1. THREE CLASSICAL RESULTS

We start out with three results due to the english mathematician John Wallis (1616–1703) evolving around π.

THM 1 (Wallis' integral) For $m \in \mathbb{N} = \{0,1,2,\dots\}$ define

$$A(m) := \int_0^{\pi/2} \sin^m \theta \, d\theta.$$

Then

$$A(m) = \begin{cases} \pi/2 \; 2^{-2n}\binom{2n}{n}, & m = 2n, \\ 2^{2n}/_{2n+1} \binom{2n}{n}^{-1}, & m = 2n+1. \end{cases}$$

Convention!

Proof (integration by parts):

$$A(m+2) = \int_0^{\pi/2} \underbrace{\sin^{m+1}\theta}_{u} \; \underbrace{\sin\theta}_{v'} \, d\theta \qquad (m \geq 0)$$

$$= \Big[\underbrace{-\sin^{m+1}\theta \cos\theta}_{\theta=0:\ 0,\ \ \theta=\pi/2:\ 0}\Big]_0^{\pi/2} + (m+1)\int_0^{\pi/2} \sin^m\theta \, \underbrace{\cos^2\theta}_{1-\sin^2\theta} \, d\theta$$

$$\Rightarrow A(m+2) = (m+1)A(m) - (m+1)A(m+2) \qquad (m \geq 0)$$

$$\Rightarrow A(m+2) = \frac{m+1}{m+2} A(m), \quad (m \geq 0)$$

↳ Initial values: $A(0) = \int_0^{\pi/2} d\theta = \pi/2$

$$A(1) = \int_0^{\pi/2} \sin\theta \, d\theta = [-\cos\theta]_0^{\pi/2} = 1$$

□

An immediate consequence of Wallis' integral formula is an inequality (coming up next) that in turn ~~##~~ leads ~~##~~ to a product formula for computing π.

Note that for $\boxed{\theta \in [0, \frac{\pi}{2}]}$, one has

$$0 \leq \sin^{2n+2}\theta \leq \sin^{2n+1}\theta \leq \sin^{2n}\theta \,, \quad n \geq 0$$

Integrate over $\theta \in [0, \frac{\pi}{2}]$:

$$A(2n+2) \leq A(2n+1) \leq A(2n) \qquad (n \geq 0)$$

$(: A(2n))$ $\Rightarrow$

$$\frac{A(2n+2)}{A(2n)} \leq \frac{A(2n+1)}{A(2n)} \leq 1 \qquad (n \geq 0)$$

$$\boxed{\frac{2n+1}{2n+2} \leq \frac{2^{4n}}{2n+1} \frac{2}{\pi} \binom{2n}{n}^{-2} \leq 1}$$

By these routine calculations we have established

THM2 (Wallis' inequality) Let $\boxed{c(n) = \frac{2^{4n}}{2n+1}\binom{2n}{n}^{-2}}$.

Then

$$\frac{2n+1}{2n+2} \leq c(n) \Big/ \frac{\pi}{2} \leq 1$$

Since $\boxed{\frac{2n+1}{2n+2} \nearrow 1}$ for $n \to \infty$ it follows right away that the sequence $c(n)$ converges towards $\pi/2$.

Furthermore, because $\boxed{\frac{c(n+1)}{c(n)} = \frac{2n+2}{2n+1}\,\frac{2n+2}{2n+3}}$

this yields the well-known product formula

follows also from product formula for $\sin x$

$$\boxed{\textbf{THM 3 (Wallis' product)} \quad \frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\cdot\frac{4}{3}\frac{4}{5}\cdot\frac{6}{5}\frac{6}{7}\cdot\ldots = \prod_{n\geq 1}\frac{2n}{2n-1}\,\frac{2n}{2n+1}}$$

The rate of convergence for this product formula, however, is very slow:

$$2\,c(1000) \approx 3.14081,$$

which is still far away from the best approximation most people know by heart, i.e., $\pi \simeq 3.141592\,(654)$ (truncation)

2. THE ARITHMETIC-GEOMETRIC MEAN

Let $x, y \in \mathbb{R}$: $\quad (x-y)^2 \geq 0 \iff \frac{1}{2}(x^2+y^2) \geq xy$

So, for $x, y > 0$: $\quad \boxed{\sqrt{xy} \leq \frac{1}{2}(x+y)}$

which is the arithmetic-geometric mean inequality in $\mathbb{R}^2$.

Now let $a, b > 0$ and define the sequences of arithmetic and geometric means $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ as follows:

$$\left|\begin{array}{l} a_0 = a \\ a_{n+1} = \frac{1}{2}(a_n + b_n) \end{array}\right. \qquad \left|\begin{array}{l} b_0 = b \\ b_{n+1} = \sqrt{a_n b_n} \end{array}\right. \qquad (n \geq 0)$$

The AGM was first discovered by Joseph-Louis Lagrange (1736-1813) and independently later rediscovered by Carl Friedrich Gauss (1777-1855) at the age of 14.

Ex (Gauss):

$$
\begin{aligned}
a_0 &= a = 1 & b_0 &= b = 0.8 \\
a_1 &= 0.9 & b_1 &= 0.894427\ldots \\
a_2 &= 0.89721359\ldots & b_2 &= 0.897209\ldots \\
a_3 &= 0.89721432\ldots & b_3 &= 0.89721432\ldots
\end{aligned}
$$

(for more digits see e.g. the survey paper by Almkvist + Berndt

Observations:

- a_n monotonically decreasing ✓
- b_n monotonically increasing ✓
- both sequences converge to the same limit ✓
- the convergence is fast ✓

By the AGM-inequality we have

$$\boxed{a_{n+1} = \frac{1}{2}(a_n + b_n) \geq \sqrt{a_n b_n} = b_{n+1}}$$

This obviously yields

$$a_{n+1} \leq \frac{1}{2}(a_n + a_n) = a_n \quad \hookrightarrow \quad \boxed{a_n \searrow}$$

and

$$b_{n+1} \geq \sqrt{b_n b_n} = b_n \quad \hookrightarrow \quad \boxed{b_n \nearrow}$$

Thus, we have

$$\boxed{b = b_0} \leq b_1 \leq b_2 \leq \ldots \leq b_n \leq a_n \leq \ldots \leq a_2 \leq a_1 \leq \boxed{a_0 = a},$$

(if not stated otherwise $b_0 = b$ and $a_0 = a$)

i.e. both sequences are $\boxed{\text{monotic and bounded}}$ and therefore they $\boxed{\text{converge}}$!

$$a_{n+1}^2 - b_{n+1}^2 \overset{DEF}{=} \frac{1}{4}(a_n^2 + 2a_nb_n + b_n^2) - a_nb_n$$
$$= \frac{1}{4}(a_n - b_n)^2 \qquad (*)$$

(trivial) $\boxed{2(b_n + b_{n+1}) \geq 0}$

$$\Rightarrow \frac{a_{n+1} - b_{n+1}}{a_n - b_n} = \frac{1}{4} \frac{a_n - b_n}{a_{n+1} + b_{n+1}} = \frac{1}{2} \underbrace{\frac{a_n - b_n}{a_n + b_n + 2b_{n+1}}}_{\leq 1} \leq \frac{1}{2}$$

$$\Rightarrow a_{n+1} - b_{n+1} \leq \frac{1}{2}(a_n - b_n) \leq \ldots \leq \underbrace{\left(\frac{1}{2}\right)^n}_{\xrightarrow[n\to\infty]{} 0} (a_0 - b_0)$$

So, $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ converge to the $\boxed{\text{same limit}}$

sometimes in lit. denoted by $M(a,b)$

$$\boxed{AG(a,b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n} \quad \left(\text{with } a_0 = a,\ b_0 = b\right)$$

Define $\boxed{c_n^2 := a_n^2 - b_n^2}$, then c_n tends to zero $\boxed{\text{quadratically:}}$

$$c_{n+1} \overset{(*)}{=} \frac{1}{2}(a_n - b_n) = \frac{1}{2}\frac{a_n^2 - b_n^2}{a_n + b_n} = \frac{1}{4}\frac{c_n^2}{a_{n+1}} \overset{a_n \downarrow}{\leq} \frac{1}{4AG(a,b)} c_n^2$$

Two further elementary, but useful, observations regarding the AGM: We have:

- $AG(a,b) = AG\left(\frac{1}{2}(a+b), \sqrt{ab}\right)$
- $AG(\lambda a, \lambda b) = \lambda\, AG(a,b) \qquad (\lambda > 0)$

2.1 Elliptic integrals

DEF 1: Let $a, b > 0$. Then we define the complete elliptic integral of the first kind as
$$I(a,b) := \int_0^{\pi/2} \left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{-1/2} d\theta$$
and the complete elliptic integral of the second kind as
$$J(a,b) := \int_0^{\pi/2} \left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{1/2} d\theta.$$

$J(a,b)$ pops up in the rectification of an ellipse, i.e. when computing the elliptic arc length, hence the names "elliptic integrals". Historically the primary interest was to compute the elliptical orbits of planets.

Since
$$a^2\underbrace{\cos^2\theta}_{1-\sin^2\theta} + b^2\sin^2\theta = a^2 - (a^2-b^2)\sin^2\theta = a^2\left(1 - \frac{a^2-b^2}{a^2}\sin^2\theta\right)$$
we have that
$$a\,I(a,b) = K(x) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} \qquad \boxed{x^2 = \frac{a^2-b^2}{a^2}}$$
and
$$\frac{1}{a}J(a,b) = E(x) := \int_0^{\pi/2} \sqrt{1-x^2\sin^2\theta}\, d\theta$$
K and E are also referred to as complete elliptic integrals

of first and second kind, respectively. With the help of Wallis' integral formula, we can find a series representation for $K(x)$ and $E(x)$.

First some notation / recall:

- Pochhammer symbol (rising factorial):

$$(a)_n := a\cdot(a+1)\cdot\ldots\cdot(a+n-1) \qquad n\geq 1$$

$$(a)_0 := 1$$

- the hypergeometric series (function):

$$F\left({a\, b \atop c}; z\right) = {}_2F_1\left({a\, b \atop c}; z\right) := \sum_{k\geq 0} \frac{(a)_k\,(b)_k}{(c)_k\, k!}\, z^k$$

- binomial theorem: $(1-z)^\alpha = \sum_{k\geq 0} \frac{(-\alpha)_k}{k!} z^k \left(= {}_1F_0\left({-\alpha \atop -}; z\right)\right.$

Note that for $n\in\mathbb{N}$: $\sum_{k\geq 0} \frac{(-n)_k}{k!} z^k = \sum_{k=0}^{n} \binom{n}{k}(-z)^k$

$$\left[\ \binom{a}{k} := (-1)^k \frac{(-a)_k}{k!} \qquad \text{for } k\in\mathbb{N}\ \right]$$

Expanding the integrand of $K(x)$ using the binomial theorem we obtain:

$$K(x) = \int_0^{\pi/2} (1-x^2\sin^2\theta)^{-1/2}\, d\theta = \int_0^{\pi/2} \sum_{k\geq 0} \frac{(\frac{1}{2})_k}{k!}\, x^{2k} \sin^{2k}\theta\, d\theta$$

exchange the order of integration and summation:

$$K(x) = \sum_{k\geq 0} \frac{(\frac{1}{2})_k}{k!} x^{2k} \underbrace{\int_0^{\pi/2} \sin^{2k}\theta \, d\theta}_{\underset{\text{THM}}{=} \frac{(1/2)_k}{k!}\frac{\pi}{2}} = {}_2F_1\left(\begin{matrix}1/2, 1/2\\ 1\end{matrix}; x^2\right)\frac{\pi}{2}$$

In the same way one obtains $E(x) = {}_2F_1\left(\begin{matrix}1/2, -1/2\\ 1\end{matrix}; x^2\right)\frac{\pi}{2}$

Back to $I(a,b)$: By means of the substitution $\sin\theta = \frac{2a\sin\theta_1}{a+b+(a-b)\sin^2\theta_1}$

and a series of (cumbersome) calculations it can be shown that

$$\boxed{I(a,b) = I\left(\frac{a+b}{2}, \sqrt{ab}\right).}$$

This is also referred to as Landen's transformation.
(John Landen, 1719–1790). Thus

$$I(a,b) = I(a_1,b_1) = \ldots = \lim_{n\to\infty} I(a_n,b_n) \quad \boxed{\text{independent of } n}$$

$(a_0 = a, b_0 = b)$

But

$$\lim_{n\to\infty} I(a_n,b_n) = \lim_{n\to\infty} \int_0^{\pi/2} \frac{d\theta}{\sqrt{a_n^2\cos^2\theta + b_n^2\sin^2\theta}}$$

$$\underset{a_n,b_n\to AG(a,b)}{=} \int_0^{\pi/2} \frac{d\theta}{AG(a,b)} = \boxed{\frac{\pi}{2AG(a,b)} = I(a,b)}$$

~~For K(x) this translates e.g. to the identity~~

$$\cancel{K(x) = \frac{\pi}{2AG(1+x,1-x)}} \quad \text{~~(and many more)~~}$$

For $K(x)$ this translates to a series of identities:

$$K(x) = \int_0^{\pi/2} \left(1-x^2\sin^2\theta\right)^{-1/2} d\theta \qquad (1=\cos^2\theta+\sin^2\theta)$$

$$= \int_0^{\pi/2} \left(\cos^2\theta + (1-x^2)\sin^2\theta\right)^{-1/2} d\theta$$

$$\overset{\text{Def}}{=} I(1,\sqrt{1-x^2}) = \frac{\pi}{2AG(1,\sqrt{1-x^2})} = \frac{\pi}{2AG(1+x,1-x)} \; (=\ldots)$$

$$\left.\begin{aligned} a_0 &= 1+x \\ b_0 &= 1-x \end{aligned}\right\} \Rightarrow \begin{aligned} a_1 &= 1 \\ b_1 &= \sqrt{1-x^2} \end{aligned}$$

2.2 AGM, elliptic integrals and fast computation of π

(very sketchy: for details and references see survey articles by Almkvist + Berndt, or Borwein + Borwein or BB + Bailey)

Thm 4 (Legendre, 1752–1833): $(|x|<1)$

$$E(x)K(\sqrt{1-x^2}) + E(\sqrt{1-x^2})K(x) - K(x)K(\sqrt{1-x^2}) = \frac{\pi}{2}$$

Salamin and Brent (1976) independently combined Legendre's relation with the AGM to find an algorithm for approximating π. With $c_n^2 = a_n^2 - b_n^2$ as before, one can show that

$$\boxed{E(x) = \left(1 - \sum_{n=0}^{\infty} 2^{n-1} c_n^2\right) K(x).}$$

$(a_0 = 1,\ b_0 = \sqrt{1-x^2})$

Now let $x=\sqrt{\frac{1}{2}}$, then $\sqrt{1-x^2}=\sqrt{\frac{1}{2}}$ and Legendre's relation becomes:

$$2\underbrace{E\left(\sqrt{\tfrac{1}{2}}\right)}K\left(\sqrt{\tfrac{1}{2}}\right)-K\left(\tfrac{1}{\sqrt{2}}\right)^2=\frac{\pi}{2}$$

$$=\left(1-\sum_{n=0}^{\infty}2^{n-1}c_n^2\right)K\left(\sqrt{\tfrac{1}{2}}\right)$$

$$\Rightarrow\left(1-\sum_{n=0}^{\infty}2^{n}c_n^2\right)K\left(\sqrt{\tfrac{1}{2}}\right)^2=\frac{\pi}{2}$$

But $K(x)=\frac{\pi}{2}\frac{1}{AG(1,\sqrt{1-x^2})}$, so with $x=\sqrt{\frac{1}{2}}$

Thus: $$\left(1-\sum_{n=0}^{\infty}2^{n}c_n^2\right)\left(\frac{\pi}{2}\right)^2\frac{1}{AG(1,\frac{1}{\sqrt{2}})^2}=\frac{\pi}{2}$$

$$\Rightarrow\quad\pi=\frac{\left(\sqrt{2}\,AG(1,\frac{1}{\sqrt{2}})\right)^2}{1-\sum_{n=0}^{\infty}2^{n}c_n^2}$$

THM5: $$\pi=\frac{AG(\sqrt{2},1)^2}{1-\sum_{n=0}^{\infty}2^{n}c_n^2}$$

where $c_n^2=a_n^2-b_n^2$ and $a_0=1$, $b_0=\sqrt{\frac{1}{2}}$

An algorithm to ~~compute~~ approximate π based on this theorem proceeds by defining the sequence:

$$\boxed{\pi_m:=\frac{2a_{m+1}^2}{1-\sum_{n=0}^{m}2^{n}c_n^2}}\quad(m\geq 0)\qquad\left(a_0=1,\ b_0=\sqrt{\tfrac{1}{2}}\right)$$

Then π_m increases monotonically towards π and already

$$\pi_2 \approx 3.141592(646)$$
$$\pi_3 \approx 3.141592\,654$$

π_{10} accurate up to (>) 10^{-10}.

Exponential Convergence

A fast algorithm to compute this sequence is due to Jonathan and Peter Borwein that is described e.g. in their 1984 paper "The AGM and fast computation of elementary functions". In this paper they also note that:

"The calculation of π to great accuracy has had a mathematical import that goes far beyond the dictates of utility. It requires a mere 39 digits of π in order to compute the circumference of a circle of radius 2×10^{25} meters (an upper bound on the distance travelled by a particle moving at the speed of light for 20 billion years, and as such an upper bound on the radius of ~~the~~ a universe) with an error of less than 10^{-12} meters (a lower bound for the radius of a hydrogen atom).

$\boxed{\text{next time: HT177F}}$