On the Propositional Realizability

Valery Plisko
Moscow State University

INTAS Meeting
Kiyv, May 28-30, 2008

Informal intuitionistic semantics: a sentence is tru
verification. If A and B are sentences, then

a verification of $A \& B$ is a text containing a verificat
a verification of B;

a verification of $A \vee B$ is a text containing a verifica
a verification of B and indicating what of them is v

a verification of $A \to B$ is a text describing a gene
operation for obtaining a verification of B from every
of A;

a verification of $\neg A$ is a verification of the sentence A
$\perp$ is a certainly absurd sentence having no verificati

If $A(x)$ is a predicate with a parameter x over a dom
in appropriate way, then

- a verification of $\forall x A(x)$ is a text describing a gen
  operation which allows to obtain a verification
  every given $m \in M$;

- a verification of sentence $\exists x A(x)$ is a text indic
  crete $m \in M$ and containing a verification of t
  $A(m)$.

Intuitionistic propositional calculus

A propositional formula $A(p_1, \ldots, p_n)$ is called intu
valid if it is a scheme of intuitionistically true sent
first axiomatization of intuitionistic propositional lo
posed by Kolmogorov in 1925. Other more wide ax
of intuitionistic logic were proposed by Glivenko, H
Gentzen. They are all equivalent: the same form
ducible from each of them. Intuitionistic propositio
is denoted by IPC.

The problem of completeness of the calculus IPC
stated in a precise mathematical form because intu
mantics is very informal. It can be made more prec
fine in a mathematical mode two key notions used
description of the informal semantics, namely the
verification and a general effective operation.

Kleene's recursive realizability

In 1945 S. C. Kleene proposed a variant of intuition
tics based on interpreting general effective operati
rithms. He introduced the notion of *recursive rea*
the first-order arithmetic sentences. The main ide
was to consider natural numbers as the codes of
and partial recursive functions as effective operati
recursive functions are coded by natural numbers
the Gödel enumeration. The unary function whos
will be denoted by $\{e\}$. A code of a verification of a
called *a realization* of the sentence.

The relation $e\,\mathbf{r}\,\Phi$ (a natural number e realizes a close
formula Φ) is defined inductively.

If Φ is an atomic sentence $t_1 = t_2$, then $e \, \mathbf{r} \, \Phi \rightleftharpoons [e =$
true].

$e \, \mathbf{r} \, (\Phi \,\&\, \Psi) \rightleftharpoons [e = 2^a \cdot 3^b$ and $a \, \mathbf{r} \, \Phi, b \, \mathbf{r} \, \Psi]$.

$e \, \mathbf{r} \, (\Phi \vee \Psi) \rightleftharpoons [e = 2^0 \cdot 3^a$ and $a \, \mathbf{r} \, \Phi$ or $e = 2^1 \cdot 3^b$ and

$e \, \mathbf{r} \, (\Phi \rightarrow \Psi) \rightleftharpoons$ [for any a, if $a \, \mathbf{r} \, \Phi$, then $\{e\}(a) \, \mathbf{r} \, \Psi$].

$e \, \mathbf{r} \, \neg\Phi \rightleftharpoons [e \, \mathbf{r} \, (\Phi \rightarrow 0 = 1)]$.

$e \, \mathbf{r} \, \forall x \Phi(x) \rightleftharpoons [\{e\}(n) \, \mathbf{r} \, \Phi(n)$ for every $n]$.

$e \, \mathbf{r} \, \exists x \Phi(x) \rightleftharpoons [e = 2^n \cdot 3^a$ and $a \, \mathbf{r} \, \Phi(n)]$.

A propositional formula $A(p_1, \ldots, p_n)$ is

1. *weakly realizable* if every its closed arithmetic realizable;

2. *irrefutable* if every its arithmetic instance is real

3. *effectively realizable* if there exists an algorithm realization of any closed arithmetic instance;

4. *uniformly realizable* if there exists a natural num every closed arithmetic instance.

Problem: Are the notions 2, 3, and 4 equivalent?

The corresponding notions of realizability for predic
are defined in a natural way.

- *There exists a weakly realizable predicate form not irrefutable.*

- *There exists an irrefutable predicate formula w effectively realizable.*

- *There exists an effectively realizable predicate fo is not uniformly realizable.*

- *The set of realizable (in any sense) predicate for arithmetical.*

Consider a language L obtained by adding an una
symbol T to the language of arithmetic. We genera
tion of realizability to this extended language by lettin
$e\,\mathbf{r}\,\Phi_n$, where Φ_n is an arithmetic sentence with the
ber n. It is proved that there are uniformly realizab
formulas which are refutable in the language L. Th
cept of a realizable predicate formula depends on t
in which we formulate the predicates admissible for
predicate variables.

Constructive meaning of a sentence can be identif
set of (the Gödel numbers of) the objects verifying th
Thus we come to the idea of interpreting propositio
as arbitrary sets of naturals, the logical operations b
according to recursive realizability:

- $A \mathbin{\&} B \leftrightharpoons \{2^a \cdot 3^b | a \in A,\, b \in B\}$;

- $A \vee B \leftrightharpoons \{2^0 \cdot 3^a | a \in A\} \cup \{2^1 \cdot 3^b | b \in b\}$;

- $A \to B \leftrightharpoons \{x | \forall a (a \in A \Rightarrow \{x\}(a) \in B)\}$;

- $\neg A \leftrightharpoons A \to \emptyset$.

A k-place generalized predicate is a function defined o
values are sentences (sets of naturals), i.e., a func
$\mathbf{N}^k \to 2^{\mathbf{N}}$. Analogs of irrefutable and uniformly rea
icate formulas are defined in a natural way. The
absolutely irrefutable and *absolutely uniformly realiz*

Theorem 1 *A closed predicate formula is absolute*
if and only if it is absolutely uniformly realizable.

Absolutely irrefutable (absolutely uniformly realizab
formulas are called *absolutely realizable*.

Thus in the context of the absolute realizability the
coincidence of various notions of realizability for p
formulas is solved positively.

Problem: Is every irrefutable propositional formul
realizable?

Let $\mathcal{P}$ be the set of one-place generalized predicate

Define $P \leq Q$ iff there exists a two-place general rec
tion f such that

$$\forall n, x\, (x \in P(n) \Leftrightarrow f(n, x) \in Q(n)).$$

$\leq$ is a preorder. Let $\sim$ be an equivalence induced
$\mathcal{P}/\sim$ is a Heyting algebra. This algebra is an exact
propositional logic of the absolute realizability.

Problem: Is there an arithmetic Heyting algebra $\forall$-e
the algebra $\mathcal{P}/\sim$?

F. L. Varpakhovskii introduced two additional propo
nectives called *strong implication* and *conditional d*

Strong implication is denoted by $\Phi \Rightarrow \Psi$.

If θ is a list of arithmetic formulas $\Phi_1, \ldots, \Phi_m$, θ_i $(i =$
its sublists $\Phi_{i,1}, \ldots, \Phi_{i,m_i}$, and $\Psi_1, \ldots, \Psi_n$ are arithm
las, then $(\theta(\theta_1\Psi_1\nabla \ldots \nabla\theta_n\Psi_n))$ is called conditiona
of the formulas $\Psi_1, \ldots, \Psi_n$ with the conditions θ, θ_1

The notion of recursive realizability is generalized
connectives in the following way.

$e\,\mathbf{r}\,(\Phi \Rightarrow \Psi)$ iff for any a such that $a\,\mathbf{r}\,\Phi$ the value $\{e\}($
and for any a, if $\{e\}(a)$ is defined, then $\{e\}(a)\,\mathbf{r}\,\Psi$.

Let $\Phi_1, \ldots, \Phi_m, \Psi_1, \ldots, \Psi_n$ be closed formulas. Then
iff

1) e is of the form $\prod_{i=0}^{n} \pi_i^{e_i}$ (π_i is the ith prime numb

2) for any sequence $\bar{a} = a_1, \ldots, a_m$ there exists $i \in \{1$
that if $a_j \, \mathbf{r} \, \Phi_j$ for any $j = 1, \ldots, m$, then $\{e_0\}(\bar{a}) = i$ a
$\{e_i\}(a_{i,1}, \ldots, a_{i,m_i})$ is defined,

3) for any $i \in \{1, \ldots, n\}$ and any $\bar{a} = a_1, \ldots, a_m$, if
any $j = 1, \ldots, m_i$, and the value $\{e_i\}(a_{i,1}, \ldots, a_{i,m_i}$
then

$$\{e_i\}(a_{i,1}, \ldots, a_{i,m_i}) \, \mathbf{r} \, \Psi_i.$$

The notion of an uniformly realizable propositional fo
extended language is defined in an obvious way. V
proposed a propositional calculus in the extended la
proved that any deducible formula is uniformly reali
over, all the known realizable propositional formu
ducible. Varpakhovskii observes that his calculus g
form formalization of the principles used in the pr
ability of propositional formulas. The problem of c
of Varpakhovskii's calculus is still open.

For an axiomatic theory T let $\mathcal{PL}$(T) be the set of p
formulas such that all their arithmetic instances a
in T. Let S be HA with additional axioms $\Phi \equiv \exists x$
arithmetic formula Φ and the Markov Principle

MP: $\forall x(\Phi(x) \vee \neg\Phi(x)) \,\&\, \neg\neg\exists x\Phi(x) \rightarrow \exists x\Phi(x)$

It was proved that all the known realizable propositio
are in the logic $\mathcal{PL}$(S).

Extended Church's Thesis is the scheme

ECT: $\forall x(\Psi(x) \to \exists y \Phi(x,y)) \to$

$$\to \exists e \forall x(\Psi(x) \to \exists y(\{e\}(x) = y \,\&\, \Phi(x,y))$$

where $\Psi(x)$ is an almost negative formula.

The system S is equivalent to the system of "Russia
tivism HA+MP+ECT.

A. Visser called this system Markov's Arithmetic an
by MA. Thus any known realizable propositional forn
logic $\mathcal{PL}$(MA).

Theorem 2 *Every propositional formula deducible ir calculus is in the logic* $\mathcal{PL}$(MA)*.*