Herbrand Theorems: the Classical and Intuitionistic Cases

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Abstract. A unified approach is applied for the construction of sequent forms of the famous Herbrand theorem for first-order classical and intuitionistic logics without equality. The forms do not explore skolemization, have wording on deducibility, and as usual, provide a reduction of deducibility in the first-order logics to deducibility in their propositional fragments. They use the original notions of admissibility, compatibility, a Herbrand extension, and a Herbrand universe being constructed from constants, special variables, and functional symbols occurring in the signature of a formula under investigation. The ideas utilized in the research may be applied for the construction and theoretical investigations of various computer-oriented calculi for efficient logical inference search without skolemization in both classical and intuitionistic logics and provide some new technique for further development of methods for automated reasoning in non-classical logics.

1 Introduction

Herbrand's paper [1] contains a theorem called now the Herbrand theorem. This theorem permits to reduce the question of the deducibility (validity) of a formula F of first-order classical logic to the question of the deducibility (validity) of a quantifier-free ("propositional") formula F', at that, the deducibility of F' can be established by means of using only propositional "calculations". When making the reduction of F to F', a certain set of terms (so-called Herbrand universe) is constructed. Different ways of construction of Herbrand universe(s) lead to different forms of the Herbrand theorem. In particular, three forms are given in [1]: A, B, and C. The Herbrand universes for B and C are defined as minimal sets of terms containing constants and functional symbols occurring in the Skolem functional form of F, and the unique difference between B and C consists in the ways of the skolemization of F[1].

The form A does not need the skolemization of F, and the Herbrand universe for it uses only constants, functional symbols, and certain quantifier variables from F. But its application requires checking a large number of quantifier sequences constructed from "schemes" (in the terminology of [1]) in order to find at least one sequence satisfying a certain condition that guarantees the validity of the formula F.

Since intuitionistic logic does not keep the skolemization transformations in general, it is impossible to obtain the forms B and C for intuitionistic logic. Therefore, for intuitionistic logic, there can be made an attempt to construct a Herbrand theorem similar to the form A only, i.e. when preliminary skolemization is not obligatory and reduction of first-order investigations to propositional "calculations" is performed. Besides, it is desired to give such forms of Herbrand's theorem for classical and intuitionistic cases providing the clear-cut distinction between them.

This research gives a possible decision of the problem under consideration: *the unified forms of Herbrand's theorem are formulated for classical and intuitionistic logics in a sequent form*[2]. It does not explore skolemization, have wordings on deducibility, and develops the approach suggested in [5–7] for classical logic and modified in a certain way for a tableau treatment of intuitionistic logic in [8], which permits to achieve the objective just reminded on the base of the original notions of admissibility and compatibility. Note that a similar style of inference search (not requiring skolemization and not giving Herbrand theorems at once) was exploited by the author of the paper and his coauthors in a number of sequent calculi for classical logic

[1] For example, see [2] and [3] for some details relating to such a type of Herbrand's theorem.
[2] The announcement of the main results of the research was made in the slightly different form at the Kurt Goedel Centenary Symposium, Vienna, Austria, 2006 [4].

(see, for example, [9, 10]) and in the tableau method for intuitionistic logic from [11] in order to optimize an item-by-item examination arising when quantifier rules applications satisfying Gentzen's admissibility – i.e. to the eigenvariable condition [12] – are made[3].

Additionally note that our approach based on the notions of admissibility and compatibility shares some ideas with the papers [16] and [17] being exploited in the original way for the construction of various computer-oriented methods for classical and non-classical logics such as matrix characterization methods [17], different modifications of the connection method (see, for example, [16], [18], [19], [20]), and ordinal sequent and tableau methods (see, for example, [21], [22]). But all these papers do not contain any direct instructions how to construct both classical and intuitionistic forms (not requiring skolemization) of Herbrand's theorem.

2 Preliminaries

We use standard terminology of first-order sequent logic without equality. The basic signature Sig_0 of the first-order language consists of a (possibly empty) set of functional symbols (including constants), a (non-empty) set of predicate symbols, and logical connectives: the quantifier symbols for the universal character $\forall$ and for existential character $\exists$ as well as the propositional symbols for the implication ($\supset$), disjunction ($\vee$), conjunction ($\wedge$), and negation ($\neg$). At that, $\forall x$ and $\exists x$ are called quantifiers; they are considered as a single whole. A countable set of variables is denoted by Var.

The notions of terms, atomic formulas, literals, formulas, sequents, free and bound variables, are defined in the usual way [23] and assumed to be known to the reader.

As usual, we assume that no two quantifiers in any formula or in any sequent have a common variable, which can be achieved by renaming bound variables.

Without loss of generality, an *initial sequent* (i.e. a sequent being investigated on deducibility) always is considered to have the form $\rightarrow F$, where F is a *closed formula*.

If the *principal connective* of a formula F is $\odot$ (i.e. F has the form $F'\odot F''$ or $\odot F'$, where $\odot$ is $\supset$, $\vee$, $\wedge$, $\neg$, $\forall$, or $\exists$), then F is called $\odot$*-formula*.

As in [2], we say that an occurrence of a subformula F in a formula G is

- *positive* if F is G;
- *positive* (*negative*) if G is of the form: $G_1 \wedge G_2$, $G_2 \wedge G_1$, $G_1 \vee G_2$, $G_2 \vee G_1$, $G_2 \supset G_1$, $\forall x G_1$, or $\exists x G_1$ and F is positive (negative) in G_1;
- *negative* (*positive*) if G is of the form $G_1 \supset G_2$ or $\neg G_1$ and F is *positive (negative)* in G_1.

Further, a formula F has a *positive (negative)* occurrence in a sequent $\Gamma \rightarrow \Delta$ if F has a positive occurrence in a formula from Δ (from Γ) or if F has a negative occurrence in a formula from Γ (from Δ). Moreover, if F has the form $\forall x F'$ ($\exists x F'$) and F has a positive (negative) occurrence in a formula G or in a sequent S, then $\forall x$ ($\exists x$) is called a *positive quantifier* in G or in S, respectively; $\exists x$ ($\forall x$) is called a *negative quantifier* in G or in S, if $\exists x F'$ ($\forall x F'$) has a positive (negative) occurrence in G or in S, respectively.

In what follows, the variable of a positive quantifier occurring in a formula G or in a sequent S is called a *parameter* in G or in S, respectively; the variable of a negative quantifier occurring in a formula G or in a sequent S is called a *dummy* in G or in S, respectively.

Remark. The terms "parameters" and "dummies" are taken from [13], where they are used in the analogous sense.

The way of the extension of the notions of dummies and parameters to sequents and sets of formulas or of sequents is obvious.

Since the property "to be a dummy" ("to be a parameter") is invariant w.r.t. logical rules applications in sequent calculi, any parameter (dummy) x in a formula (in a sequent, in a set of formulas or of sequents) is convenient to be written as $\overline{x}$ ($\underline{x}$).

For a formula F (for a sequent S), $\mu(F)$ ($\mu(S)$) denotes the result of the elimination of all the quantifiers from F (from S).

[3] S. Kanger introduced his definition of admissibility [13] which has an advantage over Gentzen's one; its modified forms were used in a number of papers concerning inference search in classical logic (see, for example, [14]) and in intuitionistic logic (see, for example, [15]).

If F (S) is a formula (a sequent) and x is its parameter or its dummy then x considered to be a parameter or a dummy in $\mu(F)$ (in $\mu(S)$).

In what follows, we suppose a reader to know all the notions relating to deducibility in Gentzen (sequent) calculi LK and LJ. Draw you attention to the fact that all the *inference trees* in calculi under consideration are understood in the usual sense and grow "from top to bottom" by applying inference rules to an input sequent and afterwards to its "heirs", and so on. Additionally remind that any inference tree having only leaves with axioms is called a *proof tree.*

3 Admissibility and compatibility

Let F be a formula. By (i, F), we denote the i-th occurrence of its subformula if F is read from left to right. We write $(i, F) \sqsubseteq_F (j, F)$ if, and only if (j, F) is a subformula of (i, F). Obviously, the relation $\sqsubseteq_F$ is a partial order.

If (i, F) is the occurrence of a $\odot$-formula, where $\odot$ is a logical connective (a propositional connective or a quantifier), we also refer to this occurrence as to ${}_i\odot$-occurrence in F.

If a formula F has ${}_i\odot$- and ${}_j\odot$-occurrences of its subformulas ($i \neq j$) and ${}_i\odot$-occurrence $\sqsubseteq_F$ ${}_j\odot$-occurrence, then ${}_j\odot'$ is said to be in the *scope* of ${}_i\odot$; this fact is denoted by ${}_i\odot \lessdot_F {}_j\odot'$. If $\odot$ ($\odot'$) is $\forall x$ or $\exists x$, we always write ${}_ix \lessdot_F {}_j\odot'$ (${}_i\odot \lessdot_F {}_jx$); at that, when $\odot'$ ($\odot$) is $\forall y$ or $\exists y$, we write ${}_ix \prec_F {}_jy$ (${}_iy \prec_F {}_jx$) underlining the fact that $\lessdot_F$ is restricted only in the case of the consideration of quantifiers variables. Moreover, any occurrence ${}_i\odot$ of a symbol $\odot$ in a formula F is treated as a new symbol. Therefore, ${}_i\odot$ and ${}_j\odot$ ($i \neq j$) are different symbols denoting the same logical "operation" $\odot$. That is why ${}_i\odot$- and ${}_j\odot$-occurrences in F can be considered as different subformulas of F ($i \neq j$), if needed.

Obviously, for any formula F, $\prec_F \subset \lessdot_F$ and the relations $\prec_F$ and $\lessdot_F$ are irreflexive and transitive.

The *extensions of* $\prec_F$ and *of* $\lessdot_F$ to the case of a sequent S ($\prec_S$ and $\lessdot_S$) are obvious. The same relates to the case of a set Ξ of formulas or of sequents ($\prec_\Xi$ and $\lessdot_\Xi$).

All the above-given extensions do not lead to confusion since we already begin all our investigations with a closed formula without common bound variables or with a sequent containing only such formulas.

A *substitution*, σ, is a finite mapping from variables to terms denoted by $\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, where variables $x_1, \ldots, x_n$ are pairwise different and x_i is distinguished from t_i for all $i = 1 \ldots n$. For an expression Ex and a substitution σ, the result of the application of σ to the expression of Ex is understood in the usual sense; it is denoted by $Ex \cdot \sigma$.

For any set Ξ of expressions, $\Xi \cdot \sigma$ denotes the set obtained by the application of σ to every expression in Ξ. If Ξ is a set of (at least two) expressions and $\Xi \cdot \sigma$ is a singleton, then σ is called a *unifier* of Ξ. The notion of a *most general simultaneous unifier (mgsu)* of a set of expressions also is understood in the usual sense.

For any formula F (for any sequent S, for any set Ξ of formulas or sequents), each substitution σ induces a (possibly empty) relation $\ll_{F,\sigma}$ ($\ll_{S,\sigma}, \ll_{\Xi,\sigma}$) as follows: $y \ll_{F,\sigma} x$ ($y \ll_{S,\sigma} x$, $y \ll_{\Xi,\sigma} x$) if, and only if, there exists $x \mapsto t \in \sigma$ such that x is a dummy in F (S, Ξ), the term t contains y, and y is a parameter in F (S, Ξ). Obviously, $\ll_{F,\sigma}$ ($\ll_{S,\sigma}$, $\ll_{\Xi,\sigma}$) is an irreflexive relation.

In what follows, for a substitution σ and for a formula F (for a sequent S, for a set Ξ of expressions), $\lhd_{F,\sigma}$ ($\lhd_{S,\sigma}$, $\lhd_{\Xi,\sigma}$) denotes the transitive closure of $\prec_F \cup \ll_{\{F\},\sigma}$ (of $\prec_S \cup \ll_{S,\sigma}$, of $\prec_\Xi \cup \ll_{\Xi,\sigma}$). At that time, $\blacktriangleleft_{F,\sigma}$ ($\blacktriangleleft_{S,\sigma}$, $\blacktriangleleft_{\Xi,\sigma}$) denotes the transitive closure of $\lessdot_F \cup \ll_{\{F\},\sigma}$ (of $\lessdot_S \cup \ll_{S,\sigma}$, of $\lessdot_\Xi \cup \ll_{\Xi,\sigma}$).

A substitution σ is *admissible* (cf. [17]) for a formula F (for a sequent S, for a set Ξ of expressions) if, and only if, for every $x \mapsto t \in \sigma$, x is a dummy in F (in S, in Ξ), and $\lhd_{F,\sigma}$ ($\lhd_{S,\sigma}$, $\lhd_{\Xi,\sigma}$) is an irreflexive relation.

Obviously, $\lhd_{F,\sigma} \subseteq \blacktriangleleft_{F,\sigma}$ ($\lhd_{S,\sigma} \subseteq \blacktriangleleft_{S,\sigma}$, $\lhd_{\Xi,\sigma} \subseteq \blacktriangleleft_{\Xi,\sigma}$). Therefore, the following facts hold on the base of the definitions.

Proposition 1. *The relation $\blacktriangleleft_{F,\sigma}$ as well as $\blacktriangleleft_{S,\sigma}$ and $\blacktriangleleft_{\Xi,\sigma}$ are irreflexive (antisymmetric) if, and only if, $\lhd_{F,\sigma}$ as well as and $\lhd_{S,\sigma}$ and $\lhd_{\Xi,\sigma}$ are an irreflexive (antisymmetric) relations. Moreover, the irreflexivity of $\blacktriangleleft_{F,\sigma}$ (of $\blacktriangleleft_{S,\sigma}$, of $\blacktriangleleft_{\Xi,\sigma}$) implies the antisymmetry of $\blacktriangleleft_{F,\sigma}$ (of $\blacktriangleleft_{S,\sigma}$, of $\blacktriangleleft_{\Xi,\sigma}$) and vise versa.*

This proposition permits the investigation of the irreflexivity (or the antisymmetry) of $\lhd$ to replace by the investigation of the irreflexivity (or the antisymmetry) of $\blacktriangleleft$ and vice versa.

Let F be a formula and ${}_{j_1}\odot_1, \ldots, {}_{j_r}\odot_r$ a sequence of all its logical connectives occurrences. Let Tr_F be an inference tree for the initial sequent $\rightarrow F$ such that if $\alpha_{Tr_F}({}_{j_1 i}\odot_1)$ denotes an inference rule application eliminating the occurrence ${}_{j_1}\odot_1$ in F then Tr can be constructed in accordance with the order determined by the sequence $\alpha_{Tr_F}({}_{j_1}\odot_1), \ldots, \alpha_{Tr_F}({}_{j_r}\odot_r)$. In this case, ${}_{j_1}\odot_1, \ldots, {}_{j_r}\odot_r$ is called a *proper sequence* for Tr_F. (It is obvious that there may exist a connectives occurrences sequence for a formula F such that the sequence is not proper for any Tr_F. Besides, it must be clear that there may exist more than one proper sequence for an inference tree Tr_F in the case of the existence of one for F.)

Let F be a formula and Tr_F an inference tree for the initial sequent $\rightarrow F$. The tree Tr_F is called *compatible* with a substitution σ if, and only if, there exists a proper sequence ${}_{j_1}\odot_1, \ldots, {}_{j_r}\odot_r$ for Tr_F such that for any natural numbers m and n, the property $m < n$ implies that the ordered pair $\langle {}_{j_n}\odot_n, {}_{j_m}\odot_m \rangle$ does not belong to $\blacktriangleleft_{F,\sigma}$.

The results of the next section demonstrate the importance of the notion of compatibility for the intuitionistic case, while it is redundant for classical one as a whole and must be "transformed" into the notion of admissibility.

4 Herbrand theorems

This section contains the main results of the paper, which condense the investigations presented in [7, 11, 8] in a unified form. Additionally note that without loss of generality, we are interested in establishing the deducibility of an *initial sequent* of the form $\rightarrow F$, where F is a closed formula.

Let F be a formula and $F_1, \ldots, F_n$ its variants. If $F_1, \ldots, F_n$ does not have any bound variables in pairs, then $F_1 \wedge \ldots \wedge F_n$ ($F_1 \vee \ldots \vee F_n$) is called a *variant* $\wedge$*-duplication* (a *variant* $\vee$*-duplication*).

Herbrand extension. Let G be a formula, F its subformula, and H a variant $\wedge$-duplication ($\vee$-duplication) of F not having common variables with G. Then the result of the replacement of F by H in G is called a *one-step Herbrand extension of* G. Further, the result $HE(G)$ of a finite sequence of one-step extensions consequently applied to G, then to a one-step Herbrand extension of G, and so on is called a *Herbrand extension* of G. If $HE(G)$ is generated by means of only $\wedge$-extensions, H is called an *intuitionistic Herbrand extension* of G.

Herbrand quasi-universe. Let F be a formula. Then $HQ(F)$ denotes the following minimal set of terms (called a *Herbrand quasi-universe*): (i) every constant and every parameter occurring in F belong to $HQ(F)$ (if there is no constant in F then the special constant $c_0 \in HQ(F)$); (ii) if f is a k-ary functional symbol and terms $t_1, \ldots, t_k \in HQ(F)$ then $f(t_1, \ldots, t_k) \in HQ(F)$.

In other words, $HQ(F)$ can be considered as a minimal set of terms constructing from constants and parameters occurring in F with the help of functional symbols of F with arities more that 0.

In what follows, pLK and pLJ denote the propositional parts of LK and LJ, respectively, which means that pLK and pLJ do not contain quantifier rules, as well as $(Con \rightarrow)$ and $(\rightarrow Con)$ (see the next section) when antecedents and succedents of sequents are identified with multisets.

Theorem 1. *(Sequent form of Herbrand's theorem for classical logic.) For a formula F, the sequent $\rightarrow F$ is deducible in the calculus LK (in any sequent calculus coextensive with LK) if, and only if, there are an Herbrand extension $HE(F)$ and a substitution σ of terms from the Herbrand quasi-universe $HQ(F)$ for all the dummies of $HE(F)$ such that*

(i) there exists a proof tree $Tr_{\mu(HE(F))\cdot\sigma}$ for $\rightarrow \mu(HE(F)) \cdot \sigma$ in pLK and

(ii) σ is an admissible substitution for $HE(F)$.

For intuitionistic logic, Theorem 1 transforms to the following form.

Theorem 2. *(Sequent form of Herbrand's theorem for intuitionistic logic.) For a formula F, the sequent $\rightarrow F$ is deducible in the calculus LJ (in any sequent calculus coextensive with LJ) if, and only if, there are an intuitionistic Herbrand extension $HE(F)$ and a substitution σ of terms from the Herbrand quasi-universe $HQ(HE(F))$ for all the dummies of $HE(F)$ such that*

(i) there exists a proof tree $Tr_{\mu(HE(F))\cdot\sigma}$ for $\rightarrow \mu(HE(F)) \cdot \sigma$ in pLJ,

(ii) σ is an admissible substitution for $HE(F)$, and

(iii) $Tr_{\mu(HE(F))\cdot\sigma}$ is compatible with σ.

Draw your attention to the fact that Theorems 1 and 2 are distinguished by only the existence of (iii) in Theorem 2 (and by the calculi LK and LJ). The requirement (iii) is essential for intuitionistic logic. It is easy to check this fact with the help of the following simple examples.

Example 1. Let we have the sequent S: $\rightarrow F$, where F is $\neg\forall x P(x) \supset \exists y \neg P(y)$ ($\rightarrow F$ is deducible in LK and is not deducible in LJ). Obviously, for any intuitionistic Herbrand extension $HE(F)$, $\mu(HE(F))$ has the form $\neg(P(\overline{x}_{1,1}) \wedge \ldots \wedge P(\overline{x}_{1,p_1})) \wedge \ldots \wedge \neg(P(\overline{x}_{r,1}) \wedge \ldots \wedge P(\overline{x}_{r,p_r})) \supset \neg P(y)$ and Herbrand quasi-universe for it is equal to $\{c_0, \overline{x}_{1,1}, \ldots, \overline{x}_{r,p_r}\}$.

For $\rightarrow \mu(HE(F))$, any substitution $\sigma_{i,j} = \{y \mapsto \overline{x}_{i,j}\}$, where i and j are any natural numbers not exceeding r and p_r respectively, leads to a possibility to construct a proof tree $Tr_{i,j}$ for the selected extension $\mu(HE(F))$. (Obviously, the substitution $\{y \mapsto c_0\}$ does not have such a property.) It is easy to check the admissibility of $\sigma_{i,j}$ for $HE(F)$ and the absence of compatibility of $Tr_{i,j}$ with $\sigma_{i,j}$ regardless of the selection of $Tr_{i,j}$ and $\sigma_{i,j}$. As result, we have the deducibility of S in LK by Theorem 1 and the non-deducibility of S in LJ by Theorem 2. (When constructing any proof tree for S in LJ, any relation $\blacktriangleleft_{HE(F),\sigma_{i,j}}$ requires the application of the rule eliminating the first negation of F on the second step of deducing the proof tree, which is impossible to do in LJ for S.)

Example 2. If we slightly modify Example 1, taking $\rightarrow \exists x \neg P(x) \supset \neg\forall y P(y)$ as S, we have for S: $HQ(S) = \{c_0, \overline{x}\}$ and the substitution $\{y \mapsto \overline{x}\}$ is admissible for S. In this case, any proof tree for $\rightarrow \neg P(\overline{x}) \supset \neg P(\overline{x})$ is compatible with $\{y \mapsto \overline{x}\}$. Thus, S is deducible in LJ (and, of course, in LK).

Example 3. If we take $\rightarrow \forall y \exists x P'(y,x) \supset \exists y_1 \forall x_1 P'(x_1, y_1)$ as a sequent S, we have: $HQ(S) = \{x, x_1\}$ and for the substitution $\sigma = \{y \mapsto x_1, y_1 \mapsto x\}$, the sequent $\rightarrow P'(x_1, x) \supset P'(x_1, x)$ is deducible in pLK. Unfortunately, σ is not admissible for S and we cannot say anything about the deducibility of S even in LK. But it is easy to show that the construction of any Herbrand extension of $\forall y \exists x P'(y,x) \supset \exists y_1 \forall x_1 P'(x_1, y_1)$ cannot lead to an admissible substitution for any Herbrand extension and any its proof tree. Therefore, S is not deducible neither in LK nor in LJ.

As you can see the above-given examples demonstrate that in comparison with Theorem 1, the grave disadvantage of Theorem 2 consists in the existence of the condition (iii) requiring a certain form of a proof tree for a sequent $\mu(\rightarrow F)\cdot\sigma$ in pLJ (or in its any analogue coextensive with pLJ): it must be *compatible* with σ, which does not permit any order of propositional rules applications leading to Tr while, in the classical case, any proof tree Tr for a sequent $\mu(\rightarrow F)\cdot\sigma$ in pLK admits any order of propositional rules applications leading to Tr.

Finally, note that since LK and LJ are sound and complete calculi, the obtained results permit to reduce the investigation the semantic characterization of classical and/or intuitionistic validity (of first-order formulas) to propositional deducibility satisfying certain conditions. Additionally, pay your attention to the fact that Theorem 1 can easily be transformed into some of sequent forms of Herbrand's theorem given in [7] for classical logic.

5 Conclusion

This paper presents the author's results on Herbrand-type theorems for the sequent treatment of first-order classical and intuitionistic logics. The sequent formalism under consideration gave a possibility to develop the unified approach to wording and proving the Herbrand forms suggested here. Besides, it also allowed us to achieve enough general considerations: many famous variants of the Herbrand theorem for classical logic known to the authors can be produced as its application. Additionally, note that obtained theorems wording has a transparent character and is connected with deducibility only. This feature and unified approach to the Herbrand theorems may be used as a theoretical basis for the construction of computer-oriented methods for enough efficient inference search in classical and intuitionistic logics. Such examples can be found in [7] for the classical case and in [8] for the intuitionistic one.

In this connection, it is interest to note than the certain calculi used in [7, 8] for the obtaining the Herbrand-type theorems can serve as examples of enough efficient sequent calculi for their computer implementation.

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