Symbolic and Algebraic Methods for LPDOs (DIFFOP)

The solution of Partial Differential Equations (PDEs) is one of the most important problems of mathematics, and has an enormous area of applications. As is the case for many other types of mathematical problems, solution methods for PDEs can be classified into symbolic (or analytical) and numerical methods. Of course, an analytical solution is to be preferred. Indeed, using an analytical solution, one can compute a numerical solution to any precision and on any segment of the domain, analyze the solution's behavior at infinity and at extremal points, explore dependence on parameters, etc. Whereas some simple Ordinary Differential Equations (ODEs) can be solved analytically, this happens more and more rarely as the complexity of the equations increases. One of the methods for extending the range of analytically solvable PDEs consists in transformations of PDEs and the corresponding transformations of their solutions. Thus, based on the fact that a second-order equation can be solved if one of its factorizations is known, the famous method of Laplace Transformations suggests a certain sequence of transformations of a given equation. Then, if at a certain step in this transformation process an equation becomes factorizable, an analytical solution of this transformed equation — and then of the initial one — can be found. Nowadays the search for analytical solutions of PDEs can benefit tremendously from the use of modern software systems in computer algebra and symbolic computation.

The aim of this project is the further development and generalization of analytical approaches to the solution of PDEs and the corresponding algebraic theory of differential operators. In our previous work we have introduced the notion of *obstacle* for the factorization of a differential operator, i.e. conditions preventing a given operator from being factorizable. These obstacles give rise to a *ring of obstacles* and furthermore to a classification of operators w.r.t. to their factorization properties. From obstacles we can also get (Laplace) invariants of operators w.r.t. to certain (gauge) transformations. We have shown how such systems of invariants can be extended to full systems of invariants for certain low order operators. Another related problem is the description of the structure of families of factorizations. For operators of order 3 it has been shown that a family of factorizations depends on at most 3 or 2 parameters, each of these parameters being a function on one variable.

In this project we plan to generalize the idea of obstacles to the case of systems of PDEs, and study their properties. At least for the case of linear partial differential operators (LPDOs) with coprime factors of the symbol, this seems to be achievable. For partial operators of order 4 the description of the structure of corresponding families of factorizations remains open. Generalizations to LPDOs with arbitrary symbols (without the complete factorization assumption), to high order LPDOs, and to those in multiple-dimensional space are of interest also. We will also work on methods for the determination of invariants for operators of order 3 or more. This should make it possible to classify operators in terms of full systems of invariants. The classical Laplace method for transforming PDEs of order 2 has been generalized in various ways: to nonlinear PDEs, to PDEs of higher order, etc. We plan to continue this work, which has obvious important applications for the analytical solution of PDEs. The theoretical results achieved in this project will be implemented in a symbolic computation program package.