$$\frac{c^k}{(c;q)_k}\sum_{n=k}^{\infty}\frac{(-1)^n(1-q^{2n+1})(q^{-n};q)_k(c;q)_n}{c^n(q/c;q)_{n+1}}\begin{bmatrix}n+k\\k\end{bmatrix}_q q^{\binom{n+1}{2}} = 1 \qquad (k \geq 0),$$

which for $k = 0$ reduces to the $m \to \infty$ case of

$$\sum_{n=0}^{m-1}\frac{(-1)^n(1-q^{2n+1})(c;q)_n}{c^n(q/c;q)_{n+1}}q^{\binom{n+1}{2}} = 1 + \frac{(-1)^m(c;q)_m}{c^m(q/c;q)_m}q^{\binom{m+1}{2}}.$$

The dual identity of (2.11) for $a = q^{-n}$ reads as

$$\sum_{k=0}^{n}\frac{(cq^n - cq^{2k} - q^{2k+1} - q^{2k+n+1} + 2q^{3k+1})(q^{k+2}/c;q)_{k-1}}{(q^2/c;q)_k}\begin{bmatrix}n\\k\end{bmatrix}_q^2 q^{-k} = 0,$$

which for $c = 0$ becomes

$$\sum_{k=0}^{n}(1+q^n-2q^k)\begin{bmatrix}n\\k\end{bmatrix}_q^2 q^{k(k-1)} = 0.$$

For $n \to \infty$ this identity turns into

$$\sum_{k=0}^{\infty}\frac{q^{k(k-1)}}{(q;q)_k^2} = 2\sum_{k=0}^{\infty}\frac{q^{k^2}}{(q;q)_k^2}. \tag{2.15}$$

Observing that the right hand side is summable by a result due to Euler (cf. Andrews [**7**], or Slater [**41**]) we get

$$\sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(q;q)_k^2} = \frac{2}{(q;q)_\infty}.$$

Moreover, equation (2.15) turns out to be the $m \to \infty$ case of the Gosper-summable identity

$$\sum_{k=0}^{m} \frac{1-2q^k}{(q;q)_k^2}\, q^{k(k-1)} = -\frac{q^{m(m+1)}}{(q;q)_m^2}.$$

The companion identity of (2.11) for $a = q^{-n}$ is

$$\frac{c^{k+1}}{(c,q;q)_k} \sum_{n=k}^{\infty} \frac{(2q^k - q^n - cq^{n+k} - q^{n+k+1} + cq^{3n+1})\,(q^{-n};q)_k^2\,(c;q)_n}{(cq^n;q)_{n+2}}\, q^{n(2k+1)} = 1 - (c;q)_\infty \sum_{j=0}^{k} \frac{c^j\, q^{j(j-1)}}{(c,q;q)_j} \qquad (k \ge 0),$$

which for $k = 0$ reduces to the $m \to \infty$ case of

$$c \sum_{n=0}^{m-1} \frac{(2 - q^n - cq^n - q^{n+1} + cq^{3n+1})\,(c;q)_n}{(cq^n;q)_{n+2}}\, q^n = 1 - \frac{(c;q)_m}{(cq^m;q)_m}.$$

The dual identity of (2.11) for $c = q^n$ reads as

$$\sum_{k=0}^{n} (-1)^k\,(1 - aq^{k+1} - q^{k+n+2} + aq^{3k+3} + aq^{3k+n+3} - aq^{4k+3}) \times \frac{(q;q)_{n+k+1}\,(aq^{k+2};q)_{k-1}}{a^k\,(q;q)_{2k+2}} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2}} = (q/a;q)_n.$$

For $n \to \infty$ and $a = 1$ this identity becomes

$$\sum_{k=0}^{\infty} \frac{(-1)^k\,(1 - q^{k+1} + q^{3k+3} - q^{4k+3})}{(1-q^{2k+1})\,(1+q^{k+1})\,(q;q)_{k+1}^2}\, q^{\binom{k+1}{2}} = 1.$$

The companion identity of (2.11) for $c = q^n$ does not exist.

2.4.4 The Bailey-Daum Summation Formula

From the Bailey-Daum summation formula [**20**, (II.9)],

$${}_2\phi_1(a, q^{-n}; aq^{n+1}; q, -q^{n+1}) = \frac{(-q;q)_n\,(aq;q^2)_n}{(aq^{n+1};q)_n}, \qquad (2.16)$$

we obtain the dual identity

$$\sum_{k=0}^{n} \frac{(-1)^k\,(-1, q^{n+1}/a;q)_k}{(q^2/a;q^2)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k+1}{2} - nk} = (-1)^n,$$

a special case of the q-Saalschütz formula (2.19) below, which for $a = 0$ turns into the following known q-analogue of the binomial theorem

$$\sum_{k=0}^{n} (-1)^k \, (-1;q)_k \begin{bmatrix} n \\ k \end{bmatrix}_q = (-1)^n.$$

The companion identity of (2.16) is

$$\frac{(-1)^k \, (a;q)_{k+1} \, q^{k+1}}{(q;q)_k} \sum_{n=k}^{\infty} \frac{(q^{-n};q)_k \, (aq^{n+2};q)_n}{(-q;q)_{n+1} \, (aq;q^2)_{n+1} \, (aq^{n+2};q)_k} \, q^{n(k+1)}$$
$$= 1 - \frac{1}{(-q;q)_\infty \, (aq;q^2)_\infty} \sum_{j=0}^{k} \frac{(a;q)_j \, q^{\binom{j+1}{2}}}{(q;q)_j} \qquad (k \geq 0),$$

which for $k \to \infty$ turns into Ex. 1.16 of Gasper and Rahman [**20**],

$$\sum_{j=0}^{\infty} \frac{(a;q)_j \, q^{\binom{j+1}{2}}}{(q;q)_j} = (-q;q)_\infty \, (aq;q^2)_\infty,$$

and for $k = 0$ reduces to the $m \to \infty$ case of

$$(1-a) \sum_{n=0}^{m-1} \frac{(aq^{n+2};q)_n \, q^{n+1}}{(-q;q)_{n+1} \, (aq;q^2)_{n+1}} = 1 - \frac{(aq^{m+1};q)_m}{(-q;q)_m \, (aq;q^2)_m}.$$

For $a = 0$ this identity becomes

$$\sum_{n=1}^{m} \frac{q^n}{(-q;q)_n} = 1 - \frac{1}{(-q;q)_m},$$

the $a = -q$ case of equation (2.12).

2.4.5 The q-Analogue of Bailey's ${}_2F_1(-1)$ Sum

From the terminating q-analogue of Bailey's ${}_2F_1(-1)$ sum [**20**, (II.10)],

$${}_2\phi_2(q^{-2n}, q^{2n+1}; -q, b; q, -b) = \frac{(bq^{-2n};q^2)_n}{(bq;q^2)_n}, \tag{2.17}$$

we obtain the dual identity

$$\sum_{k=-n}^{n} \frac{(-1)^k \, (1-q^{4k+1}) \, (b;q^2)_k}{b^k \, (q^3/b;q^2)_k} \begin{bmatrix} 4n+1 \\ 2n-2k \end{bmatrix}_q q^{k(3k+1)} = \frac{(q;q^2)_{2n+1}}{(q^2/b;q)_{2n}},$$

which for $b = 0$ becomes

$$\sum_{k=-n}^{n} \frac{1-q^{4k+1}}{1-q} \begin{bmatrix} 4n+1 \\ 2n-2k \end{bmatrix}_q q^{k(2k-1)} = \delta_{n,0}.$$

The companion identity does not exist. The dual identity of (2.17) for $b = q^{2n+1}$ reads as

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1 + q^{2k+n+1} - q^{4k+1} - q^{4k+2}}{(q^2;q^2)_{2k+1} \, (q;q)_{n-2k}} \, q^{4k^2} = \frac{1}{(q^2;q^2)_n}.$$

The companion identity for $b = q^{2n+1}$ is

$$\frac{q^{k(k+3)/2}}{(q^2;q^2)_k} \sum_{n=0}^{\infty} (q^k + q^{k+1} - q^{2n+1} - q^{4n+k+3}) \, \frac{(q^{-2n};q)_{k-1}\,(q^{2n+2};q^2)_n}{(q;q^2)_n} \, q^{2nk} = 1 - \frac{1}{(q;q^2)_\infty} \sum_{j=0}^{k} \frac{(-1)^j\, q^{j^2}}{(q^2;q^2)_j} \qquad (k \geq 0),$$

which for $k = 0$ reduces to the $m \to \infty$ case of

$$\sum_{n=0}^{m-1} (1 + q + q^{2n+2}) \, \frac{(q^{2n+2};q^2)_n}{(q;q^2)_n} \, q^{2n+1} = \frac{(q^{2m+2};q^2)_m}{(q;q^2)_m} - 1.$$

2.4.6 The q-Analogue of Gauss' ${}_2F_1(-1)$ Sum

From the terminating q-analogue of Gauss' ${}_2F_1(-1)$ sum [**20**, (II.11)],

$$ {}_2\phi_2(q^{-2n}, b; q^{1/2-n}\sqrt{b}, -q^{1/2-n}\sqrt{b}; q, -q) = \frac{(q^{1-2n};q^2)_n}{(bq^{1-2n};q^2)_n}, \tag{2.18}$$

we obtain the dual identity

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k\,(q;q^2)_k\,(q/b;q^2)_{n-k}}{b^k} \begin{bmatrix} n \\ 2k \end{bmatrix}_q q^{k(k+1)} = (q/b;q)_n,$$

a special case of the q-Chu-Vandermonde identity, which for $b = 0$ reduces to

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (q;q^2)_k \begin{bmatrix} n \\ 2k \end{bmatrix}_q q^{k(2k-2n+1)} = q^{-\binom{n}{2}},$$

and for $n \to \infty$ turns into Euler's q-analogue of the exponential function (cf. Andrews [**7**])

$$\sum_{k=0}^{\infty} \frac{b^k\, q^{\binom{k}{2}}}{(q;q)_k} = (-b;q)_\infty.$$

The companion identity of (2.18) is

$$\frac{(b;q)_{k+1}}{(q;q)_k} \, q^{k(k+3)/2} \sum_{n=0}^{\infty} \frac{b^n\,(q^{-2n-1};q)_k\,(q/b;q^2)_n}{(bq^{1-2n};q^2)_k\,(q;q^2)_{n+1}} = 1 \qquad (k \geq 0, |b| < 1),$$

which for $k = 0$ reduces to the $m \to \infty$ case of

$$(1-b) \sum_{n=0}^{m-1} \frac{b^n\,(q/b;q^2)_n}{(q;q^2)_{n+1}} = 1 - \frac{b^m\,(q/b;q^2)_m}{(q;q^2)_m}.$$

For $b = 0$ this identity becomes

$$\sum_{n=0}^{m-1} \frac{(-1)^n\, q^{n^2}}{(q;q^2)_{n+1}} = 1 - \frac{(-1)^m\, q^{m^2}}{(q;q^2)_m}.$$

Letting $m \to \infty$ we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, q^{n^2}}{(q;q^2)_{n+1}} = 1,$$

the $s = q$ and $x = -1$ case of Andrews' [**5**] transformation formula

$$\sum_{n=0}^{\infty} \frac{x^n\, q^{n^2}}{(s;q^2)_{n+1}} = \sum_{n=0}^{\infty} (-xq/s;q^2)_n\, s^n.$$

The dual identity of (2.18) for $b = q^{-2n}$ reads as

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (2q^{2n} - q^{2k+n} - q^{2k+n+1} - q^{2k+2n} + q^{6k+1})\,(q;q^2)_{n-2k-1}\,(q;q^2)_k^2 \begin{bmatrix} n \\ 2k \end{bmatrix}_q^2 q^{-4k} = 0,$$

which for $n \to \infty$ becomes

$$(q;q^2)_\infty \sum_{k=0}^{\infty} \frac{(1+q-2q^{2k})\,(q;q^2)_{2k-1}}{(q;q)_{2k}^2}\, q^{2k} = q \sum_{k=0}^{\infty} \frac{q^{2k}}{(q^2;q^2)_k^2}.$$

The companion identity does not exist in this case. The dual identity of (2.18) for $b = q^{2n}$ reads as

$$\sum_{k=-n-1}^{n} (-1)^k \begin{bmatrix} 4n+2 \\ 2n-2k \end{bmatrix}_q q^{k(3k+1)} = (1-q^{4n+2})\,(-q;q)_{2n},$$

which for $n \to \infty$ again turns into Euler's pentagonal number theorem. The companion identity of (2.18) for $b = q^{2n}$ is

$$\frac{q^{k(k+3)/2}}{(q;q)_k\,(q;q^2)_k} \sum_{n=0}^{\infty} (-1)^n\,(1+q^{2n+1})\,(q^{-1-2n}, q^{2n+1}; q)_k\, q^{n^2} = 1 \qquad (k \ge 0),$$

which for $k = 0$ reduces to the $m \to \infty$ case of the trivial identity

$$\sum_{n=0}^{m-1} (-1)^n\,(1+q^{2n+1})\,q^{n^2} = (-1)^{m-1}\,q^{m^2} + 1.$$

2.4.7 The q-Saalschütz Formula

The q-Saalschütz (or q-Pfaff-Saalschütz) formula [**20**, (II.12)],

$$ {}_3\phi_2(a, b, q^{-n}; c, abc^{-1}q^{1-n}; q, q) = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}, \tag{2.19}$$

turns out to be self-dual. The companion identity is

$$\frac{c}{ab}\,\frac{(a,b;q)_{k+1}}{(c,q;q)_k} \sum_{n=k}^{\infty} \frac{(q^{-n};q)_k\,(c, c/ab;q)_n}{(c/a, c/b;q)_{n+1}\,(abq^{1-n}/c;q)_k}\, q^n$$

$$= 1 - \frac{(c, c/ab;q)_\infty}{(c/a, c/b;q)_\infty} \sum_{j=0}^{k} \frac{c^j\,(a,b;q)_j}{a^j\,b^j\,(c,q;q)_j} \qquad (k \ge 0),$$

which for $k \to \infty$ turns into the q-Gauss sum [**20**, (II.8)] and for $k = 0$ reduces to the $m \to \infty$ case of

$$\frac{c\,(1-a)\,(1-b)}{ab} \sum_{n=0}^{m-1} \frac{(c, c/ab; q)_n}{(c/a, c/b; q)_{n+1}}\, q^n = 1 - \frac{(c, c/ab; q)_m}{(c/a, c/b; q)_m}.$$

The dual identity of (2.19) for $a = q^n$ reads as

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\,(bq/c, c; q)_k}{b^k\,(q^2/c, cq/b; q)_k} \begin{bmatrix} 2n+1 \\ n-k \end{bmatrix}_q q^{\binom{k+1}{2}} = \frac{(q/b; q)_n\,(q^{n+1}; q)_{n+1}}{(cq/b, q^2/c; q)_n},$$

a generalization of identity (2.13) but still a special case of (2.14). The companion identity does not exist in this case. The dual identity of (2.19) for $a = q^{-n}$ reads as

$$\sum_{k=0}^{n} (2c^2q^n - c^2q^k - c^2q^{k+n} - cq^{k+n+1} - bcq^{k+n+1} + cq^{3k+1} + bcq^{3k+1} + bq^{3k+2} + \\ bq^{3k+n+2} - 2bq^{4k+2})\, \frac{c^{2k}\,(c/bq; q)_{n-2k}\,(bq/c)_k^2\,(q^{k+2}/c; q)_{k-1}}{b^{2k}\,(q^2/c; q)_k} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 q^{k(k-3)} = 0.$$

The companion identity of (2.19) for $a = q^{-n}$ is

$$\frac{c}{b^2}\,\frac{(b; q)_{k+1}}{(c, q; q)_k} \sum_{n=k}^{\infty} (-2bq^k + bq^n + cq^{n+k} + bcq^{n+k} + bq^{n+k+1} - c^2q^{3n} - cq^{3n+1} - \\ bcq^{3n+1} - c^2q^{3n+k+1} + 2c^2q^{4n+1})\, \frac{(q^{-n}; q)_k^2\,(c; q)_n\,(cq^{n+1}/b; q)_{n-1}}{(c/b; q)_{n+1}\,(cq^n; q)_{n+2}\,(bq^{1-2n}/c; q)_k}\, q^n \\ = 1 - \frac{(c; q)_\infty}{(c/b; q)_\infty} \sum_{j=0}^{k} \frac{(-1)^j\,c^j\,(b; q)_j}{b^j\,(c; q)_j}\, q^{\binom{j}{2}} \qquad (k \geq 0),$$

which for $k = 0$ reduces to the $m \to \infty$ case of

$$\frac{c\,(1-b)}{b^2} \sum_{n=0}^{m-1} (-2b + bq^n + cq^n + bcq^n + bq^{n+1} - c^2q^{3n} - cq^{3n+1} - bcq^{3n+1} - c^2q^{3n+1} + 2c^2q^{4n+1}) \\ \times \frac{(c; q)_n\,(cq^{n+1}/b; q)_{n-1}}{(c/b; q)_{n+1}\,(cq^n; q)_{n+2}}\, q^n = 1 - \frac{(c, cq^m/b; q)_m}{(c/b, cq^m; q)_m}.$$

Since the case $c = q^n$ leads to a rather lengthy dual identity involving a polynomial of degree 7, we do not state the result here.

2.4.8 The q-Dixon Formula

For the q-Dixon sum [**20**, (II.14)],

$${}_4\phi_3 \left[\begin{matrix} a^2, -aq, b, q^{-n} \\ -a, a^2q/b, a^2q^{1+n} \end{matrix}; q, \frac{aq^{n+1}}{b} \right] = \frac{(a^2q, aq/b; q)_n}{(aq, a^2q/b; q)_n}, \tag{2.20}$$

the dual identity is a special case of the q-Saalschütz identity. The companion identity is

$$\frac{(aq/b)^{k+1}}{(b;q)_{k+1}\,(q,a^2q/b;q)_k}\sum_{n=k}^{\infty}\frac{(q^{-n};q)_k\,(a;q)_{n+1}\,(a^2q/b;q)_n}{(a^2q^{k+1},aq/b;q)_{n+1}}\,q^{n(k+1)}$$
$$=\frac{(a,a^2q/b;q)_\infty}{(a^2,aq/b;q)_\infty}\sum_{j=0}^{k}\frac{(-1)^j\,a^j\,(1+aq^j)\,(a^2,b;q)_j}{b^j\,(q;q)_j}\,q^{\binom{j+1}{2}}-1 \qquad (k\ge 0),$$

which for $k=0$ leads to the same identity as the q-Saalschütz companion identity. The dual identity of (2.20) for $b=q^n$ reads as

$$\sum_{k=-n-1}^{n}\frac{(-1)^k\,a^k\,(aq,q^{n+1}/a^2;q)_k}{(q/a,a^2q^{1-n};q)_k}\begin{bmatrix}2n+1\\ n-k\end{bmatrix}_q q^{\binom{k}{2}-nk}=(aq)^{-n}\,\frac{(q^{n+1};q)_{n+1}}{(1/a^2;q)_n},$$

which is again a special case of identity (2.14). For $a=0$ this identity becomes

$$\sum_{k=-n-1}^{n}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_{n-k}\,(q;q)_{n+k+1}}=\delta_{n,0},$$

the unsymmetric counterpart of the Bailey pair identity (cf. Paule [**30**])

$$\sum_{k=-n}^{n}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_{n-k}\,(q;q)_{n+k}}=\delta_{n,0}.$$

The companion identity of (2.20) for $b=q^n$ is again a special case of the ${}_6\phi_5$ summation formula (2.21) below,

$$a^{k+1}\,q^k\sum_{n=k}^{\infty}\frac{a^n\,(1-q^{2n+1})\,(q^{-n};q)_k\,(aq,1/a^2;q)_n}{(a^2q^{k+1};q)_{n+1}\,(q/a;q)_n\,(a^2q^{1-n};q)_k}\begin{bmatrix}n+k\\ k\end{bmatrix}_q=-1 \qquad (k\ge 0,|a|<1),$$

which for $k=0$ reduces to the $m\to\infty$ case of

$$\sum_{n=0}^{m-1}\frac{a^{n+1}\,(1-q^{2n+1})\,(aq,1/a^2;q)_n}{(a^2q;q)_{n+1}\,(q/a;q)_n}=\frac{a^m\,(aq,1/a^2;q)_m}{(a^2q,1/a;q)_m}-1.$$

2.4.9 The Sum of a ${}_6\phi_5$ Series

For the terminating sum of a very-well-poised ${}_6\phi_5$ series [**20**, (II.21)],

$${}_6\phi_5\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,c,q^{-n}\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq^{n+1}\end{matrix};q,\frac{aq^{n+1}}{bc}\right]=\frac{(aq,aq/bc;q)_n}{(aq/b,aq/c;q)_n}, \qquad (2.21)$$

we obtain the dual identity

$$\sum_{k=0}^{n}\frac{(-1)^k\,(q^{n+1}/a,bc/a;q)_k}{(bq/a,cq/a;q)_k}\begin{bmatrix}n\\ k\end{bmatrix}_q q^{\binom{k+1}{2}-nk}=(bc/a)^n\,\frac{(q/b,q/c;q)_n}{(bq/a,cq/a;q)_n},$$

a special case of the q-Saalschütz formula, which for $a=0$ turns into the following well-known q-analogue of the binomial theorem

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\ k\end{bmatrix}_q q^{\binom{k}{2}}=\delta_{n,0}.$$

For $a = 1$ and $b = 0 = c$ we get

$$\sum_{k=0}^{n} \frac{(-1)^k \, (q;q)_{n+k}}{(q;q)_{n-k} \, (q;q)_k} \, q^{\binom{k+1}{2}-nk} = q^{n(n+1)},$$

which is a special case of the "reversed" q-Chu-Vandermonde sum [**20**, (II.6)]. The companion identity of (2.21) is

$$(aq/bc)^{k+1} \, \frac{(a,b,c;q)_{k+1}}{(aq/b,aq/c,q;q)_k} \sum_{n=k}^{\infty} \frac{(aq/b,aq/c;q)_n \, (q^{-n};q)_k}{(aq/bc;q)_{n+1} \, (a;q)_{n+k+2}} \, q^{n(k+1)}$$
$$= \frac{(aq/b,aq/c;q)_\infty}{(a,aq/bc;q)_\infty} \sum_{j=0}^{k} \frac{(-1)^j \, a^j \, (1-aq^{2j}) \, (a,b,c;q)_j}{b^j \, c^j \, (aq/b,aq/c,q;q)_j} \, q^{\binom{j+1}{2}} - 1 \qquad (k \geq 0),$$

which for $k = 0$ leads to the same identity as the q-Saalschütz companion identity.

2.4.10 Jackson's q-Analogue of Dougall's ${}_7F_6$ Sum

For Jackson's q-analogue of Dougall's ${}_7F_6$ sum [**20**, (II.22)],

$${}_8\phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix} ; q, q \right] = \frac{(aq, aq/bc, aq/bd, aq/cd;q)_n}{(aq/b, aq/c, aq/d, aq/bcd;q)_n},$$

where $a^2 q = bcdeq^{-n}$, we find rather lengthy dual and companion identities. Since simple special cases of them can also be derived from other identities, we do not present the results here.

2.4.11 Ramanujan's Bilateral Sum

For the terminating version of Ramanujan's bilateral sum [**20**, (II.29)],

$${}_1\psi_1(q^{-n}; q^{n+1}; q, z) = \frac{(q, zq^{-n}, q^{n+1}/z;q)_n}{(q^{n+1};q)_n}, \tag{2.22}$$

we obtain the dual identity

$$\sum_{k=0}^{n} \frac{(q^n - q^k + zq^{3k} - zq^{4k+n+1}) \, (z;q)_{2k}}{z^k \, (1-q^{2k+1})} \begin{bmatrix} n+k \\ 2k \end{bmatrix}_q q^{-k(2n+1)} = q^n \, z^{n+1}.$$

The companion identity is

$$z^{k-1} \sum_{n=k}^{\infty} \frac{(-1)^n \, (z - zq^{n+k+1} + q^{3n+k+3} - q^{4n+3}) \, (q^{-n};q)_k}{z^n \, (q^{n+1};q)_{k+1} \, (q/z;q)_{2n+2}} \begin{bmatrix} 2n \\ n \end{bmatrix}_q q^{\binom{n+1}{2}} = 1 \qquad (k \geq 0).$$

For this identity, the $k = 0$ case is not Gosper-summable!

2.4.12 Bailey's Sum of a ${}_3\psi_3$ Series

For Bailey's sum of a well-poised ${}_3\psi_3$ series [**20**, (II.31)],

$${}_3\psi_3 \left[\begin{matrix} q^{-n}, c, d \\ q^{n+1}, q/c, q/d \end{matrix} ; q, \frac{q^{n+1}}{cd} \right] = \frac{(q, q/cd;q)_n}{(q/c, q/d;q)_n},$$

we obtain (with creative symmetrizing) the same dual identity as for the terminating sum of a very-well-poised ${}_6\phi_5$ series (2.21) above. The companion identity is

$$(q/cd)^{k+1}\,\frac{(c,d;q)_{k+1}}{(q/c,q/d;q)_k}\sum_{n=k}^{\infty}\frac{(q/c,q/d;q)_n\,(q^{-n};q)_k}{(q/cd;q)_{n+1}\,(q;q)_{n+k+1}}\;q^{n(k+1)}$$
$$=\frac{(q/c,q/d;q)_\infty}{(q,q/cd;q)_\infty}\sum_{j=-\infty}^{k}\frac{(-1)^j\,(1+q^j)\,(c,d;q)_j}{c^j\,d^j\,(q/c,q/d;q)_j}\;q^{\binom{j+1}{2}}-2 \qquad (k\geq 0).$$

Note that for $k=0$ the left hand side is Gosper summable by the $k=0$ case of the q-Saalschütz companion identity.

Chapter 3

Walking Along Bailey Chains

Based on a fundamental q-series transform due to Bailey [**15**], whose potential for iteration was independently observed by Andrews [**9**, **10**] and Paule [**27**, **28**], we shall describe in this chapter how the concept of Bailey pairs and Bailey chains can be used to easily prove q-identities by reducing them to more elementary ones as well as — by reversing this process — to successively construct (infinitely many) identities from a certain class of existing ones, so-called Bailey pair identities. In particular we shall demonstrate how the author's implementation of a q-analogue of Gosper's and Zeilberger's algorithm (cf. Paule and Riese [**33**]) together with the new extension package `Bailey` can be used both to produce computer proofs for certain (classical and new) results and to find new identities.

3.1 Basic Definitions and Tools

Let $\mathbb{Z}$ denote the set of all integers, $\mathbb{N}$ the set of all non-negative integers, and let F be a field of characteristic 0. In the following we consider q as an indeterminate which could be specialized to a non-zero complex number (with $|q| < 1$ for limit considerations). Let the *q-shifted factorial* of $a \in F$ be defined as usual (see, e.g., Gasper and Rahman [**20**]) by

$$(a;q)_k := \begin{cases} (1-a)\,(1-aq)\cdots(1-aq^{k-1}), & \text{if } k>0,\\ 1, & \text{if } k=0,\\ \left[(1-aq^{-1})\,(1-aq^{-2})\cdots(1-aq^{k})\right]^{-1}, & \text{if } k<0,\end{cases}$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$

where products of q-shifted factorials will be abbreviated by

$$(a_1,a_2,\ldots,a_m;q)_k := (a_1;q)_k\,(a_2;q)_k\cdots(a_m;q)_k.$$

The *q-binomial coefficients* (also called *Gaussian polynomials*) are then given by

$$\begin{bmatrix} n\\ k\end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & \text{if } 0\le k\le n,\\ 0, & \text{otherwise.}\end{cases}$$

Without proof we state the following well-known version of the q-binomial theorem.

Theorem 3.1. *For $x \neq 0$ we have*

$$\sum_{k=-n}^{n} (-1)^k x^k \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q q^{\binom{k}{2}} = (x, q/x; q)_n. \tag{3.1}$$

If we put $x = -z\sqrt{q}$ in equation (3.1), replace q by q^2, and let $n \to \infty$, we are led to Jacobi's triple product identity (see, e.g., Andrews [**6**, **7**]).

Theorem 3.2. *For $z \neq 0$ we have*

$$\sum_{k=-\infty}^{\infty} q^{k^2} z^k = (q^2, -qz, -q/z; q^2)_\infty. \tag{3.2}$$

3.2 Bailey Pairs and Bailey Chains

The notion of Bailey pairs and Bailey chains was introduced in full generality 1984 by Andrews [**9**], inspired by Bailey's [**14**, **15**] work on Rogers-Ramanujan type identities. Important special cases were discovered independently by Paule [**27**]. In our presentation of the fundamental notions we shall follow Paule's survey article [**30**].

3.2.1 Ordinary and Bilateral Bailey Pairs

Definition 3.1. We say that two sequences $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ form an *(ordinary) Bailey pair relative to x* if

$$\sum_{k=0}^{n} \frac{a_k}{(q;q)_{n-k}\,(xq;q)_{n+k}} = b_n \tag{3.3}$$

for all $n \geq 0$.

With this notation Bailey's [**15**] fundamental result can be stated as following.

Lemma 3.3 (Bailey's Lemma). *For any Bailey pair (a, b) relative to x we have*

$$\sum_{k=0}^{\infty} (\rho_1, \rho_2; q)_k \left(\frac{xq}{\rho_1\rho_2}\right)^k b_k = \frac{(xq/\rho_1, xq/\rho_2; q)_\infty}{(xq, xq/\rho_1\rho_2; q)_\infty} \sum_{k=0}^{\infty} \frac{(\rho_1, \rho_2; q)_k}{(xq/\rho_1, xq/\rho_2; q)_k} \left(\frac{xq}{\rho_1\rho_2}\right)^k a_k. \tag{3.4}$$

Proof. See Andrews [**10**]. □

With Bailey's Lemma in hands a straightforward way to prove identities is then to find a suitable Bailey pair (a, b) and parameters ρ_1, ρ_2 such that substituting (a, b) into equation (3.4) gives the desired identity. This is exactly how Slater [**40**] skillfully constructed a list of 130 identities of this type. However, the full power of Bailey's Lemma lies in its potential for iteration, which was obviously missed by Bailey himself, but observed independently by Andrews [**9**, **10**] and Paule [**27**, **28**, **30**]. For this we consider the special case $\rho_1 = q^{-m}$ and $\rho_2 = q^{-n}$ with $m, n \in \mathbb{N}$ in Bailey's Lemma,

$$\sum_{k=0}^{\infty} (q^{-m}, q^{-n}; q)_k \left(xq^{m+n+1}\right)^k b_k = \frac{(xq^{m+1}; q)_n}{(xq; q)_n} \sum_{k=0}^{\infty} \frac{(q^{-m}, q^{-n}; q)_k}{(xq^{m+1}, xq^{n+1}; q)_k} \left(xq^{m+n+1}\right)^k a_k,$$

which in the limit $m \to \infty$ turns into

$$\sum_{k=0}^{n} \frac{q^{k^2} x^k}{(q;q)_{n-k}}\, b_k = \sum_{k=0}^{n} \frac{q^{k^2} x^k}{(q;q)_{n-k}\,(xq;q)_{n+k}}\, a_k,$$

where we made use of the rewriting rule (see, e.g., Gasper and Rahman [**20**] or Slater [**41**])

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}\,(-1)^k\, q^{\binom{k}{2}-nk}.$$

Since a and b form a Bailey pair we finally obtain after replacing a_k by $q^{-k^2} x^{-k}\, a_k$

$$\sum_{k=0}^{n} \frac{a_k}{(q;q)_{n-k}\,(xq;q)_{n+k}} = \sum_{j=0}^{n} \frac{q^{j^2} x^j}{(q;q)_{n-j}} \sum_{k=0}^{j} \frac{q^{-k^2} x^{-k}\, a_k}{(q;q)_{j-k}\,(xq;q)_{j+k}}. \tag{3.5}$$

A short proof of key-equation (3.5) using merely an operator version of the q-binomial theorem was given by Paule [**29**].

Now the iteration mechanism becomes evident. Since the inner sum on the right hand side is more or less of the same form as the sum on the left hand side, we may successively replace the inner sum on the right by the corresponding result of (3.5) with a_k replaced by $q^{-k^2} x^{-k}\, a_k$ until we end up with a simpler or known identity. Vice versa, equation (3.5) tells us how to pass from one Bailey pair to another as formulated in the following theorem.

Theorem 3.4. *If (a,b) form a Bailey pair relative to x then so do (a',b'), where*

$$a'_n = q^{n^2} x^n\, a_n \quad \text{and} \quad b'_n = \sum_{j=0}^{n} \frac{q^{j^2} x^j}{(q;q)_{n-j}}\, b_j$$

for all $n \geq 0$.

Proof. The assertion follows immediately from equation (3.5). □

In practice Bailey pair identities frequently appear in symmetrized form. This gives rise to the following definition (see Paule [**30**]).

Definition 3.2. We say that two sequences $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{N}}$ form a *d-bilateral Bailey pair* if

$$\sum_{k=-n-d}^{n} \frac{a_k}{(q;q)_{n-k}\,(q;q)_{n+k+d}} = b_n$$

for $d \in \{0,1\}$ and all $n \geq 0$.

Andrews and Hickerson [**11**] call pairs arising in this symmetrized form simply "bilateral", we use the name "d-bilateral" instead to explicitly distinguish between both cases.

The corresponding iteration rules for $d = 0$ and $d = 1$ can be derived by specializing $x = 1$, $a_0 = c_0$ and $a_k = c_k + c_{-k}$ for $k \geq 1$, respectively $x = q$ and $a_k = (c_k + c_{-k-1})/(1-q)$ for $k \geq 0$, in equation (3.5) which then reduces to

$$\sum_{k=-n-d}^{n} \frac{c_k}{(q;q)_{n-k}\,(q;q)_{n+k+d}} = \sum_{j=0}^{n} \frac{q^{j^2+dj}}{(q;q)_{n-j}} \sum_{k=-j-d}^{j} \frac{q^{-k^2-dk}\, c_k}{(q;q)_{j-k}\,(q;q)_{j+k+d}}. \tag{3.6}$$

Thus, we have:

Corollary 3.5. *Theorem 3.4 holds true for d-bilateral Bailey pairs (a, b) and (a', b') with $x = q^d$, where the relation between a_n and a'_n is extended to hold for all $n \in \mathbb{Z}$.*

In the limit $n \to \infty$, equation (3.6) becomes

$$\frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} c_k = \sum_{j=0}^{\infty} q^{j^2+dj} \sum_{k=-j-d}^{j} \frac{q^{-k^2-dk}\, c_k}{(q;q)_{j-k}\,(q;q)_{j+k+d}}. \tag{3.7}$$

Example 3.1. Let us apply the iteration machinery to $c_k = (-1)^k\, q^{k(5k-1)/2}$ with $d = 0$:

$$\begin{aligned}
\frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k\, q^{k(5k-1)/2} &\overset{(3.7)}{=} \sum_{j=0}^{\infty} q^{j^2} \sum_{k=-j}^{j} \frac{(-1)^k\, q^{k(3k-1)/2}}{(q;q)_{j-k}\,(q;q)_{j+k}} \\
&\overset{(3.6)}{=} \sum_{j=0}^{\infty} q^{j^2} \sum_{l=0}^{j} \frac{q^{l^2}}{(q;q)_{j-l}} \sum_{k=-l}^{l} \frac{(-1)^k\, q^{k(k-1)/2}}{(q;q)_{l-k}\,(q;q)_{l+k}} \\
&\overset{(3.1)}{=} \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q;q)_j}.
\end{aligned}$$

Using Jacobi's triple product identity (3.2) we finally obtain

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q;q)_j} = \frac{1}{(q;q^5)_\infty\,(q^4;q^5)_\infty}. \tag{3.8}$$

This is an easy proof of the famous *first Rogers-Ramanujan identity*. For more information about these celebrated identities see, for instance, Andrews [**7**]. □

3.2.2 Bailey Chains

Suppose we are given a Bailey pair. By repeated application of Theorem 3.4 one is able to construct a sequence of Bailey pairs forming the constituents of a so-called Bailey chain.

Definition 3.3. The sequence of ordinary Bailey pairs

$$\left(a^{(0)}, b^{(0)}\right) \to \left(a^{(1)}, b^{(1)}\right) \to \left(a^{(2)}, b^{(2)}\right) \to \cdots$$

is called an *(ordinary) Bailey chain*, where $(a^{(i)}, b^{(i)})$ is constructed from $(a^{(i-1)}, b^{(i-1)})$ by applying Theorem 3.4 once.

Since d-bilateral Bailey pairs can be viewed as special cases of ordinary Bailey pairs, we call a sequence of d-bilateral Bailey pairs constructed by applications of Corollary 3.5 also a Bailey chain.

Moving in a Bailey chain as described above — we say that in this case one moves to the right — can be done algorithmically with the `Bailey` package, an add-on for the author's Mathematica implementation of the q-Zeilberger algorithm (cf. Paule and Riese [**33**]). The corresponding command is

```
BaileyForw[{a, b}, n, x, opts]
```

where the ordinary Bailey pair (a_n, b_n) relative to x is specified by a, b, n, and the optional parameter x with default value 1. For d-bilateral Bailey pairs, x must be set to q^d. The only option we describe here is `Base->`q^c, where c is a non-zero integer. In this case, all expressions are considered to be in base q^c instead of the default value q. Further options will be presented below.

Example 3.2. Let us again consider the first Rogers-Ramanujan identity above and its corresponding Bailey chain. We start with the 0-bilateral Bailey pair $(a^{(0)}, b^{(0)})$, where

$$a_n^{(0)} = (-1)^n\, q^{\binom{n}{2}} \quad \text{and} \quad b_n^{(0)} = \delta_{n,0},$$

which forms a Bailey pair because of equation (3.1) with $x = 1$. Here $\delta_{n,0}$ denotes the Kronecker symbol. Then, by Corollary 3.5, $(a^{(1)}, b^{(1)})$ is given by

$$a_n^{(1)} = q^{n^2} a_n^{(0)} = (-1)^n\, q^{n(3n-1)/2} \quad \text{and} \quad b_n^{(1)} = \sum_{j=0}^{n} \frac{q^{j^2}}{(q;q)_{n-j}}\, b_j^{(0)} = \frac{1}{(q;q)_n},$$

and $(a^{(2)}, b^{(2)})$ by

$$a_n^{(2)} = q^{n^2} a_n^{(1)} = (-1)^n\, q^{n(5n-1)/2} \quad \text{and} \quad b_n^{(2)} = \sum_{j=0}^{n} \frac{q^{j^2}}{(q;q)_{n-j}}\, b_j^{(1)} = \sum_{j=0}^{n} \frac{q^{j^2}}{(q;q)_{n-j}\,(q;q)_j}.$$

Proceeding algorithmically, i.e., by using the `Bailey` package, we get:

```
In[1]:= (* first of all load the package *)
        <<qZeil.m

Out[1]= Axel Riese's qZeilberger implementation version 1.8 loaded

In[2]:= BaileyForw[{(-1)^n q^(n(n-1)/2), Delta[n,0]}, n]

                                 2
                n  -n/2 + (3 n )/2        1
Out[2]= {(-1)  q                  , -------------}
                                    qfac[q, q, n]

In[3]:= BaileyForw[%, n]
                                                             2
                                 2                         jj
                n  -n/2 + (5 n )/2                        q
Out[3]= {(-1)  q                  , Sum[---------------------------------, {jj, 0, n}]}
                                          qfac[q, q, jj] qfac[q, q, -jj + n]
```

The corresponding Bailey pair identities for $(a^{(i)}, b^{(i)})$, $i \in \{0, 1, 2\}$, are then

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{k(k-1)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \delta_{n,0}, \tag{3.9}$$

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{k(3k-1)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{1}{(q;q)_n}, \tag{3.10}$$

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{k(5k-1)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \sum_{j=0}^{n} \frac{q^{j^2}}{(q;q)_{n-j}\,(q;q)_j}, \tag{3.11}$$

where the last one turns into the first Rogers-Ramanujan identity for $n \to \infty$ and using Jacobi's triple product identity (3.2). □

Next, we shall show that an ordinary Bailey pair (a, b) is uniquely determined by only one of the sequences a or b. Indeed, if a is given, then b is explicitly determined by (3.3). Conversely, by inverting relation (3.3) the following theorem tells us how to compute a when b is given.

Theorem 3.6. *If (a, b) form an ordinary Bailey pair relative to x then*

$$a_n = (1 - xq^{2n}) \sum_{k=0}^{n} \frac{(-1)^{n-k} q^{\binom{n-k}{2}} (xq; q)_{n+k-1}}{(q; q)_{n-k}} b_k \tag{3.12}$$

for all $n \geq 0$, and vice versa. (For $x = 1$ and $n = 0$, eq. (3.12) should be interpreted as $a_0 = b_0$.)

Proof. For a "classical" proof see Andrews [**8**]. We give a computer proof here, where we use the fact that for proving an assertion of the form

$$(\forall n)\ \ a_n = \sum_{k=0}^{n} c_{n,k}\, b_k \quad \text{if and only if} \quad (\forall n)\ \ b_n = \sum_{k=0}^{n} d_{n,k}\, a_k$$

it is sufficient to show that the (infinite) triangular matrices $C = (c_{n,k})$ and $D = (d_{n,k})$ are inverse to each other, or in other words

$$\sum_{k=m}^{n} c_{n,k}\, d_{k,m} = \delta_{n,m}.$$

However, this can be done with q-hypergeometric telescoping (cf. Paule and Riese [**33**]):

```
In[4]:= qTelescope[(1-x q^(2n)) (-1)^(n-k) q^Binomial[n-k,2] qfac[x q,q,n+k-1] /
                  (qfac[q,q,n-k] qfac[q,q,k-m] qfac[x q,q,k+m]), {k, m, n}]

                                  2                  2
                 -k + n  k/2 + k /2 - n/2 - k n + n /2          2 n
Out[4]= {Sum[((-1)       q                               (1 - x q   )

             qfac[x q, q, -1 + k + n]) / (qfac[q, q, k - m] qfac[q, q, -k + n]

               qfac[x q, q, k + m]), {k, m, n}] == 0, {m - n != 0}}

In[5]:= (* check the m = n case using that the input summand
           of the last computation has been assigned to FF *)
        qSimplify[FF /. {k -> m, n -> m}]

Out[5]= 1
```

□

For d-bilateral Bailey pairs (a, b), note that a is not uniquely determined by b, but — for instance in case of $d = 0$ — only the sequence of sums $(a_n + a_{-n})$ together with a_0. Trivially, for any d-bilateral Bailey pair (a, b), also $(\tilde{a}, b)$ is a d-bilateral Bailey pair, where

$\tilde{a}_n = a_{-n-d}$ for all $n \in \mathbb{Z}$. However, in most applications it is obvious how to construct a from $(a_n + a_{-n-d})$ such that the a_n neither vanish for all $n \leq 0$ nor for all $n \geq 0$.

The fact that Bailey's transform (3.4) can be viewed as a matrix inversion was observed by Gessel and Stanton [**23**] in the more general context of q-Lagrange inversion. One of the results given there contains Theorem 3.6 as a special case.

Computing a from b as described above can be done algorithmically with the `Bailey` via the command

`BaileyInv[`b`, `n`, `x`, `*opts*`]`,

as shown in the following example. Note that if $x = 1$, the output of `BaileyInv` is of the form `{`a_0`, `a_n`}`, $n \geq 1$, since a_0 must be treated separately then. This will happen also with other Bailey pair related functions below. Also be aware of the fact that the `Bailey` package in general does not evaluate sums automatically, unless the summand contains the Kronecker symbol.

Example 3.3. We again consider the Bailey chain from Example 3.2 above. From $b_n^{(0)} = \delta_{n,0}$ we obtain

```
In[6]:= BaileyInv[Delta[n,0], n]

                                2
                 n  -n/2 + n /2       n
Out[6]= {1, (-1)  q              (1 + q )}
```

which is simply the unsymmetrized version of $a_n^{(0)}$. For $b_n^{(1)} = (q;q)_n^{-1}$ we are led to the following result:

```
In[7]:= BaileyInv[1/qfac[q,q,n], n]

                                2
                 n  -n/2 + n /2       2 n
Out[7]= {1, (-1)  q              (1 - q   )

                                       2
                     jj  jj/2 + jj /2 - jj n
                 (-1)   q                            qfac[q, q, -1 + jj + n]
             Sum[--------------------------------------------------, {jj, 0, n}]}
                           qfac[q, q, jj] qfac[q, q, -jj + n]
```

Thus, the corresponding inverse relation is given by

$$(1-q^n)\sum_{k=0}^{n}\frac{(-1)^k\,q^{\binom{n-k}{2}}\,(q;q)_{n+k-1}}{(q;q)_k\,(q;q)_{n-k}} = q^{n(3n-1)/2} \qquad (n \geq 1),$$

which could be checked independently with `qZeil`. Finally, for $b_n^{(2)}$ we get

```
In[8]:= BaileyInv[Sum[q^(k^2) / (qfac[q,q,k] qfac[q,q,n-k]), {k, 0, n}], n]

                                2                                        2
                 n  -n/2 + n /2       2 n               jj  jj/2 + jj /2 - jj n
Out[8]= {1, (-1)  q              (1 - q   ) Sum[((-1)   q
```

```
                                                2
                                               k
                                              q
     qfac[q, q, -1 + jj + n] Sum[-------------------------------, {k, 0, jj}]) /
                                  qfac[q, q, jj - k] qfac[q, q, k]

    qfac[q, q, -jj + n], {jj, 0, n}]}
```

with the inverse double sum relation

$$(1-q^n)\sum_{j=0}^{n}\frac{(-1)^j\,q^{\binom{n-j}{2}}\,(q;q)_{n+j-1}}{(q;q)_{n-j}}\ \sum_{k=0}^{j}\frac{q^{k^2}}{(q;q)_{j-k}\,(q;q)_k}=q^{n(5n-1)/2}\qquad(n\geq 1).\qquad\square$$

So far we have seen how to find new identities from existing ones by walking along a Bailey chain. However, to find a proof for a given Bailey pair identity we have to work backwards. This means, we need a way to move back in a Bailey chain, i.e., to extend our Bailey chain also to the left as

$$\cdots\leftarrow\left(a^{(-2)},b^{(-2)}\right)\leftarrow\left(a^{(-1)},b^{(-1)}\right)\leftarrow\left(a^{(0)},b^{(0)}\right).$$

From Theorem 3.4, respectively Corollary 3.5, we see that a can be uniquely (and easily) reconstructed from a'. Hence, also b is uniquely determined. The following inversions explicitly tell us how to move to the left in a Bailey chain.

Theorem 3.7. *For Bailey pairs (a,b) and (a',b') as in Theorem 3.4 we have for all $n\geq 0$*

$$a_n=q^{-n^2}x^{-n}\,a'_n\qquad\text{and}\qquad b_n=q^{-n^2}x^{-n}\sum_{j=0}^{n}\frac{(-1)^{n-j}\,q^{\binom{n-j}{2}}}{(q;q)_{n-j}}\,b'_j.$$

The same holds true for d-bilateral Bailey pairs (a,b) and (a',b') as in Corollary 3.5 with $x=q^d$, where the relation between a_n and a'_n is extended to hold for all $n\in\mathbb{Z}$.

Proof. The relation between a and a' is obvious. For constructing b from b' we again give an algorithmic proof of the underlying inverse relation.

```
In[9]:= qTelescope[(-1)^(k-m) q^Binomial[k-m,2] / (qfac[q,q,n-k] qfac[q,q,k-m]),
                 {k, m, n}]

                                   2                  2
                 k - m  -k/2 + k /2 + m/2 - k m + m /2
             (-1)      q
Out[9]= {Sum[-----------------------------------------, {k, m, n}] == 0,
               qfac[q, q, k - m] qfac[q, q, -k + n]

         {-m + n != 0}}

In[10]:= (* check the m = n case; FF is the input summand of the last computation *)
         qSimplify[FF /. {k -> m, n -> m}]

Out[10]= 1
```

$\square$

The command in the `Bailey` package for moving to the left in a Bailey chain according to Theorem 3.7 is

`BaileyBack[{`a`, `b`}, `n`, `x`, `$opts$`].`

Example 3.4. Let us consider identity (3.9) and its associated Bailey pair $(a^{(0)}, b^{(0)})$. Moving to the left, by Theorem 3.7 with $x = 1$, then leads to

$$a_n^{(-1)} = q^{-n^2} a_n^{(0)} = (-1)^n \, q^{-n(n+1)/2},$$
$$b_n^{(-1)} = q^{-n^2} \sum_{j=0}^{n} \frac{(-1)^{n-j} \, q^{\binom{n-j}{2}}}{(q;q)_{n-j}} \, b_j^{(0)} = \frac{(-1)^n \, q^{-n(n+1)/2}}{(q;q)_n},$$

and

$$a_n^{(-2)} = q^{-n^2} a_n^{(-1)} = (-1)^n \, q^{-n(3n+1)/2},$$
$$b_n^{(-2)} = q^{-n^2} \sum_{j=0}^{n} \frac{(-1)^{n-j} \, q^{\binom{n-j}{2}}}{(q;q)_{n-j}} \, b_j^{(-1)} = (-1)^n \, q^{-n(n+1)/2} \sum_{j=0}^{n} \frac{q^{-nj}}{(q;q)_{n-j} \, (q;q)_j},$$

which is comfortably checked with the computer:

```
In[11]:= BaileyBack[{(-1)^n q^(n(n-1)/2), Delta[n,0]}, n]

                                                  2
                         2          n  -n/2 - n /2
             n  -n/2 - n /2  (-1)  q
Out[11]= {(-1)  q           , ------------------}
                                 qfac[q, q, n]

In[12]:= BaileyBack[%, n]

                            2                     2
             n  -n/2 - (3 n )/2       n  -n/2 - n /2
Out[12]= {(-1)  q                , (-1)  q

                                1
          Sum[--------------------------------------, {jj, 0, n}]}
               jj n
              q     qfac[q, q, jj] qfac[q, q, -jj + n]
```

The corresponding identities are

$$\sum_{k=-n}^{n} \frac{(-1)^k \, q^{-k(k+1)/2}}{(q;q)_{n-k} \, (q;q)_{n+k}} = \frac{(-1)^n \, q^{-n(n+1)/2}}{(q;q)_n}, \tag{3.13}$$

$$\sum_{k=-n}^{n} \frac{(-1)^k \, q^{-k(3k+1)/2}}{(q;q)_{n-k} \, (q;q)_{n+k}} = (-1)^n \, q^{-n(n+1)/2} \sum_{j=0}^{n} \frac{q^{-nj}}{(q;q)_{n-j} \, (q;q)_j}. \tag{3.14}$$

□

3.3 From Bailey Chains to Bailey Lattices

For effective use of our iteration mechanism we shall now present other techniques to construct new Bailey pairs from given ones. This enables us to walk along Bailey chains in several ways which leads to the more general concept of a Bailey lattice introduced by Agarwal, Andrews, and Bressoud [**2**] (cf. also Bressoud [**17**]). More precisely, this means that one leaves a Bailey chain at a certain step (respectively pair) and continues walking along a different Bailey chain.

3.3.1 Binomial Bailey Pairs

As we saw above, d-bilateral Bailey pairs correspond more or less directly to taking $x = q^d$ in ordinary Bailey pairs. More generally, we shall now treat the case $x = q^d$ for $d \in \mathbb{N}$. Clearly, for arbitrary $d \in \mathbb{N}$ we are dealing with sums of $(2n + d + 1)$ terms of the form

$$\sum_{k=-n-d}^{n} \frac{c_k}{(q;q)_{n-k}\,(q;q)_{n+k+d}} = b_n.$$

However, the d-bilateral Bailey pair (c, b) cannot be transformed into an ordinary Bailey pair (a, b) relative to q^d in general, because after identifying a_n with $(c_{-n-d} + c_n)/(q;q)_d$, a_{n-1} with $(c_{-n-d+1} + c_{n-1})/(q;q)_d$, etc., the remaining terms $c_0, c_{-1}, \ldots, c_{-d}$ need to be covered all by a_0, which is impossible for $d \geq 2$. For instance, for $d = 2$ we obtain

$$\frac{c_0}{(q;q)_n\,(q;q)_{n+2}} + \frac{c_{-1}}{(q;q)_{n+1}\,(q;q)_{n+1}} + \frac{c_{-2}}{(q;q)_{n+2}\,(q;q)_n} = \frac{a_0}{(q;q)_n\,(q^3;q)_n},$$

or equivalently

$$a_0 = \frac{1}{(1-q)(1-q^2)}\Big(c_0 + \frac{1-q^{n+2}}{1-q^{n+1}}\,c_{-1} + c_{-2}\Big).$$

Since a_0 is not free of n, this approach fails. Therefore we introduce the following definition (cf. Paule [**30**]) which avoids this problem completely.

Definition 3.4. We say that two sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ form a *binomial Bailey pair relative to d* if

$$\sum_{k=0}^{n} \begin{bmatrix} 2n+d \\ n-k \end{bmatrix}_q A_k = B_n$$

for some fixed $d \in \mathbb{N}$ and all $n \geq 0$.

Trivially, for any ordinary Bailey pair (a, b) relative to $x = q^d$ the corresponding binomial Bailey pair (A, B) relative to d is given by $A_n = a_n$ and $B_n = (q^{d+1};q)_{2n}\,b_n$, and vice versa. Thus, the corresponding inverse relation for binomial Bailey pairs follows immediately from Theorem 3.6.

Corollary 3.8. *If (A, B) form a binomial Bailey pair relative to d then*

$$A_n = (1 - q^{2n+d}) \sum_{k=0}^{n} \frac{(-1)^{n-k}\,q^{\binom{n-k}{2}}}{1 - q^{n+k+d}} \begin{bmatrix} n+k+d \\ n-k \end{bmatrix}_q B_k \tag{3.15}$$

for all $n \geq 0$, and vice versa. (For $n = 0 = d$, eq. (3.15) should be interpreted as $A_0 = B_0$.)

The analogue of the function `BaileyInv` for binomial Bailey pairs is

`BaileyInvBinom[`B`, `n`, `d`, `$opts$`]`,

as demonstrated in the following example.

Example 3.5. From the q-binomial theorem, Theorem 3.1, we obtain:

```
In[13]:= BaileyInvBinom[qfac[x,q,n] qfac[q/x,q,n], n, 0]

                                  2
                     n   -n/2 + n /2       2 n
Out[13]= {1, (-1)  q             (1 - q   )
                                     2
                   jj  jj/2 + jj /2 - jj n
          Sum[((-1)   q                    qBinomial[jj + n, -jj + n, q]

                  q                                jj + n
             qfac[-, q, jj] qfac[x, q, jj]) / (1 - q      ), {jj, 0, n}]}
                  x
```

The corresponding inverse identity is therefore given by

$$(1-q^{2n})\sum_{k=0}^{n}\frac{(-1)^k\,q^{\binom{k+1}{2}-nk}}{1-q^{n+k}}\begin{bmatrix}n+k\\2k\end{bmatrix}_q(x,q/x;q)_k=(1+q^nx^{-2n})\,x^n\qquad(n\geq 1).\qquad\square$$

Furthermore, it is possible to pass from a binomial Bailey pair relative to d to a binomial Bailey pair relative to $d-1$, i.e., to change the parameter x to x/q.

Lemma 3.9. *For all $n,k,d\in\mathbb{N}$ with $d\geq 1$ we have*

$$\begin{bmatrix}2n+d\\n-k\end{bmatrix}_q=\frac{1-q^{2n+d}}{1-q^{2k+d}}\left(\begin{bmatrix}2n+d-1\\n-k\end{bmatrix}_q-q^{2k+d}\begin{bmatrix}2n+d-1\\n-k-1\end{bmatrix}_q\right).$$

Proof.

$$\begin{aligned}\frac{1-q^{2n+d}}{1-q^{2k+d}}&\left(\begin{bmatrix}2n+d-1\\n-k\end{bmatrix}_q-q^{2k+d}\begin{bmatrix}2n+d-1\\n-k-1\end{bmatrix}_q\right)\\&=\frac{1-q^{2n+d}}{1-q^{2k+d}}\,\frac{(q;q)_{2n+d-1}}{(q;q)_{n-k}\,(q;q)_{n+k+d}}\,\big[(1-q^{n+k+d})-q^{2k+d}(1-q^{n-k})\big]\\&=\frac{(q;q)_{2n+d}}{(q;q)_{n-k}\,(q;q)_{n+k+d}}=\begin{bmatrix}2n+d\\n-k\end{bmatrix}_q.\qquad\square\end{aligned}$$

Theorem 3.10. *If (A,B) form a binomial Bailey pair relative to $d\geq 1$, then (A',B') form a binomial Bailey pair relative to $d-1$, where*

$$A'_n=\begin{cases}A_0/(1-q^d), & \text{if } n=0,\\ \dfrac{A_n}{1-q^{2n+d}}-\dfrac{q^{2n+d-2}\,A_{n-1}}{1-q^{2n+d-2}} & \text{if } n\geq 1,\end{cases}\qquad\text{and}\qquad B'_n=\frac{B_n}{1-q^{2n+d}}$$

for all $n\geq 0$.

Proof. From the assumption on (A, B) and Lemma 3.9 we obtain

$$\begin{aligned}
B'_n = \frac{B_n}{1-q^{2n+d}} &= \frac{1}{1-q^{2n+d}} \sum_{k=0}^{n} \begin{bmatrix} 2n+d \\ n-k \end{bmatrix}_q A_k \\
&= \sum_{k=0}^{n} \begin{bmatrix} 2n+d-1 \\ n-k \end{bmatrix}_q \frac{A_k}{1-q^{2k+d}} - \sum_{k=1}^{n} \begin{bmatrix} 2n+d-1 \\ n-k \end{bmatrix}_q \frac{q^{2k+d-2}\, A_{k-1}}{1-q^{2k+d-2}} \\
&= \sum_{k=0}^{n} \begin{bmatrix} 2n+d-1 \\ n-k \end{bmatrix}_q A'_k.
\end{aligned}$$

□

Invoking the `Bailey` package with the command

`BaileyBackBinom[{`A`, `B`}, `n`, `d`, `*opts*`]`

we obtain, for instance, for $d = 1$, $A_n = (-1)^n\,(1-q^{2n+1})\,q^{(c+1)n^2+cn}$, $c \in \mathbb{R}$, and arbitrary B_n:

```
In[14]:= BaileyBackBinom[{(-1)^n (1-q^(2n+1)) q^((c+1)n^2+c n), B}, n, 1]


                              2      2
                  n  -(c n) + n  + c n        2 c n         B
Out[14]= {{1, (-1)  q                    (1 + q     )}, -----------}
                                                         1 + 2 n
                                                    1 - q
```

From this we may conclude after symmetrizing the corresponding identity that any 0-bilateral Bailey pair (a, b), where $a_n = (-1)^n\,q^{(c+1)n^2+cn}$ is also a 1-bilateral Bailey pair and vice versa, or in other words:

Corollary 3.11. *For all $c \in \mathbb{R}$ we have*

$$\sum_{k=-n}^{n} \frac{(-1)^k\,q^{(c+1)k^2+ck}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \sum_{k=-n-1}^{n} \frac{(-1)^k\,q^{(c+1)k^2+ck}}{(q;q)_{n-k}\,(q;q)_{n+k+1}}. \tag{3.16}$$

This relation, for instance, was used by Paule [**30**] for proving multiple series generalizations of the Rogers-Ramanujan identities.

Example 3.6. The second Rogers-Ramanujan identity can be computed as follows ($d = 1$):

$$\begin{aligned}
\frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k\,q^{k(5k+3)/2} &\overset{(3.7)}{=} \sum_{j=0}^{\infty} q^{j^2+j} \sum_{k=-j-1}^{j} \frac{(-1)^k\,q^{k(3k+1)/2}}{(q;q)_{j-k}\,(q;q)_{j+k+1}} \\
&\overset{(3.6)}{=} \sum_{j=0}^{\infty} q^{j^2+j} \sum_{l=0}^{j} \frac{q^{l^2+l}}{(q;q)_{j-l}} \sum_{k=-l-1}^{l} \frac{(-1)^k\,q^{k(k-1)/2}}{(q;q)_{l-k}\,(q;q)_{l+k+1}} \\
&\overset{(3.16)}{=} \sum_{j=0}^{\infty} q^{j^2+j} \sum_{l=0}^{j} \frac{q^{l^2+l}}{(q;q)_{j-l}} \sum_{k=-l}^{l} \frac{(-1)^k\,q^{k(k-1)/2}}{(q;q)_{l-k}\,(q;q)_{l+k}} \\
&\overset{(3.1)}{=} \sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q;q)_j}.
\end{aligned}$$

□

Next, we shall show that it is also possible to change the parameter x to xq, i.e., to switch from a binomial Bailey pair relative to d to a binomial Bailey pair relative to $d+1$.

Theorem 3.12. *If (A', B') form a binomial Bailey pair relative to d, then (A, B) form a binomial Bailey pair relative to $d+1$, where*

$$A_n = (1 - q^{2n+d+1})\, q^{n^2+dn} \sum_{j=0}^{n} q^{-j^2-dj}\, A'_j \quad \text{and} \quad B_n = (1 - q^{2n+d+1})\, B'_n$$

for all $n \geq 0$.

Proof. The relation between B and B' is obvious from Theorem 3.10 with d replaced by $d+1$. For the computation of A we define $f_n := q^{-n^2-dn}\, A'_n$ and $g_n := q^{-n^2-dn}\, A_n/(1 - q^{2n+d+1})$ with $g_{-1} := 0$, since then the relation between A' and A in Theorem 3.10 can be written as $f_n = g_n - g_{n-1}$. However, this is equivalent to $g_n = \sum_{j=0}^{n} f_j$, which completes the proof. □

The algorithmic counterpart of Theorem 3.12 is the function

`BaileyForwBinom[{`A`, `B`}, `n`, `d`, `*opts*`]`

as demonstrated in the following example.

Example 3.7. From the trivial binomial Bailey pair identity

$$\sum_{k=0}^{n} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q \delta_{k,0} = \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

we are led to:

```
In[15]:= BaileyForwBinom[{Delta[n,0], qBinomial[2n,n,q]}, n, 0]

             2
            n        1 + 2 n   qfac[q, q, 1 + 2 n]
Out[15]= {q    (1 - q       ), -------------------}
                                                2
                                qfac[q, q, n]
```

It turns out that the corresponding identity

$$\sum_{k=0}^{n} (1 - q^{2k+1}) \begin{bmatrix} 2n+1 \\ n-k \end{bmatrix}_q q^{k^2} = (1 - q^{2n+1}) \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

was found to be a dual identity of the q-Chu-Vandermonde identity in Chapter 2. □

Since for binomial Bailey pairs (A, B), the value of A_0 often has to be defined separately, the functions `BaileyBackBinom` and `BaileyForwBinom` can be called with the option `A0->`a_0 for explicitly specifying $A_0 := a_0$.

3.3.2 Dual Bailey Pairs

As Andrews [**9**] pointed out, another important way to produce new Bailey pairs is to switch from x and q to x^{-1} and q^{-1}, respectively.

Theorem 3.13. *If (a, b) form an ordinary Bailey pair relative to x with $a_n = a_n(x, q)$ and $b_n = b_n(x, q)$, then so do (a', b'), where*

$$a'_n(x, q) = q^{n^2} x^n \, a_n(x^{-1}, q^{-1}) \quad \text{and} \quad b'_n(x, q) = q^{-n^2-n} \, x^{-n} \, b_n(x^{-1}, q^{-1})$$

for all $n \geq 0$.

Proof. The assertion is an immediate consequence of the rewriting rule

$$(a; q^{-1})_k = (a^{-1}; q)_k \, (-a)^k \, q^{-\binom{k}{2}}.$$ □

The pair (a', b') constructed as above is called the *dual Bailey pair*. Note that dualization in this context is completely different from qWZ-dualization introduced in Chapter 2. While 0-bilateral Bailey pairs fit perfectly well into Theorem 3.13 with $x = 1$, for 1-bilateral Bailey pairs with $x = q$ we additionally have to divide the resulting b'_n by $(1 - q)/(1 - q^{-1}) = -q$. This is again easily seen by applying the rewriting rule used in the proof of Theorem 3.13.

Corollary 3.14. *If (a, b) form a d-bilateral Bailey pair with $a_n = a_n(q)$ and $b_n = b_n(q)$, then so do (a', b'), where*

$$a'_m(q) = q^{m^2+dm} \, a_m(q^{-1}) \quad \text{and} \quad b'_n(q) = (-1)^d \, q^{-n^2-(d+1)n-d} \, b_n(q^{-1})$$

for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Passing from a Bailey pair (a, b) to the dual Bailey pair (a', b') as described in Theorem 3.13 and Corollary 3.14 can be carried out automatically with the `Bailey` package by calling the function

`BaileyDual[{`*a*`, `*b*`}, `*n*`, `*x*`, `*opts*`]`,

where x is either an integer power of q or an indeterminate (i.e., a certain Mathematica symbol). For 1-bilateral Bailey pairs, `BaileyDual` must be called with the option `Bilateral1->True`. In this case the program automatically assumes $x = q$.

Example 3.8. Recall the Bailey pair identities from Example 3.2 and Example 3.4, which were obtained by walking to the right, respectively to the left in the corresponding Bailey chain. Alternatively, identities (3.13) and (3.14) could be derived directly from (3.10) and (3.11), respectively, by application of Corollary 3.14. For instance, running `BaileyDual` on the pair $(a^{(1)}, b^{(1)})$ from Example 3.2 gives:

```
In[16]:= BaileyDual[{(-1)^n q^(n(3n-1)/2), 1/qfac[q,q,n]}, n]

                                                     2
                            2         n  -n/2 - n /2
                n  n/2 - n /2  (-1)  q
Out[16]= {(-1)  q           , -------------------}
                                   qfac[q, q, n]
```

This is one of the underlying Bailey pairs of identity (3.13) from Example 3.4. □

Now we are able to give an alternative proof for another Bailey pair generation due to Paule [**28**] by a 4-step Bailey lattice walk.

Theorem 3.15. *For all $c \in \mathbb{R}$ we have*

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{ck^2+ck}}{(q;q)_{n-k}\,(q;q)_{n+k}} = q^n \sum_{k=-n}^{n} \frac{(-1)^k\, q^{ck^2+(c-1)k}}{(q;q)_{n-k}\,(q;q)_{n+k}}. \tag{3.17}$$

Proof. Let $b_n(c,q)$ denote the left hand side of equation (3.17). Then, by Corollary 3.14 and Corollary 3.11, we have

$$q^{-n^2-n}\, b_n(c,q^{-1}) = \sum_{k=-n}^{n} \frac{(-1)^k\, q^{(-c+1)k^2-ck}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \sum_{k=-n-1}^{n} \frac{(-1)^k\, q^{(-c+1)k^2-ck}}{(q;q)_{n-k}\,(q;q)_{n+k+1}},$$

which after dualizing once more turns into

$$-q^{-n-1}\, b_n(c,q) = \sum_{k=-n-1}^{n} \frac{(-1)^k\, q^{ck^2+(c+1)k}}{(q;q)_{n-k}\,(q;q)_{n+k+1}}.$$

Replacing k by $-k-1$ in the sum on the right hand side and applying Corollary 3.11 again yields

$$q^{-n}\, b_n(c,q) = \sum_{k=-n-1}^{n} \frac{(-1)^k\, q^{ck^2+(c-1)k}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \sum_{k=-n}^{n} \frac{(-1)^k\, q^{ck^2+(c-1)k}}{(q;q)_{n-k}\,(q;q)_{n+k}},$$

which completes the proof. □

3.3.3 c-Step Bailey Pairs

Finally we once more extend the notion of Bailey pairs.

Definition 3.5. We say that two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ form a *c-step Bailey pair relative to x* if

$$\sum_{k\geq 0} \frac{a_k}{(q;q)_{n-ck}\,(xq;q)_{n+ck}} = b_n$$

for some fixed $c \in \mathbb{N}$ with $c \geq 1$ and all $n \geq 0$.

The iteration mechanism in this case works as follows.

Theorem 3.16. *If (a,b) form a c-step Bailey pair relative to x then so do (a',b'), where*

$$a'_n = q^{(cn)^2} x^{cn}\, a_n \quad \text{and} \quad b'_n = \sum_{j=0}^{n} \frac{q^{j^2} x^j}{(q;q)_{n-j}}\, b_j$$

for all $n \geq 0$.

Proof. We define $\tilde{a}_k := a_m$ if $k = m \cdot c$ for some $m \in \mathbb{N}$, and $\tilde{a}_k := 0$ otherwise. Since (a,b) form a c-step Bailey pair, it is immediately clear that $(\tilde{a},b)$ form an ordinary Bailey pair. Thus, by Theorem 3.4, the same holds true for $(\tilde{a}',b')$, where

$$\tilde{a}'_n = q^{n^2} x^n\, \tilde{a}_n \quad \text{and} \quad b'_n = \sum_{j=0}^{n} \frac{q^{j^2} x^j}{(q;q)_{n-j}}\, b_j.$$

The proof is completed by observing that

$$b'_n = \sum_{k=0}^{n} \frac{\tilde{a}'_k}{(q;q)_{n-k}\,(xq;q)_{n+k}} = \sum_{k\geq 0} \frac{q^{(ck)^2}\,x^{ck}\,a_k}{(q;q)_{n-ck}\,(xq;q)_{n+ck}}. \qquad \square$$

From Theorem 3.16 and Theorem 3.7 it is easily seen how to move to the left in a c-step Bailey chain.

Corollary 3.17. *For c-step Bailey pairs (a,b) and (a',b') as in Theorem 3.16 we have*

$$a_n = q^{-(cn)^2}\,x^{-cn}\,a'_n \quad \text{and} \quad b_n = q^{-n^2}\,x^{-n} \sum_{j=0}^{n} \frac{(-1)^{n-j}\,q^{\binom{n-j}{2}}}{(q;q)_{n-j}}\,b'_j$$

for all $n \geq 0$.

The functions `BaileyForw` and `BaileyBack` can be applied also to c-step Bailey pairs with the option `Step->`c, i.e.,

`BaileyForw[{`a`, `b`}, `n`, `x`, Step->`c`, `*opts*`]`

and

`BaileyBack[{`a`, `b`}, `n`, `x`, Step->`c`, `*opts*`]`,

respectively.

The next step in extending the Bailey pair database is to combine the notion of c-step and binomial Bailey pairs.

Definition 3.6. We say that two sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ form a *c-step binomial Bailey pair relative to d* if

$$\sum_{k\geq 0} \begin{bmatrix} 2n+d \\ n-ck \end{bmatrix}_q A_k = B_n$$

for some fixed $c, d \in \mathbb{N}$ with $c \geq 1$ and all $n \geq 0$.

Unfortunately, the only useful generalization of Lemma 3.9 seems to exist for $c = 2$. In other cases it turns out that the resulting recurrence relation for $\begin{bmatrix} 2n+d \\ n-ck \end{bmatrix}_q$ does not factor appropriately. Thus, we will restrict ourselves to this case here, where again the corresponding functions

`BaileyForwBinom[{`A`, `B`}, `n`, `d`, Step->2, `*opts*`]`

and

`BaileyBackBinom[{`A`, `B`}, `n`, `d`, Step->2, `*opts*`]`

are available.

Lemma 3.18. *For all $n, k, d \in \mathbb{N}$ with $d \geq 2$ we have*

$$\begin{bmatrix} 2n+d \\ n-2k \end{bmatrix}_q = \frac{1-q^{2n+d}}{1-q^{4k+d}} \left(\begin{bmatrix} 2n+d-2 \\ n-2k \end{bmatrix}_q - q^{4k+d} \begin{bmatrix} 2n+d-2 \\ n-2k-2 \end{bmatrix}_q \right).$$

Proof.

$$\frac{1-q^{2n+d}}{1-q^{4k+d}}\left(\begin{bmatrix}2n+d-2\\ n-2k\end{bmatrix}_q - q^{4k+d}\begin{bmatrix}2n+d-2\\ n-2k-2\end{bmatrix}_q\right)$$
$$= \frac{1-q^{2n+d}}{1-q^{4k+d}}\,\frac{(q;q)_{2n+d-2}}{(q;q)_{n-2k}\,(q;q)_{n+2k+d}}$$
$$\times\left[(1-q^{n+2k+d-1})(1-q^{n+2k+d}) - q^{4k+d}(1-q^{n-2k-1})(1-q^{n-2k})\right]$$
$$= \frac{1-q^{2n+d}}{1-q^{4k+d}}\,\frac{(q;q)_{2n+d-2}}{(q;q)_{n-2k}\,(q;q)_{n+2k+d}}\,(1-q^{4k+d})(1-q^{2n+d-1})$$
$$= \frac{(q;q)_{2n+d}}{(q;q)_{n-2k}\,(q;q)_{n+2k+d}} = \begin{bmatrix}2n+d\\ n-2k\end{bmatrix}_q. \qquad \square$$

Theorem 3.19. *If (A,B) form a 2-step binomial Bailey pair relative to $d\geq 2$, then (A',B') form a 2-step binomial Bailey pair relative to $d-2$, where*

$$A'_n = \begin{cases} A_0/(1-q^d), & \text{if } n=0,\\ \dfrac{A_n}{1-q^{4n+d}} - \dfrac{q^{4n+d-4}\,A_{n-1}}{1-q^{4n+d-4}} & \text{if } n\geq 1,\end{cases} \qquad \text{and} \qquad B'_n = \frac{B_n}{1-q^{2n+d}}$$

for all $n\geq 0$.

Proof. From the assumption on (A,B) and Lemma 3.18 we obtain

$$B'_n = \frac{B_n}{1-q^{2n+d}} = \frac{1}{1-q^{2n+d}}\sum_{k\geq 0}\begin{bmatrix}2n+d\\ n-2k\end{bmatrix}_q A_k$$
$$= \sum_{k\geq 0}\begin{bmatrix}2n+d-2\\ n-2k\end{bmatrix}_q \frac{A_k}{1-q^{4k+d}} - \sum_{k\geq 1}\begin{bmatrix}2n+d-2\\ n-2k\end{bmatrix}_q \frac{q^{4k+d-4}\,A_{k-1}}{1-q^{4k+d-4}}$$
$$= \sum_{k\geq 0}\begin{bmatrix}2n+d-2\\ n-2k\end{bmatrix}_q A'_k. \qquad \square$$

Example 3.9. In Chapter 2 a dual identity of the q-Gauss sum was found to be

$$\sum_{k=-\infty}^{\infty}(-1)^k\begin{bmatrix}2n+2\\ n-2k\end{bmatrix}_q q^{k(3k+1)} = (1-q^{2n+2})\,(-q;q)_n.$$

The underlying 2-step binomial Bailey pair relative to 2 is given by

$$A_n = (-1)^n\,(1-q^{4n+2})\,q^{n(3n+1)} \qquad \text{and} \qquad B_n = (1-q^{2n+2})\,(-q;q)_n.$$

Hence, by Theorem 3.19, we have:

```
In[17]:= BaileyBackBinom[{(-1)^n (1-q^(4n+2)) q^(n(3n+1)), (1-q^(2n+2)) *
                          qfac[-q,q,n]}, n, 2, Step->2]

                                    2
                  n  -n + 3 n        2 n
Out[17]= {{1, (-1)  q          (1 + q   )}, qfac[-q, q, n]}
```

The corresponding identity reads as

$$\sum_{k=-\infty}^{\infty} (-1)^k \begin{bmatrix} 2n \\ n-2k \end{bmatrix}_q q^{k(3k-1)} = (-q;q)_n. \tag{3.18}$$

□

As with ordinary (1-step) binomial Bailey pairs also in this case it is possible to walk into the opposite direction, i.e., to switch from a 2-step binomial Bailey pair relative to d to a 2-step binomial Bailey pair relative to $d+2$.

Theorem 3.20. *If (A', B') form a 2-step binomial Bailey pair relative to d, then (A, B) form a 2-step binomial Bailey pair relative to $d+2$, where*

$$A_n = (1-q^{4n+d+2})\, q^{2n^2+dn} \sum_{j=0}^{n} q^{-2j^2-dj} A'_j \quad \textit{and} \quad B_n = (1-q^{2n+d+2})\, B'_n$$

for all $n \geq 0$.

Proof. We apply the same argumentation as in the proof of Theorem 3.12, where now $f_n := q^{-2n^2-dn} A'_n$ and $g_n := q^{-2n^2-dn} A_n/(1-q^{4n+d+2})$ with $g_{-1} := 0$. □

Analogously to the proof of Theorem 3.16 it is easily shown how dualization works for c-step Bailey pairs.

Theorem 3.21. *If (a, b) form a c-step Bailey pair relative to x with $a_n = a_n(x, q)$ and $b_n = b_n(x, q)$, then so do (a', b'), where*

$$a'_n(x,q) = q^{(cn)^2} x^{cn}\, a_n(x^{-1}, q^{-1}) \quad \textit{and} \quad b'_n(x,q) = q^{-n^2-n}\, x^{-n}\, b_n(x^{-1}, q^{-1})$$

for all $n \geq 0$.

Example 3.10. Recall identity (3.18) from the example above. The corresponding 2-step Bailey pair relative to 1 is given by $A_0 = 1$, $A_n = (-1)^n (1+q^{2n})\, q^{n(3n-1)}$, $n \geq 1$, and $B_n = (-q;q)_n/(q;q)_{2n}$.

```
In[18]:= BaileyDual[{(-1)^n (1+q^(2n)) q^(n(3n-1)), qfac[-q,q,n] / qfac[q,q,2n]},
                   n, Step->2]

                                                        2
                          2               -n/2 + n /2
                  n  n + n        -2 n   q             qfac[-q, q, n]
Out[18]= {(-1)  q       (1 + q    ), --------------------------}
                                              qfac[q, q, 2 n]
```

Therefore, we obtain the dual identity

$$\sum_{k=-\infty}^{\infty} (-1)^k \begin{bmatrix} 2n \\ n-2k \end{bmatrix}_q q^{k(k+1)} = (-q;q)_n\, q^{\binom{n}{2}}.$$

□

3.4 Slater's Table of Bailey Pairs

In 1951, Slater [**39**] (see also Slater [**40**]) derived a list of 70 Bailey pairs by specializing parameters in a sum of a very-well-poised ${}_6\psi_6$ series due to Bailey [**13**]. The following table illustrates that many of them are related to other ones by short walks in the Bailey lattice. Furthermore, errata in the original list are corrected in the footnotes. To illustrate how to read the table, let us consider, for instance, the fifth line. It states that Bailey pair B (1), which corresponds to

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{k(3k-1)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{1}{(q;q)_n}$$

is the underlying Bailey pair of identity (3.10) above, whereas Bailey pair B (2) corresponding to

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{3k(k-1)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{q^n}{(q;q)_n}$$

can be derived from B (1) by application of Theorem 3.15. Entries with a "—" in both the second and third column, such as A (1)–A (4), serve as "starting" Bailey pairs, i.e., we start a walk in the Bailey lattice from such a pair but cannot derive it from another one.

#	from	by using	#	from	by using
A (1)	—	—	A (2)	—	—
A (3)	—	—	A (4)	—	—
A (5)	A (1)	Thm. 3.13	A (6)	A (4)	Thm. 3.13
A (7)	A (3)	Thm. 3.13	A (8)	A (2)	Thm. 3.13
B (1)	—	eq. (3.10)	B (2)	B (1)	Thm. 3.15
B (3)	B (1)	Cor. 3.11	B (4)	B (2)	Thm. 3.12
C (1)	—	—	C (2)	C (1), C (3)	addition
C (3)	C (1)	Thm. 3.20	C (4)	C (6)	Thm. 3.13
C (5)	C (1)	Thm. 3.13	C (6)	C (5)	Thm. 3.20
C (7)	C (3)	Thm. 3.13			
E (1)	—	—	E (2)	E (1)	Thm. 3.13
E (3)	E (1)	Cor. 3.11	E (4)	E (5)	Thm. 3.13
E (5)	H (8)	—	E (6)[1]	E (3)	Cor. 3.11, Thm. 3.15
E (7)[2]	E (3)	Thm. 3.13			
F (1)	—	—	F (2)	—	—
F (3)	F (1)	Thm. 3.13	F (4)	F (2)	Thm. 3.13

[1] $\alpha_r = (-1)^r\, q^{r^2}\,(q^r + q^{-r}), \quad x = q$
[2] $\alpha_r = (-1)^r\,(q^{-r} - q^{r+1})/(1-q)$

#	from	by using	#	from	by using
G (1)	—	—	G (2)[3]	G (1)	Cor. 3.11
G (3)	G (1)	Thm. 3.15	G (4)	G (1)	Thm. 3.13
G (5)[4]	G (2)	Thm. 3.13	G (6)	G (3)	Thm. 3.13
H (1)[5]	—	—	H (2)	—	—
H (3)	B (2)	Thm. 3.13	H (4)	H (3)	Thm. 3.15
H (5)	—	—	H (6)	—	trivial
H (7)	E (7)	Cor. 3.11, Thm. 3.15	H (8)	H (7)	Thm. 3.15
H (9)	H (5)	Thm. 3.13	H (10)	—	—
H (11)	H (10)	Thm. 3.13	H (12)	H (5)	Thm. 3.12
H (13)	—	trivial	H (14)	H (12)	Thm. 3.13
H (15)	—	—	H (16)	H (15)	Thm. 3.13
I (1)	—	—	I (2)[6]	—	—
I (3)	I (2)	Thm. 3.13	I (4)	I (1)	Thm. 3.13
I (5)	—	—	I (6)	I (5)	Thm. 3.13
I (7)	—	—	I (8)[7]	I (7)	Thm. 3.13
I (9)	—	—	I (10)	H (6)	Thm. 3.20
I (11)	I (10)	Thm. 3.13	I (12)	—	—
I (13)	I (12)	Thm. 3.13			
K (1)[8]	—	—	K (2)[8]	K (1)	Thm. 3.13
K (3)	—	—	K (4)	K (3)	Thm. 3.13
K (5)	—	—	K (6)	K (5)	Thm. 3.13

How to Find Slater's Pairs Automatically

We want to emphasize that all of these Bailey pairs can be verified algorithmically. Moreover, many of them, namely those of groups B, C, F, and H can also be found automatically with the Extended q-Zeilberger Algorithm (see Appendix B). For this we will make use of the new option `PolyMult`. With this option enabled, `qZeil` automatically finds polynomial multipliers of the input summand for which a q-Zeilberger recurrence of the specified order exists. A more detailed description of this feature can be found in Appendix B.7.6.

The reason for the fact that we cannot find all Bailey pairs in this way is that we are not able to apply our machinery to Bailey pairs (a, b) where the definition of a_n splits into several cases. For this we would have to find a rational function multiplier of the input summand that might depend on both q^k and q^n. Besides the question whether this is actually possible from theoretic point of view, one would at least end up with *huge* systems of non-linear equations which most probably exceed the capabilities of today's computer algebra systems.

[3] $\alpha_{2r+1} = q^{3r^2+11r/2+5/2}\,(1-q^{-2r-3/2})/(1-q^{1/2})$

[4] $\alpha_{2r} = q^{r^2-r/2}\,(1-q^{2r+1/2})/(1-q^{1/2}), \quad \alpha_{2r+1} = q^{r^2+5r/2+3/2}\,(1-q^{-2r-3/2})/(1-q^{1/2})$

[5] $\alpha_r = q^{r^2/2}\,(q^{r/2}+q^{-r/2})$

[6] $\alpha_{2r+1} = (-1)^{r+1}\,q^{r^2}\,(q^{r/2}-q^{(3r+1)/2})$

[7] $\alpha_{2r+1} = (-1)^r\,q^{2r^2}\,(q^{3r+1}-q^r)$

[8] $\beta_0 = 1$

As an example, let us first consider multiples of $a_n = 1$ with $x = 1$. Since the program cannot find Laurent polynomials, e.g. $(q^{-k} + q^k) = q^{-k}(1 + q^{2k})$, we simply multiply the summand by q^{-k} and then look for polynomials of degree 2. Note that in this case we succeed even with q-hypergeometric telescoping.

```
In[19]:= qTelescope[q^(-k) / (qfac[q,q,n-k] qfac[q,q,n+k]), {k, 1, n}, PolyMult->2]

                              k        k
                        (-1 + q ) (1 + q )
Out[19]= Sum[---------------------------------------, {k, 1, n}] ==
               k
              q  qfac[q, q, -k + n] qfac[q, q, k + n]

                              1
           -(----------------------------------)
               n
              q  qfac[q, q, -1 + n] qfac[q, q, n]
```

This corresponds to the Bailey pair H (5) in Slater [**39**].

Similarly, for multiples of $a_n = (-1)^n$ with $x = 1$ we obtain the following (where `SUMX[`n`, `$P(k)$`]` in the output stands for $\sum_k f_{n,k} \cdot P(k)$ with $f_{n,k}$ being the original summand and $P(k)$ the polynomial multiplier found by the program; see Appendix B).

```
In[20]:= qZeil[(-1)^k q^(-k) / (qfac[q,q,n-k] qfac[q,q,n+k]),
               {k, -Infinity, Infinity}, n, 1, PolyMult->{2,2}]

Out[20]=
                                   k                                            2 k
          k         SUMX[-1 + n, q ]                 2 k        SUMX[-1 + n, 1 + q   ]
 {SUMX[n, q ] == -(----------------), SUMX[n, 1 + q   ] == -(----------------------),
                               2 n                                            2 n
                          1 - q                                          q (1 - q   )

                                                                k        k
               k        k        SUMX[-1 + n, (-1 + q ) (1 + q )]
  SUMX[n, (-1 + q ) (1 + q )] == -(--------------------------------),
                                                  -1 + 2 n
                                            q (1 - q        )

                                                   k 2
              k 2       SUMX[-1 + n, (1 + q ) ]
  SUMX[n, (1 + q ) ] == -(------------------------),
                                     -1 + n        n
                         q (1 + q       ) (1 - q )

                                                     k 2
               k 2       SUMX[-1 + n, (-1 + q ) ]
  SUMX[n, (-1 + q ) ] == -(------------------------)}
                                       -1 + n        n
                           q (1 - q       ) (1 + q )
```

The first solution corresponds to H (7), the second one to H (8), the third one is trivial, and the last two ones are linear combinations of H (7) and H (8), which is easily seen after expanding the corresponding polynomial multipliers.

3.5 Discovering New Bailey Pairs

By proceeding as described in the previous section we are not only able to find many Bailey pairs presented in Slater [**39**], we can even go beyond this list. As we saw above, in most steps of a Bailey lattice walk, a_n is multiplied with powers of q^{n^2} *and* q^n but not with a power of q^n alone. This might be the reason why several families of Bailey pairs $\{(a^{(d)}, b^{(d)})\}_{d\in\mathbb{N}}$, whose first members $a_n^{(d)}$ differ by powers of q^n, do not appear in literature up to now. The object of this section is to list some typical examples of this kind where we will restrict ourselves to 0- and 1-bilateral Bailey pairs determined by a_n of the type

$$a_n = (\pm 1)^n \, q^{n(\alpha n+\beta)/2}, \tag{3.19}$$

with α, β being specific non-negative integers. Note that the $\beta < 0$ case is implicitly covered by changing the order of summation in the corresponding Bailey pair identity. For each choice of α we start with a sequence $a_n^{(0)}$ obtained by putting $\beta = 0$ or $\beta = 1$ in (3.19). Then we use the Extended q-Zeilberger Algorithm to check for which $d \in \mathbb{N}$ there exist closed form Bailey pairs $(a^{(d)}, b^{(d)})$, where $a_n^{(d)} = q^{dn}\, a_n^{(0)}$. For all the results shown below this seems to happen either for no d or for all d. In the latter case, the complexity of $b_n^{(d)}$ in general increases rather fast with d. Thus, we only state the results for $d \leq 3$.

Case $\alpha = 0$:

- For $a_n^{(0)} = 1$, there seems to exist no closed form Bailey pair $(a^{(d)}, b^{(d)})$. Since we cannot check for all $d \in \mathbb{N}$ whether a polynomial multiplier of the summand exists, we state this result as a conjecture.

- For $a_n^{(0)} = (-1)^n$, i.e., Slater's Bailey pair H (7) and its corresponding identity

$$\sum_{k=-n}^{n} \frac{(-1)^k}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{(-1)^n}{(q^2;q^2)_n},$$

we find that the "next" Bailey pair $(a^{(1)}, b^{(1)})$ is still in Slater's list, namely H (8) or

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^k}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{(-1)^n}{q^n\,(q^2;q^2)_n},$$

which is obtained easily from H (7) by application of Theorem 3.15. However, thanks to `qZeil` we find that $(a^{(2)}, b^{(2)})$ is characterized by

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{2k}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{(-1)^n\left(1+q-q^{2n+1}\right)}{q^{2n}\,(q^2;q^2)_n},$$

a Bailey pair no longer appearing in Slater's list. The dual identity reads as

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{k(k+2)}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{q^{2n+1}+q^{2n}-1}{q\,(q^2;q^2)_n},$$

where we again want to emphasize that dualization in the context of Bailey pairs has nothing to do with qWZ-dualization. By Theorem 3.10, the 1-bilateral counterpart of this identity is given by

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\,(1-q^{2k+1})\,q^{k(k+2)}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{(1+q^2)\,(q^{2n+1}+q^{2n}-1)}{q^3\,(q^2;q^2)_n}.$$

Similarly, for $(a^{(3)}, b^{(3)})$ we obtain

$$\sum_{k=-n}^{n} \frac{(-1)^k\,q^{3k}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{(-1)^n\,\big(1+q(1+q+q^2)\,(1-q^{2n})\big)}{q^{3n}\,(q^2;q^2)_n},$$

with the dual identity

$$\sum_{k=-n}^{n} \frac{(-1)^k\,q^{k(k+3)}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{\big(q^{3n+1}+(q^n+q^{n-1}+q^{n-2})\,(q^{2n}-1)\big)}{q\,(q^2;q^2)_n},$$

and the 1-bilateral counterpart

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\,(1-q^{2k+1})\,q^{k(k+3)}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{(1+q)\,(1-q+q^2)\,\big(q^{3n+1}+(q^n+q^{n-1}+q^{n-2})\,(q^{2n}-1)\big)}{q^4\,(q^2;q^2)_n}.$$

Summarizing, if $(a^{(d)}, b^{(d)})$ is a closed form Bailey pair, we may conclude, by dualization, that the same is true for $(\alpha^{(d)}, \beta^{(d)})$, where $\alpha_n^{(d)} = q^{n^2} a_n^{(d)}$, and also, by Theorem 3.10, for $(A^{(d)}, B^{(d)})$, where $A_n^{(d)} = (1-q^{2n+1})\,q^{n^2} a_n^{(d)}$.

- It turns out that proceeding as above also works for the corresponding 1-bilateral Bailey pairs, i.e., for the same $a_n^{(d)}$ with $x=q$ instead of $x=1$. The Bailey pairs for $d=0$ and $d=1$ are H (13) and E (7), or equivalently

$$\sum_{k=-n-1}^{n} \frac{(-1)^k}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = 0,$$

and

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\,q^k}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{(-1)^{n+1}}{q^{n+1}\,(q^2;q^2)_n},$$

respectively, whereas for $d=2$ we are led to

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\,q^{2k}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{(-1)^{n+1}\,(1+q)}{q^{2n+2}\,(q^2;q^2)_n}. \tag{3.20}$$

Moving one step to the right from this identity according to Corollary 3.5 we obtain

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\,q^{k(k+3)}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = q^{-2} \sum_{j=0}^{n} \frac{q^{j(j-1)}\,(-1)^{j+1}\,(1+q)}{(q;q)_{n-j}\,(q^2;q^2)_j},$$

which for $n \to \infty$ turns into

$$\frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k \, q^{k(k+3)} = q^{-2} \sum_{j=0}^{\infty} \frac{q^{j(j-1)}\,(-1)^{j+1}\,(1+q)}{(q^2;q^2)_j}.$$

Since the left hand side vanishes by Jacobi's triple product identity (3.2), we finally get

$$\sum_{j=0}^{\infty} \frac{(-1)^j \, q^{j(j-1)}}{(q^2;q^2)_j} = 0,$$

which is a special case of Euler's q-analogue of the exponential function (cf. Andrews [**7**]). The dual identity of (3.20) reads as

$$\sum_{k=-n-1}^{n} \frac{(-1)^k \, q^{k(k-1)}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{q^n\,(1+q)}{(q^2;q^2)_n}.$$

Again, moving one step to the right, letting $n \to \infty$ and using Jacobi's triple product identity we get

$$\sum_{j=0}^{\infty} \frac{q^{j(j+2)}}{(q^2;q^2)_j} = \frac{(q^4;q^4)_\infty^2\,(q^2;q^4)_\infty}{(q^2;q)_\infty}.$$

This identity does not appear in Slater's [**40**] list. Similarly, for $(a^{(3)}, b^{(3)})$ we obtain

$$\sum_{k=-n-1}^{n} \frac{(-1)^k \, q^{3k}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{(-1)^{n+1}\,(1+q+q^2+q^3-q^{2n+3})}{q^{3n+3}\,(q^2;q^2)_n},$$

with the dual identity

$$\sum_{k=-n-1}^{n} \frac{(-1)^k \, q^{k(k-2)}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{q^{2n}(1+q+q^2+q^3)-1}{q\,(q^2;q^2)_n}.$$

- On the other hand, if we start from Slater's Bailey pair F (3), i.e., from

$$\sum_{k=-n}^{n} \frac{q^{k/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{1}{q^{n/2}\,(q^{1/2},q;q)_n},$$

again all 0-bilateral $a_n^{(d)}$ seem to be summable. For instance, for $d = 1$ we obtain

$$\sum_{k=-n}^{n} \frac{q^{3k/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{1-q^{1/2}+q^{n+1/2}}{q^{3n/2}\,(q^{1/2},q;q)_n}, \tag{3.21}$$

and for $d = 2$

$$\sum_{k=-n}^{n} \frac{q^{5k/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{1-q^{1/2}-q^{3/2}+q^2+q^{n+1/2}-q^{n+2}+q^{2n+3/2}}{q^{5n/2}\,(q^{1/2},q;q)_n},$$

with the dual identities

$$\sum_{k=-n}^{n} \frac{q^{k(k+3/2)}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{1-q^n+q^{n+1/2}}{q^{1/2}\,(q^{1/2},q;q)_n},$$

and

$$\sum_{k=-n}^{n} \frac{q^{k(k+5/2)}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{1-q^{n-1/2}+q^{n+1}+q^{2n-1/2}-q^{2n}-q^{2n+1}+q^{2n+3/2}}{q^{3/2}\,(q^{1/2},q;q)_n},$$

respectively. After moving one step to the right from identity (3.21), letting $n\to\infty$ and using Jacobi's triple product identity we obtain

$$\sum_{j=0}^{\infty} \frac{q^{j(2j-3)}}{(q;q)_{2j}} - q\sum_{j=1}^{\infty} \frac{q^{j(2j-3)}}{(q;q^2)_j\,(q^2;q^2)_{j-1}} = \frac{(-q;q)_\infty}{q}.$$

- The corresponding 1-bilateral results starting from F (4),

$$\sum_{k=-n-1}^{n} \frac{q^{k/2}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{1}{q^{(n+1)/2}\,(q^{1/2};q)_{n+1}\,(q;q)_n},$$

are found to be

$$\sum_{k=-n-1}^{n} \frac{q^{3k/2}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{1-q^{1/2}+q^{n+1}}{q^{(3n+3)/2}\,(q^{1/2};q)_{n+1}\,(q;q)_n}, \tag{3.22}$$

and

$$\sum_{k=-n-1}^{n} \frac{q^{5k/2}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{1-q^{1/2}-q^{3/2}+q^2+q^{n+1}-q^{n+5/2}+q^{2n+5/2}}{q^{(5n+5)/2}\,(q^{1/2};q)_{n+1}\,(q;q)_n},$$

with the dual identities

$$\sum_{k=-n-1}^{n} \frac{q^{k(k-1/2)}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{1-q^{n+1/2}+q^{n+1}}{(q^{1/2};q)_{n+1}\,(q;q)_n}, \tag{3.23}$$

and

$$\sum_{k=-n-1}^{n} \frac{q^{k(k-3/2)}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{1-q^{n}+q^{n+3/2}+q^{2n+1/2}-q^{2n+1}-q^{2n+2}+q^{2n+5/2}}{q^{1/2}\,(q^{1/2};q)_{n+1}\,(q;q)_n},$$

respectively. After moving one step to the right from identity (3.22), letting $n\to\infty$ and using Jacobi's triple product identity we obtain

$$\sum_{j=0}^{\infty} \frac{q^{j(2j-1)}}{(q;q^2)_{j+1}\,(q^2,q^2)_j} - q\sum_{j=0}^{\infty} \frac{q^{j(2j-1)}}{(q;q^2)_j\,(q^2;q^2)_j} = (-q;q)_\infty.$$

The same process starting from identity (3.23) leads to

$$\sum_{j=0}^{\infty} \frac{q^{2j(j+1)}}{(q;q)_{2j}} + q^2\sum_{j=0}^{\infty} \frac{q^{2j(j+2)}}{(q;q^2)_{j+1}\,(q^2;q^2)_j} = \frac{(q^8;q^8)_\infty\,(-q^5;q^8)_\infty\,(-q^3;q^8)_\infty}{(q^2;q^2)_\infty}.$$

- Similar results can be carried out from two identities not contained in Slater's list, namely

$$\sum_{k=-n}^{n} \frac{(-1)^k\,q^{k/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{(-1)^n}{q^{n/2}\,(-q^{1/2},q;q)_n},$$

and

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\, q^{k/2}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{(-1)^{n+1}}{q^{(n+1)/2}\,(-q^{1/2};q)_{n+1}\,(q;q)_n},$$

which we have found with the help of `qZeil`.

Case $\alpha = 1$:

This case is completely covered by the q-binomial theorem (Theorem 3.1) and its 1-bilateral counterpart (cf., for instance, Paule [**30**])

$$\sum_{k=-n-1}^{n} (-1)^k\, x^k \begin{bmatrix} 2n+1 \\ n-k \end{bmatrix}_q q^{\binom{k}{2}} = (q/x;q)_n\,(x;q)_{n+1}.$$

Both identities are self-dual.

Case $\alpha = 2$:

As shown above, this case is covered by dualization of the case $\alpha = 0$.

Case $\alpha = 3$:

- Let $a_n^{(0)} = (-1)^k\, q^{k(3k+1)/2}$. Then $(a^{(0)}, b^{(0)})$ and $(a^{(1)}, b^{(1)})$ are Slater's Bailey pairs B (1) and B (2), or

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{k(3k+1)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{1}{(q;q)_n},$$

  and

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{k(3k+3)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{q^n}{(q;q)_n},$$

  respectively. For $d = 2$ we obtain

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{k(3k+5)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{q^{2n+1} + q^n - 1}{q\,(q;q)_n},$$

  and for $d = 3$

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{k(3k+7)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{q^{3n+2} + q^{2n+1} + q^{2n} - q^{n+1} - 1}{q^2\,(q;q)_n},$$

  with the dual identities

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{-k(k+5)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{(-1)^n\,(1 + q^{n+1} - q^{2n+1})}{q^{n(n+5)/2}\,(q;q)_n},$$

  and

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{-k(k+7)/2}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \frac{(-1)^n\,(1 + q^{n+1} + q^{n+2} - q^{2n+1} - q^{3n+2})}{q^{n(n+7)/2}\,(q;q)_n},$$

  respectively.

- Analogously, from the 1-bilateral counterparts B (3)

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\, q^{k(3k+1)/2}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{1}{(q;q)_n},$$

and the trivial identity

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\, q^{k(3k+3)/2}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = 0,$$

for $d = 2$ and $d = 3$ we are led to

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\, q^{k(3k+5)/2}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = -\frac{1}{q\,(q;q)_n}, \tag{3.24}$$

and

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\, q^{k(3k+7)/2}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = -\frac{1+q^{n+1}}{q^2\,(q;q)_n}, \tag{3.25}$$

respectively. The dual identities read as

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\, q^{-k(3k+3)/2}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{(-1)^n}{q^{n(n+3)/2}\,(q;q)_n},$$

and

$$\sum_{k=-n-1}^{n} \frac{(-1)^k\, q^{-k(3k+5)/2}}{(q;q)_{n-k}\,(q;q)_{n+k+1}} = \frac{(-1)^n\,(1+q^{n+1})}{q^{n(n+5)/2}\,(q;q)_n},$$

respectively. Moving one step to the right from identity (3.24), letting $n \to \infty$ and using Jacobi's triple product identity we get the second Rogers-Ramanujan identity

$$\sum_{j=0}^{\infty} \frac{q^{j(j+1)}}{(q;q)_j} = \frac{1}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty}.$$

Starting from identity (3.25) the same process leads to the first Rogers-Ramanujan identity (3.8).

Case $\alpha \geq 4$:

No (non-trivial) closed form Bailey pairs could be found in this case.

Appendix A

A Note on q-Binomial Coefficients

Since q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ play a fundamental role in q-hypergeometric summation, we shall briefly outline in this chapter how to generalize them to arbitrary integers n and k as well as to complex parameters. The former is needed, for instance, in the q-Zeilberger algorithm for evaluating q-hypergeometric sequences involving q-binomial coefficients at the boundary points of the summation interval.

Definition A.1. For $n, k \in \mathbb{Z}$, let the *q-binomial coefficient* of n and k be given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \lim_{\delta \to 1} \frac{(\delta q; q)_n}{(q; q)_k \, (\delta q; q)_{n-k}}. \tag{A.1}$$

Clearly, if $n \geq 0$, equation (A.1) reduces to the usual definition of q-binomial coefficients,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k \, (q; q)_{n-k}}.$$

Note that the symmetry property for q-binomial coefficients,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$$

only holds for non-negative n. For instance, we have $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$, but $\begin{bmatrix} n \\ n \end{bmatrix}_q = 0$ for all negative n. However, the well-known recurrences for q-binomial coefficients are still satisfied for all integers n and k as stated in the following theorem.

Theorem A.1. *For $n, k \in \mathbb{Z}$, we have*

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q q^k + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{n+1-k}.$$

Proof.

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix}_q q^k &= \lim_{\delta\to 1} \frac{(\delta q;q)_n}{(q;q)_k\,(\delta q;q)_{n-k}} \frac{1-\delta q^{n+1-k}}{1-\delta q^{n+1-k}}\, q^k \\ &= \lim_{\delta\to 1} \frac{(\delta q;q)_n}{(q;q)_k\,(\delta q;q)_{n+1-k}} \big((1-\delta q^{n+1})-(1-q^k)\big) \\ &= \begin{bmatrix} n+1 \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q;\end{aligned}$$

$$\begin{aligned}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{n+1-k} &= \lim_{\delta\to 1} \frac{(\delta q;q)_n}{(q;q)_{k-1}\,(\delta q;q)_{n+1-k}} \frac{1-q^k}{1-q^k}\, \delta q^{n+1-k} \\ &= \lim_{\delta\to 1} \frac{(\delta q;q)_n}{(q;q)_k\,(\delta q;q)_{n+1-k}} \big((1-\delta q^{n+1})-(1-\delta q^{n+1-k})\big) \\ &= \begin{bmatrix} n+1 \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k \end{bmatrix}_q.\end{aligned}$$

□

Furthermore, the q-binomial coefficients might also be generalized to complex parameters. This has been already done by Gasper and Rahman [**20**], however, with the definition presented there,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{(q^{\beta+1}, q^{\alpha-\beta+1};q)_\infty}{(q, q^{\alpha+1};q)_\infty},$$

we get into trouble if α is a negative integer, because the denominator becomes 0 then.

To overcome this problem we first introduce a q-analogue of the gamma function (see, e.g., Askey [**12**] or Gasper and Rahman [**20**]).

Definition A.2. For $x \in \mathbb{C}$, the *q-gamma function* is given by

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}\,(1-q)^{1-x},$$

when $0 < q < 1$.

From this definition it follows directly that Γ_q satisfies the functional equation

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\,\Gamma_q(x), \quad \Gamma_q(1) = 1, \tag{A.2}$$

the q-counterpart of the well-known functional equation for the gamma function

$$\Gamma(x+1) = x\,\Gamma(x), \qquad \Gamma(1) = 1.$$

Furthermore, for $n \in \mathbb{N}$ we have

$$\Gamma_q(n+1) = [n]_q!,$$

where $[n]_q!$ denotes the q-factorial defined as $[n]_q! = 1\cdot(1+q)\cdot(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$ which trivially turns into $\Gamma(n+1) = n!$ for $q = 1$. Obviously, the q-gamma function has poles at $x = 0, -1, -2$, etc. The q-binomial coefficient for complex parameters could then be defined as follows.

Definition A.3. For $\alpha, \beta \in \mathbb{C}$, the *q-binomial coefficient* of α and β is given by

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_q := \lim_{\epsilon\to 0} \frac{\Gamma_q(\alpha+1+\epsilon)}{\Gamma_q(\beta+1)\,\Gamma_q(\alpha-\beta+1+\epsilon)},$$

where it is assumed that β is an integer whenever α is a negative integer.

This is a q-analogue of the extended definition for ordinary binomial coefficients,

$$\binom{\alpha}{\beta} := \lim_{\epsilon\to 0} \frac{\Gamma(\alpha+1+\epsilon)}{\Gamma(\beta+1)\,\Gamma(\alpha-\beta+1+\epsilon)},$$

which has been used, for instance, by Wegschaider [**42**].

Theorem A.2. *For $\alpha, \beta \in \mathbb{C}$, we have*

$$\begin{bmatrix}\alpha+1\\ \beta\end{bmatrix}_q = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}_q q^{\beta} + \begin{bmatrix}\alpha\\ \beta-1\end{bmatrix}_q = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}_q + \begin{bmatrix}\alpha\\ \beta-1\end{bmatrix}_q q^{\alpha+1-\beta}.$$

Proof. This is easily seen by proceeding as in the proof of Theorem A.1 and using relation (A.2). ☐

Appendix B

How to Use qZeil

In the following we shall describe the usage of the author's Mathematica implementation of the Extended q-Zeilberger Algorithm. Parts of this manual have been taken from an article by Paule and Riese [**33**] and the author's diploma thesis [**37**]. The latest version is available by email request to the author†. Additional information can be retrieved via the World Wide Web from the qZeil homepage‡.

B.1 Package Structure and Installation

The package consists of five files named `qZeil.m`, `qTelescope.m`, `qInput.m`, `qSimplify.m`, and `LinSolve.m`, which have to be copied into one directory. After starting a Mathematica session from this directory and typing `<<qZeil.m` all files (including the add-on package `Bailey.m` described in Chapter 3) are loaded automatically.

The whole package has been adapted for Mathematica 3.0. Since notebooks are now machine independent, the collection of examples (formerly known as `qZeilExamples.txt`) is distributed in notebook format as `qZeilExamples.nb`. At the moment this notebook contains input for about 500 identities. This also means that this set serves as a test suite for each update of the package. Furthermore, the package is accompanied by the file `WhatsNew.txt` which describes all the changes since version 1.4.

The source files `qTelescope.m` and `qSimplify.m` may be renamed to `qtelesco.m` and `qsimplif.m`, respectively, if the system has troubles with file names not matching the "8.3" (MS-DOS) naming scheme.

B.2 Interfaces

The package has two interfaces. The user can invoke q-hypergeometric telescoping to find a closed form for a sum, or Zeilberger's algorithm to come up with a recurrence for a sum. The corresponding commands are given by

`qTelescope[`*summand*`,` *range*`,` *intconst*`,` *opts*`]`

and

`qZeil[`*summand*`,` *range*`,` n`,` *order*`,` *intconst*`,` *opts*`]`,

† `Axel.Riese@risc.uni-linz.ac.at`

‡ `http://www.risc.uni-linz.ac.at/research/combinat/risc/software/qZeil`

where the parameters *intconst* and *opts* are optional arguments.

B.3 The Summand

Let $K = \mathbb{Q}(\kappa_1, \ldots, \kappa_m)$ denote the field of rational functions over the rational numbers $\mathbb{Q}$ in a fixed number of indeterminates $\kappa_1, \ldots, \kappa_m$ all different from q, k, and q^k. The transcendental extension of K by the indeterminate q is denoted by F, i.e., $F = K(q)$.

In the present implementation we allow as summand for `qTelescope` any q-hypergeometric sequence $(f_k)_{k\in\mathbb{Z}}$ of the form

$$f_k = \frac{\prod_r (C_r\, q^{(c_r i_r)k+d_r}; q^{i_r})_{a_r k+b_r}}{\prod_s (D_s\, q^{(v_s j_s)k+w_s}; q^{j_s})_{t_s k+u_s}} \cdot R(q^k) \cdot q^{\alpha\binom{k}{2}} \cdot z^k, \tag{B.1}$$

with

C_r, D_s	power products in K,
a_r, t_s	specific integers (i.e., integers free of any parameters),
b_r, u_s	integers, which may depend on parameters free of k,
c_r, d_r, v_s, w_s	specific integers,
i_r, j_s	specific non-zero integers,
R	a rational function in $F(q^k)$ such that the denominator factors completely into a product of terms of the form $(1 - D\, q^{vk+w})$,
α	specific integer, and
z	a rational function in F.

As summand for `qZeil` we allow any q-hypergeometric sequence $(f_{n,k})_{n\in\mathbb{N},k\in\mathbb{Z}}$ (where we additionally assume that the κ_i are different from n and q^n) of the form

$$f_{n,k} = \frac{\prod_r (C_r\, q^{(d_r i_r)n+(e_r i_r)k+l_r}; q^{i_r})_{a_r n+b_r k+c_r}}{\prod_s (D_s\, q^{(f_s j_s)n+(g_s j_s)k+m_s}; q^{j_s})_{u_s n+v_s k+w_s}} \cdot R(q^n, q^k) \cdot q^{\alpha\binom{k}{2}+\beta nk} \cdot z^k, \tag{B.2}$$

with

C_r, D_s	power products in K,
a_r, b_r, u_s, v_s	specific integers (i.e., integers free of any parameters),
c_r, w_s	integers, which may depend on parameters free of n and k,
d_r, e_r, f_s, g_s	specific integers,
l_r, m_s	integers free of n and k,
i_r, j_s	specific non-zero integers,
R	a rational function in $F(q^n, q^k)$ such that the denominator factors completely into a product of terms of the form $(1 - D\, q^{fn+gk+m})$,
α, β	specific integers, and
z	a rational function in F.

The q-shifted factorial $(a; q^i)_m$ has to be typed as `qfac[`a`,q^`i`,`m`]`. In addition we allow terms of the form `qBrackets[`a`,q]` for $[a]_q := (1-q^a)/(1-q)$, `qFactorial[`a`,q]` for $[a]_q! := [1]_q\,[2]_q \cdots [a]_q$, and `qBinomial[`$a$`,`$b$`,q]` for $\begin{bmatrix} a \\ b \end{bmatrix}_q$, provided that those expressions can be translated correctly — with respect to (B.1) or (B.2) — into terms of q-shifted factorials. Note that also for these forms powers `q^`i are admitted.

B.4 The Summation Range

The range of summation has to be specified in the form

$$range \texttt{ := \{}k\texttt{, } low\texttt{, } upp\texttt{\}}.$$

In `qTelescope`, *low* and *upp* may be arbitrary integers free of k satisfying $low \leq upp$. In `qZeil`, *low* and *upp* are linear integer functions in the recurrence variable n being free of k such that $low \leq upp$.

In Zeilberger's algorithm the user may specify one or both bounds to be ±`Infinity`. In this case, the bounds are assumed to be naturally induced by the finite support. The algorithm runs considerably faster in this *Turbo*-mode, since no inhomogeneous part of the recurrence has to be computed.

B.5 The Optional Argument *intconst*

Since Mathematica is not able to handle typed variables, it is necessary to simulate them by telling the system explicitly which indeterminates should be treated as non-negative integer constants. If one assigns to the optional argument *intconst* a list of Mathematica symbols representing those indeterminates, the program will assume them to be non-negative integers. This also improves the simplification abilities of the program.

Consider the following example. Suppose we want to find a closed form for the indefinite sum

$$\sum_{k=0}^{n} \begin{bmatrix} m+k \\ k \end{bmatrix}_q q^k.$$

Without any knowledge about m the program is not able to recognize m and $m+k$ in $(q;q)_m$ and $(q;q)_{m+k}$, respectively, as integers. The problem disappears if we make the assignment $intconst := \{m\}$.

```
In[1]:= <<qZeil.m

Out[1]= Axel Riese's qZeilberger implementation version 1.8 loaded

In[2]:= qTelescope[qBinomial[m+k,k,q] q^k, {k, 0, n}, {m}]

Out[2]= qBinomial[1 + m + n, 1 + m, q]
```

Note that in `qZeil` and `qTelescope` all indeterminates appearing in the bounds as well as the recursion variable n (in `qZeil`) are assumed to be elements of *intconst* automatically.

B.6 Global Variables

The (simplified) certificate $cert(n, k)$, i.e., the rational function from Chapter 2 such that $g_{n,k} = cert(n, k) \cdot f_{n,k}$, is delivered by calling the function `Cert` without any parameters.

The values of the global variables `FF` and `GG` correspond to $f_{n,k}$ and $g_{n,k}$, respectively, of the last computation.

The output behavior of the program can be influenced by the global Boolean variables `Talk` and `Output`. If `Talk` is set to `True`, the user can see explicitly which step of the algorithm is executed at the moment. This is mainly thought for time-consuming examples. Default value for `Talk` is `False`. If `Output` is set to `True`, then running `qTelescope` or `qZeil` generates the file `GoOut`, where some intermediate results of the actual computation are written to. Default value for `Output` is `True`. For an example, see Paule and Riese [**33**].

As shown in Chapter 2, the program computes the companion identity, if the global variable `Companion` is set to `True`, and f and g in fact form a qWZ-pair. Default value for `Companion` is `False`, the result is assigned to the variable `CompId`.

The program computes the dual qWZ-pair, if the global variable `Dual` is set to `True`, and f and g actually form a qWZ-pair. The result is assigned to the variable `DualPair`. Default value for `Dual` is `False`. The dual identity can be computed from the dual pair by calling the function `DualId` as shown in Section B.8 below.

B.7 Options

Beginning with version 1.5, `qZeil` and `qTelescope` can be called with several options described in the following, where for sake of simplicity $f_{n,k}$ denotes the input summand for `qZeil` as well as for `qTelescope` (where the recurrence variable n is completely insignificant).

B.7.1 Option `EquationSolver`

In the setting `EquationSolver->NullSpace`, the procedures `qZeil` and `qTelescope` use the built-in Mathematica function `NullSpace` for solving systems of linear equations. Default value for `EquationSolver` is `Automatic`, which invokes the null space algorithm provided with the package (cf. Section B.9). The main difference between these methods is that `NullSpace` is faster in general but does not put the elements in the result over a common denominator. In practice one should use `NullSpace` only for showing that a system of linear equations has *no* solution, because even for rather simple examples the output of `NullSpace` cannot be brought over common denominators in reasonable time.

B.7.2 Option `OnlySummand`

With `OnlySummand->True`, calling `qZeil` only computes

$$\sum_{i=0}^{d} \sigma_i(n)\, f_{n-i,k} = f_{n,k} \cdot \sum_{i=0}^{d} \sigma_i(n)\, \frac{f_{n-i,k}}{f_{n,k}}$$

as a rational function multiple of $f_{n,k}$ with undetermined σ_i, where d is the order of the recurrence specified by the user. This feature, for instance, allows to perform and check all steps of the qWZ-method (cf. Chapter 2) "by hand": call `qZeil` with $d = 1$ and `OnlySummand->True`, replace σ_0 and σ_1 by 1 and -1, respectively, and finally run `qTelescope` on the output as demonstrated in the following example generating a qWZ-proof of a special case of the q-Chu-Vandermonde identity

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 q^{k^2} = \begin{bmatrix} 2n \\ n \end{bmatrix}_q .$$

```
In[3]:= qZeil[qBinomial[n,k,q]^2 q^(k^2) / qBinomial[2n,n,q], {k, 0, n}, n, 1,
              OnlySummand->True]

                       2
          -2 k + k                4   2 k       n 4
Out[3]= (q           qfac[q, q, n]  (q    (1 - q )  Sigma[0] +

               2 n       k - n 2       2 n        -1 + 2 n
              q    (1 - q     )  (1 - q   ) (1 - q        ) Sigma[1])) /

                 n 4              2                                   2
          ((1 - q )  qfac[q, q, k]  qfac[q, q, 2 n] qfac[q, q, -k + n] )

In[4]:= qTelescope[% /. {Sigma[0]->1, Sigma[1]->-1}, {k, 0, n}]

                            2
               -2 k + k    2 k       n 4
Out[4]= {Sum[(q           (q    (1 - q )  -

                  2 n       k - n 2       2 n        -1 + 2 n               4
                 q    (1 - q     )  (1 - q   ) (1 - q        )) qfac[q, q, n] ) /

                    n 4              2                                   2
             ((1 - q )  qfac[q, q, k]  qfac[q, q, 2 n] qfac[q, q, -k + n] ),

            {k, 0, n}] == 0, {n != 0}}
```

Checking the original identity for $n = 0$ completes the proof. Note that proceeding as above might cause problems if the rational function involved is very "large", because factorization in `qTelescope` might take several hours then.

B.7.3 Option `MagicFactor`

The option `MagicFactor` automatically applies Paule's [**31**] method of creative symmetrizing (cf. also Paule and Riese [**33**]) for decreasing the order of the recurrence in `qZeil`. Let $\mu = \mu(k)$ denote a non-constant linear integer function of k with the property that

$$\sum_k f_{n,k} = \sum_k f_{n,\mu(k)}.$$

Typical examples for μ are $-k$, $-k-1$, $-k+1$, $n-k$, $2n-k$, etc., which pop up, e.g., by symmetry reasons or by operations like reversing the order of summation. In this case we have

$$\sum_k f_{n,k} = \sum_k \frac{f_{n,k} + f_{n,\mu(k)}}{2} = \sum_k f_{n,k}\,\frac{1 + f_{n,\mu(k)}/f_{n,k}}{2} = \sum_k f_{n,k} \cdot MF(n,k),$$

for some function $MF(n,k)$. Now, if MF is actually rational in q^n and q^k satisfying the input restrictions for `qZeil` or `qTelescope`, option `MagicFactor->`μ automatically multiplies the original summand $f_{n,k}$ with $MF(n,k)$.

As an example let us consider Jackson's terminating q-analogue of Dixon's sum (see, e.g., Gasper and Rahman [**20**]),

$$ {}_3\phi_2\left[\begin{matrix} q^{-2n}, b, c \\ q^{1-2n}/b, q^{1-2n}/c \end{matrix}; q, \frac{q^{2-n}}{bc}\right] = \frac{(b,c;q)_n\,(q,bc;q)_{2n}}{(q,bc;q)_n\,(b,c;q)_{2n}}, $$

for which no recurrence of order 1 and order 2 can be found directly:

```
In[5]:= fnk = qfac[q^(-2n),q,k] qfac[b,q,k] qfac[c,q,k] (q^(2-n)/(b c))^k *
             qfac[q,q,n] qfac[b c,q,n] qfac[b,q,2n] qfac[c,q,2n] /
            (qfac[q^(1-2n)/b,q,k] qfac[q^(1-2n)/c,q,k] qfac[q,q,k] *
             qfac[b,q,n] qfac[c,q,n] qfac[q,q,2n] qfac[b c,q,2n]);

In[6]:= qZeil[fnk, {k, 0, 2n}, n, 1]

Out[6]= No solution: Increase order by 1

In[7]:= qZeil[fnk, {k, 0, 2n}, n, 2]

Out[7]= No solution: Increase order by 1
```

When trying to compute a recurrence of order 3, it turns out that the underlying system of equations cannot be solved within 90 minutes.

```
In[8]:= TimeConstrained[ qZeil[fnk, {k, 0, 2n}, n, 3], 90*60, $Failed ]

Out[8]= $Failed
```

However, observe that for $\mu(k) = 2n - k$ we have

$$ \begin{aligned} MF(n,k) &= \frac{1 + f_{n,2n-k}/f_{n,k}}{2} \\ &= \frac{1}{2}\left(1 + \frac{(q^{-2n},b,c;q)_{2n-k}\,(q^{1-2n}/b,q^{1-2n}/c,q;q)_k}{(q^{1-2n}/b,q^{1-2n}/c,q;q)_{2n-k}\,(q^{-2n},b,c;q)_k}\ \frac{b^{2k}\,c^{2k}\,q^{2(2-n)(n-k)}}{b^{2n}\,c^{2n}}\right). \end{aligned} $$

Thus, by using the rules (cf. Gasper and Rahman [**20**])

$$ (a;q)_{2n-k} = \frac{(a;q)_{2n}}{(q^{1-2n}/a;q)_k}\,(-q/a)^k\,q^{\binom{k}{2}-2nk} $$

and

$$ (aq^{-2n};q)_{2n} = (q/a;q)_{2n}\,a^{2n}\,q^{-n(2n+1)} $$

we end up with

$$ MF(n,k) = (1+q^{n-k})/2. $$

Applying `qZeil` to the new summand $(1+q^{n-k})/2 \cdot f_{n,k}$ now indeed delivers the recurrence of expected order 1.

```
In[9]:= qZeil[(1+q^(n-k))/2 fnk, {k, 0, 2n}, n, 1]

Out[9]= SUM[n] == 1
```

Moreover, calling `qZeil` with option `MagicFactor` completely saves us from computing $MF(n,k)$ by hand:

```
In[10]:= qZeil[fnk, {k, 0, 2n}, n, 1, MagicFactor->2n-k]

                              -k + n
                         1 + q
         Magic factor: -----------
                              2

Out[10]= SUM[n] == 1
```

Default value for `MagicFactor` is k.

B.7.4 Option `Shadow`

The option `Shadow` to change the default shadowing strategy for computing dual identities has already been described in Chapter 2. Thus, we will not go into the details here.

B.7.5 Option `FindAlphaBeta`

With `FindAlphaBeta->True`, the procedures `qZeil` and `qTelescope` are enforced to make suggestions for all possible choices of integers α and β such that there eventually exists a recurrence of the specified order, respectively a closed form, for

$$f_{n,k} \cdot q^{\alpha\binom{k}{2}+\beta k}.$$

The result should be interpreted in the following way. For all integer pairs not being (α,β)-candidates there in fact does not exist a solution. However, not all candidates necessarily yield a solution.

Let us briefly outline why it is possible to make such an assertion. For this, let F_k denote our recurrence-“Ansatz”, i.e., $F_k = \sum_{i=0}^{d} \sigma_i(n)\, f_{n-i,k}$, which is a rational function multiple of $f_{n,k}$ with undetermined σ_i. Since (F_k) is a q-hypergeometric sequence, there exists a rational function $\rho(x)$ such that $F_{k+1}/F_k = \rho(q^k)$ for all k. Suppose that the normal form of ρ (as described by Paule and Riese [**33**]) is given by

$$\rho = \frac{\epsilon P}{P} \cdot \frac{Q}{\epsilon R},$$

where ϵ denotes the q-shift operator defined by $(\epsilon P)(x) = P(qx)$, and the polynomials $P(x)$, $Q(x)$, and $R(x)$ are normalized in a certain way satisfying $\gcd(P,Q) = 1 = \gcd(P,R)$ and $\gcd(Q, \epsilon^k R) = 1$ for all $k \geq 1$.

Now, for $\alpha \in \mathbb{Z}$ let us define $f_{n,k}^{\alpha} := f_{n,k} \cdot A_k^{\alpha}$, where $A_k^{\alpha} = q^{\alpha\binom{k}{2}}$. To easily distinguish between the cases $\alpha \leq 0$ and $\alpha \geq 0$, we define $\alpha_+ = \max(\alpha, 0)$ and $\alpha_- = \max(-\alpha, 0)$. Clearly, the corresponding F^{α} is then given by $F_k^{\alpha} = F_k \cdot A_k^{\alpha}$. Since $A_{k+1}^{\alpha}/A_k^{\alpha} = q^{\alpha k}$, our normal form for F_k^{α} becomes

$$\rho^{\alpha} = \frac{\epsilon P^{\alpha}}{P^{\alpha}} \cdot \frac{Q^{\alpha}}{\epsilon R^{\alpha}},$$

where $P^{\alpha}(x) = P(x)$, $Q^{\alpha}(x) = Q(x) \cdot x^{\alpha_+}$, and $R^{\alpha}(x) = R(x) \cdot x^{\alpha_-} \cdot q^{-\alpha_-}$. Thus, we see that $\deg(Q^{\alpha}) = \deg(Q) + \alpha_+$ and $\deg(R^{\alpha}) = \deg(R) + \alpha_-$, whereas $\deg(P^{\alpha})$ does not depend on α. But to solve the key equation $Q^\alpha \cdot \epsilon Y - R^\alpha \cdot Y = P^\alpha$ for a polynomial Y (see Paule and Riese [**33**]), the degree of neither Q^α nor R^α must exceed the degree of P^α unless the degrees of Q^α and R^α are equal. Note that this only happens for one specific choice of α. From this it is immediately clear, that there are always only finitely many α-candidates.

On the other hand, if we define $f^\beta_{n,k} := f_{n,k} \cdot B^\beta_k$, where $B^\beta_k = q^{\beta k}$, things are more complicated, because $B^\beta_{k+1}/B^\beta_k = q^\beta$ may alter the normal form of ρ^β in several ways depending on the sign of β and the original summand $f_{n,k}$ itself (see again Paule and Riese [**33**] for further details). More precisely, the factor B^β_k might influence the degree of P^β, but never the degrees of Q^β and R^β. As a consequence, we can find at most a lower or upper bound for the β-candidates.

To illustrate this feature we consider again the special case of the q-Chu-Vandermonde identity above with the factor q^{k^2} dropped.

```
In[11]:= qZeil[qBinomial[n,k,q]^2 / qBinomial[2n,n,q], {k, 0, n}, n, 1,
           FindAlphaBeta->True]

         alpha   beta
         -----   ----
         2       Interval[{-∞, ∞}]
```

Thus, we may conclude that for input $f_{n,k} \cdot q^{\alpha\binom{k}{2}+\beta k}$ there does not exist a q-Zeilberger recurrence of order 1 for $\alpha \neq 2$ whatever the choice for β is. Indeed, there could exist a recurrence of order 1 which is missed by the algorithm. Running `qZeil` on $f_{n,k} \cdot q^{2\binom{k}{2}+\beta k}$ with concrete values for β leads to the conjecture that we find a recurrence of order 1 for all $\beta \leq 1$.

Default value for `FindAlphaBeta` is `False`.

B.7.6 Option `PolyMult`

With `PolyMult->`c, the procedure `qTelescope` computes all polynomial multipliers of input f_k with maximal degree c (w.r.t. q^k) which make the input Gosper-summable. Note that this still leads to a system of *linear* equations. For instance, consider the Bailey pair identity

$$\sum_{k=-n}^{n} \frac{(-1)^k\, q^{\binom{k}{2}}}{(q;q)_{n-k}\,(q;q)_{n+k}} = \delta_{n,0}$$

from Chapter 3 which is not Gosper-summable. However, we can find a polynomial multiplier of degree 1 such that q-hypergeometric telescoping succeeds.

```
In[12]:= qTelescope[(-1)^k q^Binomial[k,2] / (qfac[q,q,n-k] qfac[q,q,n+k]),
                   {k, -n, n}, PolyMult->1]

                                   2
                    k  -k/2 + k /2       k
                (-1)  q            (1 + q )
Out[12]= {Sum[-----------------------------------, {k, -n, n}] == 0, {n != 0}}
              qfac[q, q, -k + n] qfac[q, q, k + n]
```

Clearly, the polynomial $1 + q^k$ comes from creative symmetrizing and could have been found also with the `MagicFactor->-k` option.

Beginning with version 1.8, option `PolyMult` also works in connection with `qZeil` in the following way. Our goal is to find all polynomials $P(k)$ of maximal degree c and being free of n such that there exists a q-Zeilberger recurrence of order d for $f'_{n,k} := f_{n,k} \cdot P(k)$, i.e., to compute polynomials $\sigma_0(n), \dots, \sigma_d(n)$ in q^n and a q-hypergeometric sequence $g_{n,k}$ such that

$$\sum_{j=0}^{d} \sigma_j(n)\, f'_{n-j,k} = g_{n,k} - g_{n,k-1}. \tag{B.3}$$

For this, we first observe that Zeilberger's algorithm can be used more or less directly (as shown below) to compute recurrences with polynomial coefficients that may depend on q^k, too. More precisely, our Extended q-Zeilberger Algorithm computes in the first step polynomials $\tau_j(n,k)$ in q^n and q^k with maximal degree c (w.r.t. q^k) and a q-hypergeometric sequence $h_{n,k}$, such that

$$\sum_{j=0}^{d} \tau_j(n,k)\, f_{n-j,k} = h_{n,k} - h_{n,k-1}. \tag{B.4}$$

To illustrate how to find such a recurrence, let us first recall the main steps of the (ordinary) q-Zeilberger algorithm very briefly (for the details see Paule and Riese [**33**]). In order to determine polynomials $\sigma_0(n), \dots, \sigma_d(n)$ in q^n and a q-hypergeometric sequence $g_{n,k}$ such that a given summand $f_{n,k}$ satisfies the recurrence

$$\sum_{j=0}^{d} \sigma_j(n)\, f_{n-j,k} = g_{n,k} - g_{n,k-1}, \tag{B.5}$$

we apply q-hypergeometric telescoping to the left hand side of (B.5), a rational function multiple of $f_{n,k}$, say F_k, with undetermined σ_j. This means, we compute the normal form for the rational function F_{k+1}/F_k, solve the corresponding key equation for the σ_j and the coefficients $y_0, \dots, y_m$ of the solution polynomial $Y = y_0 + y_1 q^k + \cdots + y_m q^{mk}$, and finally compute $g_{n,k}$ from $f_{n,k}$ and Y.

To come up with a recurrence of the form (B.4) we proceed as follows. We simply put $\tau_j(n,k) = \sum_{l=0}^{c} \tau_{j,l}(n)\, q^{lk}$ with $\tau_{j,l}$ being undetermined polynomials in q^n, and solve the resulting key equation in `qTelescope` for the $\tau_{j,l}$ and the coefficients $y_0, \dots, y_m$ of the solution polynomial $Y = y_0 + y_1 q^k + \cdots + y_m q^{mk}$ being polynomials in q^n, too. This is done as usual by comparing the coefficients of each power of q^k, which still leads to a homogeneous system of *linear* equations.

Now, suppose we have found a solution and suppose in addition that the τ_j split into

$$\tau_j(n,k) = \sigma_j(n) \cdot P(k) \qquad \text{for all } 0 \le j \le d, \tag{B.6}$$

where the σ_j are polynomials in q^n and P is a polynomial in q^k. In this case we would be done, since recurrence (B.3) is satisfied for $f'_{n,k} = f_{n,k} \cdot P(k)$ and $g_{n,k} = h_{n,k}$ computed as usual from $f_{n,k}$ and Y.

Unfortunately, the separation of variables (B.6) is not possible in general for one single choice of the τ_j. However, we can overcome this problem by using a null space algorithm to compute a set of vectors for the unknowns $\tau_{j,l}$ and y_i, i.e., a set of vectors of the form

$$\left\{ \left(\tau_{0,0}^{(i)}, \dots, \tau_{0,c}^{(i)}, \tau_{1,0}^{(i)}, \dots, \tau_{1,c}^{(i)}, \dots, \tau_{d,0}^{(i)}, \dots, \tau_{d,c}^{(i)}, y_0^{(i)}, \dots, y_m^{(i)} \right)^T \right\}_{1 \le i \le I}, \tag{B.7}$$

that form a basis for the I-dimensional solution space of the underlying key equation, a vector space over the polynomials in q^n. With this basis in hands, any linear combination of its elements gives then rise to a solution $\tau_j(n,k)$ and $h_{n,k}$ of recurrence (B.4), where $h_{n,k}$ is computed from $f_{n,k}$ and the linear combination of the corresponding $Y^{(i)}$. Thus, our problem reduces to finding a linear combination of the basis elements which fulfills (B.6). In other words, we have to find polynomials $\lambda_1,\ldots,\lambda_I$, $\sigma_0,\ldots,\sigma_d$ in q^n, and a polynomial P in q^k of maximal degree c such that

$$\sum_{i=1}^{I} \lambda_i(n)\,\tau_j^{(i)}(n,k) = \sigma_j(n)\cdot P(k) \qquad \text{for all } 0 \le j \le d, \tag{B.8}$$

since then the recurrence looked for is given by

$$\sum_{j=0}^{d} \sigma_j(n)\, f'_{n-j,k} = g_{n,k} - g_{n,k-1},$$

where $f'_{n,k} = f_{n,k}\cdot P(k)$ and $g_{n,k}$ denotes the sequence computed from $f_{n,k}$ and $\sum_{i=1}^{I}\lambda_i\,Y^{(i)}$.

Finally, for solving (B.8) we need an upper bound for the degree of the σ_j and λ_i first, which has to be specified explicitly by the user. Comparing coefficients of powers of q^n and q^k in (B.8) then leads to a system of non-linear equations, though being rather close to a linear system. Nevertheless, it turns out that Mathematica is not capable of solving even small examples, whereas Maple does the job excellently. Therefore, for an implementation of a first prototype version we decided to implement an interface between both systems, where Mathematica sets up the equations, hands them over to Maple, which solves the system and returns the solution back to Mathematica. For this, Mathematica needs to know how to start Maple as a filter. This is done by assigning a string to the variable `MapleCall` specifying the command for invoking Maple and redirecting the input from the file `MapleIn`, such as the following:

```
In[13]:= MapleCall = "maple -f -q <MapleIn";
```

This extended version of the q-Zeilberger algorithm is actually applied when `qZeil` is called with the option `PolyMult->{`c_1`, `c_2`}`, where c_1 is the maximal degree of P (w.r.t. q^k) and c_2 the maximal degree of the σ_j and λ_i (w.r.t. q^n). Note that with this option enabled, the summation bounds are automatically changed to $\pm\infty$, since inhomogeneous recurrences cannot be handled at the moment. Therefore, $f_{n,k}$ must have finite support.

Let us consider a simple example, namely a terminating version of Bailey's sum of a well-poised ${}_3\psi_3$ series (see, e.g., Gasper and Rahman [**20**]),

$${}_3\psi_3\left[\begin{matrix} q^{-n}, c, d \\ q^{n+1}, q/c, q/d \end{matrix}; q, \frac{q^{n+1}}{cd}\right] = \frac{(q, q/cd; q)_n}{(q/c, q/d; q)_n}.$$

Note that for this identity we actually need creative symmetrizing to find a recurrence of order 1. Moreover, let us slightly change the argument of the ${}_3\psi_3$ series to q^n/cd. Clearly, `qZeil` does not find a recurrence of order 1 then:

```
In[14]:= fnk = qfac[q^(-n),q,k] qfac[c,q,k] qfac[d,q,k] (q^n/(c d))^k /
              (qfac[q^(1+n),q,k] qfac[q/c,q,k] qfac[q/d,q,k]);

In[15]:= qZeil[fnk, {k, -Infinity, Infinity}, n, 1]

Out[15]= No solution: Increase order by 1
```

However, for this example we know in advance that there exists a polynomial multiplier $P(k) = q^k(1 + q^k)$, because multiplying the summand by q^k restores the original argument and $1 + q^k$ comes from creative symmetrizing. Therefore, we try the following:

```
In[16]:= qZeil[fnk, {k, -Infinity, Infinity}, n, 1, PolyMult->{2,0}]

Out[16]= No solution: Increase order by 1

In[17]:= qZeil[fnk, {k, -Infinity, Infinity}, n, 1, PolyMult->{2,1}]

Out[17]= No solution: Increase order by 1

In[18]:= qZeil[fnk, {k, -Infinity, Infinity}, n, 1, PolyMult->{2,2}]

Out[18]= No solution: Increase order by 1

In[19]:= qZeil[fnk, {k, -Infinity, Infinity}, n, 1, PolyMult->{2,3}]

                                                 n
                                        n       q                  k       k
                                  (1 - q ) (1 - ---) SUMX[-1 + n, q  (1 + q )]
                  k       k                     c d
Out[19]= SUMX[n, q  (1 + q )] == --------------------------------------------
                                                     n        n
                                                    q        q
                                               (1 - --) (1 - --)
                                                    c        d
```

The last computation took about 15 seconds on a Pentium 100. Note that in the output the symbol SUMX[n, P] is used as an abbreviation for $\sum_k f_{n,k} \cdot P(k)$, where $f_{n,k}$ denotes the original summand and $P(k)$ the polynomial multiplier found by the program.

For this example we additionally stored some intermediate results and adapted them to the notation used in this section to easily check the main steps. For instance, we find that the basis (B.7) for the solution space of the key equation contains three elements:

```
In[20]:= BASIS
                     2
                1 + S
Out[20]= {{-1, ------, -1, 0, 1, 0, 0, 0},
                  S
                                          2      2    3
          -(c d q) + c q S + d q S - c S  - d S  + S
         {------------------------------------------,
                              q S
```

```
c + d - q + c d q - S + c d S - c q S - d q S  -(c d) + q S
------------------------------------------, ------------,
                     q                             q

-(c d q) + S
------------, 0, 0, 1, 0},
     q

 -(c d) + S  -1 + c + d + c d - S - c S - d S + c d S
{----------, ---------------------------------------,
    q                          q

                        2      2    3
-(c d) + c S + d S - c S  - d S  + S         -(c d) + S
-----------------------------------, 0, 0, ----------, 0, 1}}
               q S                              q
```

The symbol S stands for q^n. It turns out that the resulting system of homogeneous equations (B.8) consists of 34 equations in 23 variables. However, for some of the variables we immediately find by inspection that they have to vanish, so that we actually end up with 24 equations in 15 variables:

```
In[21]:= Length[EQS]

Out[21]= 24

In[22]:= Short[EQS, 7] (* don't show all equations *)

Out[22]//Short=

         {-(q P[0] SIG[1, 0]) == 0, -(q P[1] SIG[1, 0]) == 0,

           -(q P[2] SIG[1, 0]) == 0, -(c d q LAM[1, 1]) - q P[0] SIG[1, 1] == 0,

           q LAM[2, 1] - q P[1] SIG[1, 1] == 0,

           -(c d LAM[3, 1]) - q P[2] SIG[1, 1] == 0,

           LAM[1, 1] - q P[0] SIG[1, 2] == 0, <<2>>, <<12>>,

           q LAM[1, 1] - q LAM[2, 2] + <<2>> - q P[2] SIG[2, 2] == 0,

           LAM[1, 1] - q P[0] SIG[2, 3] == 0, q LAM[2, 2] - q P[1] SIG[2, 3] == 0,

           LAM[3, 1] - q P[2] SIG[2, 3] == 0}

In[23]:= VARS

Out[23]= {LAM[1, 1], LAM[2, 1], LAM[2, 2], LAM[3, 1], SIG[1, 0], SIG[1, 1],

           SIG[1, 2], SIG[1, 3], SIG[2, 0], SIG[2, 1], SIG[2, 2], SIG[2, 3],

           P[0], P[1], P[2]}
```

According to (B.8), the LAM[i, j] and SIG[i, j] denote the coefficients of q^{jn} in λ_i and σ_i, respectively, whereas the P[i] denote the coefficients of q^{ik} in P.

If we now try to solve the system of equations with the built-in Mathematica function Solve, we do not get an answer within 10 minutes whatever the order of the variables is, for instance:

```
In[24]:= TimeConstrained[ Solve[EQS, VARS], 600, $Failed ]

Out[24]= $Failed

In[25]:= TimeConstrained[ Solve[EQS, Reverse[VARS]], 600, $Failed ]

Out[25]= $Failed
```

On the other hand, if we first try to compute a Gröbner basis, say $\{g_1, \ldots, g_m\}$, for the original set of equations and then solve the new system $g_1 = 0, \ldots, g_m = 0$, we find that either the Gröbner basis cannot be computed within reasonable time or the resulting polynomial equations $g_i = 0$ cannot be solved then depending on the term ordering we use.

However, Maple's built-in function solve does the job in less than 10 seconds. For this example, we obtain three solutions. Two of them are useless, since all $\lambda_i = 0$, thus we concentrate on the third one.

```
In[26]:= SOLU[[3]]

Out[26]= {LAM[1, 0] -> 0, LAM[1, 1] -> 0, LAM[1, 2] -> 0, LAM[1, 3] -> 0,

          LAM[2, 0] -> 0, LAM[2, 1] -> -(c d P[2] SIG[1, 3]),

          LAM[2, 2] -> P[2] SIG[1, 3], LAM[2, 3] -> 0, LAM[3, 0] -> 0,

          LAM[3, 1] -> q P[2] SIG[1, 3], LAM[3, 2] -> 0, LAM[3, 3] -> 0,

          P[0] -> 0, P[1] -> P[2], P[2] -> P[2], SIG[0, 0] -> 0,

          SIG[0, 1] -> -(c d SIG[1, 3]), SIG[0, 2] -> SIG[1, 3],

          SIG[0, 3] -> 0, SIG[1, 0] -> -(c d SIG[1, 3]),

          SIG[1, 1] -> c SIG[1, 3] + d SIG[1, 3] + c d SIG[1, 3],

          SIG[1, 2] -> -SIG[1, 3] - c SIG[1, 3] - d SIG[1, 3],

          SIG[1, 3] -> SIG[1, 3]}
```

Looking at the coefficients P[i], it is easily seen that our polynomial multiplier is given by $P(k) = c\,q^k\,(1 + q^k)$ for an arbitrary constant c. Note that we could possibly find further polynomial multipliers of degree 2 by enlarging the parameter c_2 in PolyMult->{2, c_2}.

B.8 Additional Functions

The `qZeil` package comes with its own simplification procedure

`qSimplify[`*exp*`]`

for q-hypergeometric expressions *exp*. To make this highly non-trivial task powerful and efficient, our strategy is based on collecting several rewrite rules into blocks which are applied one after the other. Furthermore, note that this procedure does not factor at all.

In the context of qWZ-dualization (see Chapter 2), after a dual pair $(f'_{n,k}, g'_{n,k})$ has been computed by the program, calling

`DualId[{`k`, `a`, `b`}, `n`]`

evaluates $SUM'(n) := \sum_{k=a}^{b} f'_{n,k}$ provided that SUM' is constant, i.e., not depending on n. `DualId` uses the value of the (global) variable `DualPair`.

The function

`SameRec[`rec_1`, `rec_2`]`

checks whether the recurrences rec_1 and rec_2 are equal. This is useful and very fast for higher-order recurrences arising from transformation formulas which quite frequently yield several pages of output. In this case, `SameQ` does not necessarily give `True` for equal recurrences and `Together[` rec_1`[[2]] - `rec_2`[[2]] ]` can be rather time-consuming. Similarly, the function

`Check1[`*rec*`]`

checks whether the recurrence *rec* is satisfied by 1.

Finally,

`ToqHyper[`*rec*`]`

converts the recurrence *rec* into valid input for Petkovšek's package `qHyper` (see Abramov, Paule, and Petkovšek [**1**]) in terms of $SUM(S) := SUM(q^n)$. Since both packages `qHyper` and `qZeil` define the symbol q, one has to set the global variable `NoContext` to `True` and to reload the packages which will then reside in context `Global‘`. This is an inelegant but very useful hack allowing the simultaneous use of different packages defining the same symbols provided that the packages do not define functions with the same name and the same number of arguments. Note that the original behavior can be restored only by restarting Mathematica.

B.9 Speed-Up

With the latest version of `qZeil` we could achieve a significant speed-up mainly based on two new ideas, an improved pivot search in Aichinger's [**3**] null space algorithm and a new preprocessing of the recurrence looked for. As a consequence we can now, for instance, compute a recurrence of order 3 for the ${}_{12}\phi_{11}$ series on the right hand side of identity (III.25) in Gasper and Rahman [**20**] within less than one minute (this was the "out-of-memory" example listed in Paule and Riese [**33**]).

The new pivot search method is based on the observation that not only the size of the pivot element plays a fundamental role in Gaussian elimination but also the number of zeros in the corresponding row. Note that whenever we speak of the "size" of a rational function we actually mean its "complexity" which in the present implementation is determined with the

Mathematica function `LeafCount`. Thus we first look for the rows of the underlying matrix M for which the number of zeros is maximal. Let $R_M = \{r_1, \ldots, r_d\}$ denote the set of those rows. If R_M consists of only one element, we simply choose the smallest non-zero entry of row r_1 as our pivot element. Otherwise, it is obvious that the effectiveness of our method heavily depends also on the total size of the r_i from which the pivot element is chosen. Let s_i and t_i denote the size of the smallest non-zero entry of r_i, respectively the total size of r_i. A natural way to decide that the pivot element should be the smallest non-zero element of r_n instead of r_m is when

$$\left(\frac{s_n}{s_m}\right)^c < \frac{t_m}{t_n}$$

for some positive number c. It turns out that — at least for matrices appearing in q-hypergeometric summation — the choice $c = 3/4$ leads to highly satisfactory results. Additional evidence for our improvement of efficiency stems from the fact that this kind of pivot search has been also successfully integrated into Wegschaider's [**42**] package `MultiSum` for automatically proving binomial multi-sum identities resulting in a speed-up factor of 10 there. Moreover, the approach described above could be improved once more by taking the number of zeros in the columns into account, too.

The second idea which substantially helped to decrease the run-time, simply consists in filtering out all factors not depending on q^k in the polynomials $p_i(n,k)$ of the recurrence-"Ansatz"

$$\sum_{i=0}^{d} \sigma_i(n)\, f_{n-i,k} = f_{n,k} \cdot \frac{\sigma_0(n)\, p_0(n,k) + \sigma_1(n)\, p_1(n,k) + \cdots + \sigma_d(n)\, p_d(n,k)}{p_{d+1}(n,k)}.$$

Those constant factors only blow up the system of equations unnecessarily. It is sufficient to take them into consideration after the system of equations has been solved.

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