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MUG: Matching, Unification, Generalizations Part 2

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Overview

Part 1

Syntactic unification and matching

Part 2

Equational unification and matching

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Part 1

Syntactic unification and matching

Part 2

Equational unification and matching

Motivation

- Equational matching and unification algorithms are used in
  - rewriting and completion modulo equalities,
  - automated reasoning,
  - logic programming with equalities,
  - ...

Motivation

- Equational unification is a dual problem for the word problem.
- E: A given set of equalities.
- Word problem:
  Does $\forall \overline{x}.\ s \doteq t$ hold in all models of E?
- Equational unification:
  Does $\exists \overline{x}.\ s \doteq t$ hold in all nonempty models of E?

Motivation

- Equational unification generalizes syntactic unification.
- $f(x,y) \doteq^? f(a,b)$ has only one mgu $\{x \mapsto a, y \mapsto b\}$, if it is a syntactic unification problem.
- If f is commutative, then $\{x \mapsto b, y \mapsto a\}$ is another unifier.

Notation

- First-order language.
- $\mathcal{F}$: Set of function symbols.
- $\mathcal{V}$: Set of variables.
- x, y, z: Variables.
- a, b, c: Constants.
- f, g, h: Arbitrary function symbols.
- s, t, r: Terms.
- $\mathcal{T}(\mathcal{F}, \mathcal{V})$: Set of terms over $\mathcal{F}$ and $\mathcal{V}$.

Notation

- Equation: a pair of terms, written $s \doteq t$.
- $vars(t)$: The set of variables in t. This notation will be used also for sets of terms, equations, and sets of equations.
- σ, ϑ, η, ρ: Substitutions.
- ε: The identity substitution.

Equational Theory

Equational Theory

- E: a set of equations over $\mathcal{T}(\mathcal{F},\mathcal{V})$, called identities.
- Equational theory $\doteq_E$ defined by E: The least congruence relation on $\mathcal{T}(\mathcal{F},\mathcal{V})$ stable under substitution application and containing E.

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- Equational theory $\doteq_E$ defined by E: The least congruence relation on $\mathcal{T}(\mathcal{F},\mathcal{V})$ stable under substitution application and containing E.
- That means, $\doteq_E$ is the least binary relation on $\mathcal{T}(\mathcal{F},\mathcal{V})$ such that:
  - $E \subseteq \doteq_E$.
  - Reflexivity: $s \doteq_E s$ for all s.
  - Symmetry: If $s \doteq_E t$ then $t \doteq_E s$ for all s, t.
  - Transitivity: If $s \doteq_E t$ and $t \doteq_E r$ then $s \doteq_E r$ for all s, t, r.
  - Congruence: If $s_1 \doteq_E t_1, \dots, s_n \doteq_E t_n$ then $f(s_1, \dots, s_n) \doteq_E f(t_1, \dots, t_n)$ for all s, t, n and n-ary f.
  - Stability: If $s \doteq_E t$ then $s\sigma \doteq_E t\sigma$ for all s, t, σ.

Notation, Terminology

- $s \doteq_E t$:
  - The pair (s, t) belongs to the equational theory $\doteq_E$.
  - The term s is equal modulo E to the term t.
- $s \approx t$: Identities.
- $sig(E)$: The set of function symbols that occur in E.
- Sometimes E is called an equational theory as well.

Notation, Terminology

Example

- C := $\{f(x,y) \approx f(y,x)\}$: f is commutative.
  $sig(\mathsf{C}) = f$.
  $f(f(a,b),c) \doteq_{\mathsf{C}} f(c,f(b,a))$.
- AU := $\{f(f(x,y),z) \approx f(x,f(y,z)), f(x,e) \approx x, f(e,x) \approx x\}$:
  f is associative, e is unit.
  $sig(\mathsf{AU}) = \{f,e\}$
  $f(a,f(x,f(e,a))) \doteq_{\mathsf{AU}} f(f(a,x),a)$.

Notation, Terminology

E-Unification Problem, E-Unifier, E-Unifiability

- E: a given set of identities.
- E-Unification problem over $\mathcal{F}$: a finite set of equations

$$\Gamma = \{s_1 \doteq^?_E t_1, \dots, s_n \doteq^?_E t_n\},$$

where $s_i, t_i \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

- E-Unifier of Γ: a substitution σ such that

$$s_1\sigma \doteq_E t_1\sigma, \dots, s_n\sigma \doteq_E t_n\sigma.$$

- $u_E(\Gamma)$: the set of E-unifiers of Γ.
- Γ is E-unifiable iff $u_E(\Gamma) \neq \varnothing$.

E-Unification vs Syntactic Unification

- Syntactic unification: a special case of E-unification with $E = \varnothing$.
- Any syntactic unifier of an E-unification problem Γ is also an E-unifier of Γ.
- For $E \neq \varnothing$, $u_E(\Gamma)$ may contain a unifier that is not a syntactic unifier.

E-Unification vs Syntactic Unification

Example

- Terms $f(a,x)$ and $f(b,y)$:
  - Not syntactically unifiable.
  - Unifiable module commutativity of f.
  - C-unifier: $\{x \mapsto b, y \mapsto a\}$

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  - Not syntactically unifiable.
  - Unifiable module commutativity of f.
  - C-unifier: $\{x \mapsto b, y \mapsto a\}$
- Terms $f(a, x)$ and $f(y, b)$:
  - Have the most general syntactic unifier $\{x \mapsto b, y \mapsto a\}$.
  - If f is associative, then there are additional unifiers, e.g., $\{x \mapsto f(z, b), y \mapsto f(a, z)\}$.

Notions Adapted

Instantiation Quasi-Ordering (Modified)

- E: equational theory. $\mathcal{X}$: set of variables.
- A substitution σ is *more general than* ϑ *modulo* E *on* $\mathcal{X}$, written $\sigma \leq^{\mathcal{X}}_{E} \vartheta$, if there exists η such that $x\sigma\eta \doteq_E x\vartheta$ for all $x \in \mathcal{X}$.
- ϑ is called an E*-instance* of σ modulo E on $\mathcal{X}$.

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- ϑ is called an E*-instance* of σ modulo E on $\mathcal{X}$.
- The relation $\lessdot_E^{\mathcal{X}}$ is quasi-ordering, called *instantiation quasi-ordering.*
- $\doteqdot_E^{\mathcal{X}}$ is the equivalence relation corresponding to $\lessdot_E^{\mathcal{X}}$.

No MGU

- When comparing unifiers of Γ, the set $\mathcal{X}$ is $vars(\Gamma)$.
- Unifiable E-unification problems might not have an mgu.

Example

- f is commutative.
- $\Gamma = \{f(x,y) \doteq^?_{\mathsf{C}} f(a,b)\}$ has two C-unifiers:

$$\sigma_1 = \{x \mapsto a, y \mapsto b\}$$
$$\sigma_2 = \{x \mapsto b, y \mapsto a\}.$$

- On $vars(\Gamma) = \{x,y\}$, any unifier is equal to either σ_1 or σ_2.
- σ_1 and σ_2 are not comparable wrt $\leq_{\mathsf{C}}^{\{x,y\}}$.
- Hence, no mgu for Γ.

MCSU vs MGU

In E-unification, the role of mgu is taken on by a complete set of E-unifiers.

Complete and Minimal Complete Sets of E-Unifiers

- Γ: E-unification problem over $\mathcal{F}$.
- $\mathcal{X} = vars(\Gamma)$.
- $\mathcal{C}$ is a *complete set of E-unifiers* of Γ iff
  1. $\mathcal{C} \subseteq u_E(\Gamma)$: $\mathcal{C}$'s elements are E-unifiers of Γ, and
  2. For each $\vartheta \in u_E(\Gamma)$ there exists $\sigma \in \mathcal{C}$ such that $\sigma \leq_E^{\mathcal{X}} \vartheta$.

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  2. For each $\vartheta \in u_E(\Gamma)$ there exists $\sigma \in \mathcal{C}$ such that $\sigma \leq_E^{\mathcal{X}} \vartheta$.
- $\mathcal{C}$ is a *minimal complete set of E-unifiers* $(mcsu_E)$ of Γ if it is a complete set of E-unifiers of Γ and
  3. Two distinct elements of $\mathcal{C}$ are not comparable wrt $\leq_E^{\mathcal{X}}$.
- σ is an mgu of Γ iff $mcsu_E(\Gamma) = \{\sigma\}$.

MCSU's

- $mcsu_E(\Gamma) = \varnothing$ if Γ is not E-unifiable.
- Minimal complete sets of unifiers do not always exist.
- When they exist, they may be infinite.
- When they exist, they are unique up to $\doteq^{\mathcal{X}}_{E}$.

Unification Type

Unification Type of a Problem, Theory.

- E: equational theory.
- Γ: E-unification problem over $\mathcal{F}$.
- Γ has *unification type*
  - *unitary*, if $mcsu(\Gamma)$ has cardinality at most one,
  - *finitary*, if $mcsu(\Gamma)$ has finite cardinality,
  - *infinitary*, if $mcsu(\Gamma)$ has infinite cardinality,
  - *zero*, if $mcsu(\Gamma)$ does not exist.

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  - *zero*, if $mcsu(\Gamma)$ does not exist.
- Abbreviation: type unitary - 1, finitary - ω, infinitary - ∞, zero - 0.
- Ordering: $1 < \omega < \infty < 0$.

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- Abbreviation: type unitary - 1, finitary - ω, infinitary - ∞, zero - 0.
- Ordering: $1 < \omega < \infty < 0$.
- *Unification type* of E wrt $\mathcal{F}$: the maximal type of an E-unification problem over $\mathcal{F}$.

Unification Type

The unification type of an E-equational problem over $\mathcal{F}$ depends both

- on E, and
- on $\mathcal{F}$ (which function symbols are permitted in unification problems).

Unification Type

Example (Type Unitary)

Syntactic unification.

- The empty equational theory $\varnothing$: Syntactic unification.
- Unitary wrt any $\mathcal{F}$ because any unifiable syntactic unification problem has an mgu.

Unification Type

Example (Type Finitary)

Commutative unification: $\{f(x,y) \approx f(y,x)\}$

- Not unitary.
- $\{f(x,y) \doteq^?_{\mathsf{C}} f(a,b)\}$ has two unifiers $\{x \mapsto a, y \mapsto b\}$ and $\{x \mapsto b, y \mapsto a\}$.
- No mgu.
- C unification is finitary.

Unification Type

Example (Type Finitary)

C unification is finitary for any $\mathcal{F}$:

- Let $\Gamma = \{s_1 \doteq^?_{\mathsf{C}} t_1, \dots, s_n \doteq^?_{\mathsf{C}} t_n\}$ be a C-unification problem.
- Consider all possible syntactic unification problems $\Gamma' = \{s'_1 \doteq^? t'_1, \dots, s'_n \doteq^? t'_n\}$, where $s'_i \doteq_{\mathsf{C}} s_i$ and $t'_i \doteq_{\mathsf{C}} t_i$ for each $1 \le i \le n$.
- There are only finitely many such Γ's, because the C-equivalence class for a given term t is finite.
- It can be shown that collection of all mgu's of Γ's is a complete set of C-unifiers of Γ. This set if finite.
- If this set is not minimal (often the case), it can be minimized by removing redundant C-unifiers.

Unification Type

Example (Type Infinitary)

Associative unification: $\{f(f(x,y),z) \approx f(x,f(y,z))\}$.

- $\{f(x,a) \doteq^?_{\mathsf{A}} f(a,x)\}$ has an infinite $mcsu$: $\{\{x \mapsto a\}, \{x \mapsto f(a,a)\}, \{x \mapsto f(a,f(a,a))\}, \ldots\}$
- Hence, A-unification can not be unitary or finitary.
- It is not of type zero because any A-unification problem has an $mcsu$ that can be enumerated by the procedure from

  G. Plotkin.
  Building in equational theories.
  In B. Meltzer and D. Michie, editors, *Machine Intelligence*, volume 7, pages 73–90. Edinburgh University Press, 1972.

- A-unification is infinitary for any $\mathcal{F}$.

Unification Type

Example (Type Zero)

Associative-Idempotent unification:
$\{f(f(x,y),z) \approx f(x,f(y,z)), f(x,x) \approx x\}$.

- $\{f(x,f(y,x)) \doteq^{?}_{\mathsf{AI}} f(x,f(z,x))\}$ does not have a minimal complete set of unifiers, see

  F. Baader.
  Unification in idempotent semigroups is of type zero.
  J. Automated Reasoning, 2(3):283–286, 1986.

- AI-unification is of type zero.

Unification Type. Signature Matters

Unification Type depends on $\mathcal{F}$.

Example

Associative-commutative unification with unit (ACU):

- $\{f(f(x,y),z) \approx f(x,f(y,z)), f(x,y) \approx f(y,x), f(x,e) \approx x\}$.
- Any ACU problem built using only f and variables is unitary.
- There are ACU problems containing function symbols other than f and e, which are finitary, not unitary.
- For instance, $mcsu(\{f(x,y) \doteq^?_{\mathsf{ACU}} f(a,b)\})$ consists of four unifiers (which ones?).

Kinds of E-unification.

Kinds of E-Unification

One may distinguish three kinds of E-unification problems, depending on the function symbols that are allowed to occur in them.

E-Unification Problems: Elementary, with Constants, General.

- E: an equational Theory.
  Γ: an E-unification problem over $\mathcal{F}$.
- Γ is an elementary E-unification problem iff $\mathcal{F} = sig(E)$.
- Γ is an E-unification problem with constants iff $\mathcal{F} \smallsetminus sig(E)$ consists of constants.
- Γ is a general E-unification problem iff $\mathcal{F} \smallsetminus sig(E)$ may contain arbitrary function symbols.

Unification Types of Theories wrt Kinds

Unification Types Depending on Signature

- Unification type of E wrt elementary unification: Maximal unification type of E wrt all $\mathcal{F}$ such that $\mathcal{F} = sig(E)$.
- Unification type of E wrt unification with constants: Maximal unification type of E wrt all $\mathcal{F}$ such that $\mathcal{F} \setminus sig(E)$ is a set of constants.
- Unification type of E wrt general unification: Maximal unification type of E wrt all $\mathcal{F}$ such that $\mathcal{F} \setminus sig(E)$ is a set of arbitrary function symbols.

Unification Types of Theories wrt Kinds

The same equational theory can have different unification types for different kinds. Examples:

- ACU (Abelian monoids): Unitary wrt elementary unification, finitary wrt unification with constants and general unification.
- AG (Abelian groups): Unitary wrt elementary unification and unification with constants, finitary wrt general unification.

Single Equation vs Systems of Equations

- In syntactic unification, solving systems of equations can be reduced to solving a single equation.
- For equational unification, the same holds only for general unification.
- For elementary unification and for unification with constants it is not the case.

Unification Types wrt of Cardinality of Problems

There exists an equational theory E such that

- all elementary E-unification problems of cardinality 1 (single equations) have minimal complete sets of E-unifiers, but
- E is of type zero wrt to elementary unification: There exists an elementary E-unification problem of cardinality 2 that does not have a minimal complete set of unifiers.

H.-J. Bürckert, A. Herold, and M. Schmidt-Schauß.
On equational theories, unification, and decidability.
J. Symbolic Computation **8**(3,4), 3–49. 1989.

Decision and Unification Procedures

- Decision procedure for an equational theory E (wrt $\mathcal{F}$):
  An algorithm that for each E-unification problem Γ (wrt $\mathcal{F}$) returns $success$ if Γ is E-unifiable, and $failure$ otherwise.

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- (Minimal) E-unification algorithm (wrt $\mathcal{F}$): An algorithm that computes a (minimal) finite complete set of E-unifiers for all E-unification problems over $\mathcal{F}$.

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- E is decidable if it admits a decision procedure.
- (Minimal) E-unification algorithm (wrt $\mathcal{F}$): An algorithm that computes a (minimal) finite complete set of E-unifiers for all E-unification problems over $\mathcal{F}$.
- E-unification algorithm yields a decision procedure for E.
- (Minimal) E-unification procedure: A procedure that enumerates a possible infinite (minimal) complete set of E-unifiers.
- E-unification procedure does not yield a decision procedure for E.

Decidability wrt Kinds

Decidability of an equational theory might depend on the kinds of E-unification.

- There exists an equational theory for which elementary unification is decidable, but unification with constants is undecidable:

H.-J. Bürckert.
Some relationships between unification, restricted unification, and matching.
In J. Siekmann, editor, *Proc. 8th Int. Conference on Automated Deduction*, volume 230 of *LNCS*. Springer, 1986.

Decidability wrt Kinds

Decidability of an equational theory might depend on the kinds of E-unification.

- There exists an equational theory for which unification with constants is decidable, but general unification is undecidable:

J. Otop.
E-unification with constants vs. general E-unification.
Journal of Automated Reasoning, 48(3):363–390, 2012.

Decidability wrt Problem Cardinality

There exists an equational theory E such that

- unifiability of elementary E-unification problems of cardinality 1 (single equations) is decidable, but
- for elementary problems of larger cardinality it is undecidable.

P. Narendran and H. Otto.
Some results on equational unification.
In M. E. Stickel, editor, *Proc. 10th Int. Conference on Automated Deduction*, volume 449 of *LNAI*. Springer, 1990.

Summary

- Unification type depends on
  - equational theory,
  - signature (kinds),
  - cardinality of unification problems.

Summary

- Unification type depends on
  - equational theory,
  - signature (kinds),
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- Decidability depends on
  - equational theory,
  - signature (kinds),
  - cardinality of unification problems.

Three Main Questions in Unification Theory

Decidability: Is it decidable whether an E-unification problem is solvable? If yes, what is the complexity of this decision problem?

Unification type: What is the unification type of the theory E?

Unification algorithm: How can we obtain an (efficient) E-unification algorithm, or a (preferably minimal) E-unification procedure?

Summary of Results for Specific Theories

General unification:

Theory	Decidability	Type	Algorithm/Procedure
$\varnothing$, BR	Yes	1	Yes
A, AU	Yes	∞	Yes
C, AC, ACU	Yes	ω	Yes
I, CI, ACI	Yes	ω	Yes
AI	Yes	0	?
$\mathrm{D}_{\{f,g\}}\mathrm{A}_g$	No	∞	?
AG	Yes	ω	Yes
CRU	No	? (∞ or 0)	?

BR - Boolean ring, D - distributivity, CRU - commutative ring with unit.

Commutative Unification and Matching

- C-unification inference system $\mathcal{U}_{\mathsf{C}}$ can be obtained from the $\mathcal{U}$ by adding the C-Decomposition rule:

C-Decomposition:
$$\{f(s_1, s_2) \doteq^?_{\mathsf{C}} f(t_1, t_2)\} \uplus P'; S \Longrightarrow$$
$$\{s_1 \doteq^?_{\mathsf{C}} t_2, s_2 \doteq^?_{\mathsf{C}} t_1\} \cup P'; S,$$
if f is commutative.

- **C-Decomposition** and **Decomposition** transform the same system in different ways.

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if f is commutative.

- **C-Decomposition** and **Decomposition** transform the same system in different ways.
- C-matching algorithm $\mathcal{M}_{\mathsf{C}}$ is obtained analogously from $\mathcal{M}$.

C-Unification

In order to C-unify s and t:

1. Create an initial system $\{s \doteq^{?}_{\mathsf{C}} t\}; \varnothing$.
2. Apply successively rules from $\mathcal{U}_{\mathsf{C}}$, building a complete tree of derivations. **C-Decomposition** and **Decomposition** rules have to be applied concurrently and form branching points in the derivation tree.

Example. C-Unification

C-unify $g(f(x,y),z)$ and $g(f(f(a,b),f(b,a)),c)$, commutative f.

$$\{g(f(x,y),z) \doteq^{?}_{\mathsf{C}} g(f(f(a,b),f(b,a))),c)\};\varnothing$$

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$$\downarrow$$

$$\{f(x,y) \doteq^?_{\mathsf{C}} f(f(a,b),f(b,a)), z \doteq^?_{\mathsf{C}} c\};\varnothing$$

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& \{f(x,y) \doteq^?_{\mathsf{C}} f(f(a,b),f(b,a)), z \doteq^?_{\mathsf{C}} c\};\varnothing \arrow[dl] \arrow[dr] & \\
\{x \doteq^?_{\mathsf{C}} f(a,b), y \doteq^?_{\mathsf{C}} f(b,a), z \doteq^?_{\mathsf{C}} c\};\varnothing & & \{x \doteq^?_{\mathsf{C}} f(b,a), y \doteq^?_{\mathsf{C}} f(a,b), z \doteq^?_{\mathsf{C}} c\};\varnothing
\end{tikzcd}$$

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Example. C-Unification

C-unify $g(f(x,y),z)$ and $g(f(f(a,b),f(b,a)),c)$, commutative f.

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\end{tikzcd}$$

Not minimal.

Properties of the C-Unification Algorithm

Theorem

Applied to a C*-unification problem P, the* C*-unification algorithm terminates and computes a complete set of* C*-unifiers of P.*

Properties of the C-Unification Algorithm

Theorem

Applied to a C-unification problem P, the C-unification algorithm terminates and computes a complete set of C-unifiers of P.

Proof.

- Termination is proved using the same measure as for syntactic unification.
- Completeness is based on the following two facts:
  - If Γ is transformed by only one rule of $\mathcal{U}_{\mathsf{C}}$ into Γ', then $u_{\mathsf{C}}(\Gamma) = u_{\mathsf{C}}(\Gamma')$.
  - If Γ is transformed by two rules of $\mathcal{U}_{\mathsf{C}}$ into Γ_1 and Γ_2, then $u_{\mathsf{C}}(\Gamma) = u_{\mathsf{C}}(\Gamma_1) \cup u_{\mathsf{C}}(\Gamma_2)$.

□

MCSU for C-Unification/Matching Problems Can Be Large

Example

- Problem: $f(f(x_1,x_2),f(x_3,x_4)) \doteq^?_{\mathsf{C}} f(f(a,b),f(c,d))$.
- $mcsu$ contains 4! substitutions.

Properties of the C-Unification Algorithm

- The algorithm, in general, does not return a minimal complete set of C-unifiers.
- The obtained complete set can be further minimized, removing redundant unifiers.
- Not clear how to design a C-unification algorithm that computes a minimal complete set of unifiers directly.

Properties of the C-Unification Algorithm

Theorem

The decision problem of C-matching and unification is NP-complete.

Proof.

Exercise. □

ACU-Unification

ACU = $\{f(f(x,y),z) \approx f(x,f(y,z)), f(x,y) \approx f(y,x), f(x,e) \approx x\}$

1. Associativity, commutativity, unit element.
2. f is associative and commutative, e is the unit element.

Example: Elementary ACU-Unification

Elementary ACU-unification problem:

$$\Gamma = \{f(x, f(x, y)) \doteq^?_{\mathsf{ACU}} f(z, f(z, z))\}$$

Solving idea:

1. Associate with the equation in Γ a homogeneous linear Diophantine equation $2x + y = 3z$.
2. The equation states that the number of new variables introduced by a unifier σ in both sides of $\Gamma\sigma$ must be the same.

(Continues on the next slide.)

Example. Elementary ACU-Unification (Cont.)

3. Solve $2x + y = 3z$ over nonnegative integers. Three minimal solutions:

$$x = 1,\ y = 1,\ z = 1$$
$$x = 0,\ y = 3,\ z = 1$$
$$x = 3,\ y = 0,\ z = 2$$

   Any other solution of the equation can be obtained as a nonnegative linear combination of these three solutions.

(Continues on the next slide.)

Example. Elementary ACU-Unification (Cont.)

4. Introduce new variables v_1, v_2, v_3 for each solution of the Diophantine equation:

	x	y	z
v_1	1	1	1
v_2	0	3	1
v_3	3	0	2

5. Each row corresponds to a unifier of Γ:

$$\begin{aligned}
\sigma_1 =&\{x \mapsto v_1, y \mapsto v_1, z \mapsto v_1\}\\
\sigma_2 =&\{x \mapsto e, y \mapsto f(v_2, f(v_2, v_2)), z \mapsto v_2\}\\
\sigma_3 =&\{x \mapsto f(v_3, f(v_3, v_3)), y \mapsto e, z \mapsto f(v_3, v_3)\}
\end{aligned}$$

However, none of them is an mgu.

Example. Elementary ACU-Unification (Cont.)

6. To obtain an mgu, we should combine all three solutions:

	x	y	z
v_1	1	1	1
v_2	0	3	1
v_3	3	0	2

   The columns indicate that the mgu we are looking for should have

   - in the binding for x one v_1, zero v_2, and three v_3's,
   - in the binding for y one v_1, three v_2's, and zero v_3,
   - in the binding for z one v_1, one v_2, and two v_3's

7. Hence, we can construct an mgu:

$$\begin{aligned}\sigma = \{&x \mapsto f(v_1, f(v_3, f(v_3, v_3))), y \mapsto f(v_1, f(v_2, f(v_2, v_2))),\\ &z \mapsto f(v_1, f(v_2, f(v_3, v_3)))\}\end{aligned}$$

Example: ACU-Unification with constants

- ACU-unification problem with constants

$$\Gamma = \{f(x, f(x, y)) \doteq^?_{\mathsf{ACU}} f(a, f(z, f(z, z)))\}$$

reduces to inhomogeneous linear Diophantine equation

$$S = \{2x + y = 3z + 1\}.$$

- The minimal nontrivial natural solutions of S are $(0, 1, 0)$ and $(2, 0, 1)$.

Example: ACU-Unification with constants

- ACU-unification problem with constants

  $$\Gamma = \{f(x, f(x, y)) \doteq^?_{\mathsf{ACU}} f(a, f(z, f(z, z)))\}$$

  reduces to inhomogeneous linear Diophantine equation

  $$S = \{2x + y = 3z + 1\}.$$

- Every natural solution of S is obtained by as the sum of one of the minimal solution and a solution of the corresponding homogeneous LDE $2x + y = 3z$.
- One element of the minimal complete set of unifiers of Γ is obtained from the combination of one minimal solution of S with the set of all minimal solutions of $2x + y = 3z$.

Example: ACU-Unification with constants

- ACU-unification problem with constants

  $$\Gamma = \{f(x, f(x, y)) \doteq^?_{\mathsf{ACU}} f(a, f(z, f(z, z)))\}$$

  reduces to inhomogeneous linear Diophantine equation

  $$S = \{2x + y = 3z + 1\}.$$

- The minimal complete set of unifiers of Γ is $\{\sigma_1, \sigma_2\}$, where

  $$\begin{aligned}\sigma_1 = \{&x \mapsto f(v_1, f(v_3, f(v_3, v_3))),\\ &y \mapsto f(a, f(v_1, f(v_2, f(v_2, v_2))),\\ &z \mapsto f(v_1, f(v_2, f(v_3, v_3)))\}\\ \sigma_2 = \{&x \mapsto f(a, f(a, f(v_1, f(v_3, f(v_3, v_3))))),\\ &y \mapsto f(v_1, f(v_2, f(v_2, v_2)),\\ &z \mapsto f(a, f(v_1, f(v_2, f(v_3, v_3))))\}\end{aligned}$$

ACU-Unification with constants

- If an ACU-unification problem contains more than one constant, solve the corresponding inhomogeneous LDE for each constant.
- The unifiers in the minimal complete set correspond to all possible combinations of the minimal solutions of these inhomogeneous equations.

ACU-Unification with constants

Example

$xxy \doteq^?_{\mathsf{ACU}} aabbb$:

- Equation for a: $2x + y = 2$. Minimal solutions: $(1, 0)$ and $(0, 2)$.
- Corresponding unifiers: $\{x \mapsto a, y \mapsto e\}$, $\{x \mapsto e, y \mapsto aa\}$
- Equation for b: $2x + y = 3$. Minimal solutions: $(0, 3)$ and $(1, 1)$.
- Corresponding unifiers: $\{x \mapsto e, y \mapsto bbb\}$, $\{x \mapsto b, y \mapsto b\}$
- Unifiers in the minimal complete set: $\{x \mapsto a, y \mapsto bbb\}$, $\{x \mapsto ab, y \mapsto b\}$, $\{x \mapsto e, y \mapsto aabbb\}$, $\{x \mapsto b, y \mapsto aab\}$.

From ACU to AC

Example

- How to solve $\Gamma_1 = \{f(x, f(x, y)) \doteq^?_{\mathsf{AC}} f(z, f(z, z))\}$?
- We "know" how to solve $\Gamma_2 = \{f(x, f(x, y)) \doteq^?_{\mathsf{ACU}} f(z, f(z, z))\}$, but its mgu is not an mgu for Γ_1.
- Mgu of Γ_2:

$$\sigma = \{x \mapsto f(v_1, f(v_3, f(v_3, v_3))), y \mapsto f(v_1, f(v_2, f(v_2, v_2))),$$
$$z \mapsto f(v_1, f(v_2, f(v_3, v_3)))\}$$

- Unifier of Γ_1: $\vartheta = \{x \mapsto v_1, y \mapsto v1, z \mapsto v_1\}$.
- σ is not more general modulo AC than ϑ.

From ACU to AC

Example

- Idea: Take the mgu of Γ_2.
- Compose it with all possible erasing substitutions that map a subset of $\{v_1, v_2, v_3\}$ to the unit element.
- Restriction: The result of the composition should not map x, y, and z to the unit element.

From ACU to AC

Example

Minimal complete set of unifiers for Γ_1:

$$\begin{aligned}
\sigma_1 = \{&x \mapsto f(v_1, f(v_3, f(v_3, v_3))), y \mapsto f(v_1, f(v_2, f(v_2, v_2))),\\
&z \mapsto f(v_1, f(v_2, f(v_3, v_3)))\}\\
\sigma_2 = \{&x \mapsto f(v_3, f(v_3, v_3)), y \mapsto f(v_2, f(v_2, v_2)),\\
&z \mapsto f(v_2, f(v_3, v_3))\}\\
\sigma_3 = \{&x \mapsto f(v_1, f(v_3, f(v_3, v_3))), y \mapsto v_1, z \mapsto f(v_1, f(v_3, v_3))\}\\
\sigma_4 = \{&x \mapsto v_1, y \mapsto f(v_1, f(v_2, f(v_2, v_2))), z \mapsto f(v_1, v_2)\}\\
\sigma_5 = \{&x \mapsto v_1, y \mapsto v_1, z \mapsto v_1\}
\end{aligned}$$

How to Solve Systems of LDEs over Naturals?

Contejean-Devie Algorithm:

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Generalizes Fortenbacher's Algorithm for solving a single equation:

Michael Clausen and Albrecht Fortenbacher.
Efficient Solution of Linear Diophantine Equations.
J. Symbolic Computation 8(1,2): 201–216 (1989).

Homogeneous Case

Homogeneous linear Diophantine system with m equations and n variables:

$$\left\{\begin{array}{ccccc} a_{11}x_1 & +\cdots+ & a_{1n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & +\cdots+ & a_{mn}x_n & = & 0 \end{array}\right.$$

- a_{ij}'s are integers.
- Looking for nontrivial natural solutions.

Homogeneous Case

Example

$$\begin{cases} -x_1 + x_2 + 2x_3 - 3x_4 = 0 \\ -x_1 + 3x_2 - 2x_3 - x_4 = 0 \end{cases}$$

Nontrivial solutions:

- $s_1 = (0, 1, 1, 1)$
- $s_2 = (4, 2, 1, 0)$
- $s_3 = (0, 2, 2, 2)$
- $s_4 = (8, 4, 2, 0)$
- $s_5 = (4, 3, 2, 1)$
- $s_6 = (8, 5, 3, 1)$
- $\dots$

Homogeneous Case

Example

$$\begin{cases} -x_1 + x_2 + 2x_3 - 3x_4 = 0 \\ -x_1 + 3x_2 - 2x_3 - x_4 = 0 \end{cases}$$

Nontrivial solutions:

- $s_1 = (0, 1, 1, 1)$
- $s_2 = (4, 2, 1, 0)$
- $s_3 = (0, 2, 2, 2) = 2s_1$
- $s_4 = (8, 4, 2, 0) = 2s_2$
- $s_5 = (4, 3, 2, 1) = s_1 + s_2$
- $s_6 = (8, 5, 3, 1) = s_1 + 2s_2$
- ...

Homogeneous Case

Homogeneous linear Diophantine system with m equations and n variables:

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- a_{ij}'s are integers.
- Looking for a basis in the set of nontrivial natural solutions.

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- a_{ij}'s are integers.
- Looking for a basis in the set of nontrivial natural solutions.
- Does it exist?

Homogeneous Case

The basis in the set S of nontrivial natural solutions of a homogeneous LDS is the set of $\gg$-minimal elements S.

$\gg$ is the ordering on tuples of natural numbers:

$$(x_1, \ldots, x_n) \gg (y_1, \ldots, y_n)$$

if and only if

- $x_i \geq y_i$ for all $1 \leq i \leq n$ and
- $x_i > y_i$ for some $1 \leq i \leq n$.

Matrix Form

Homogeneous linear Diophantine system with m equations and n variables:

$$Ax{\downarrow} = 0{\downarrow},$$

where

$$A := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad x{\downarrow} := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad 0{\downarrow} := \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Matrix Form

- Canonical basis in $\mathbb{N}^n$: $(e_1{\downarrow}, \dots, e_n{\downarrow})$.
- $e_j{\downarrow} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, with 1 in j's row.
- Then $Ax{\downarrow} = x_1 Ae_1{\downarrow} + \cdots + x_n Ae_n{\downarrow}$.

Matrix Form

- a: The linear mapping associated to A.

$$a(x\!\downarrow) = \begin{pmatrix} a_{11}x_1 & +\cdots+ & a_{1n}x_n \\ \vdots & & \vdots \\ a_{m1}x_1 & +\cdots+ & a_{mn}x_n \end{pmatrix} = x_1 a(e_1\!\downarrow) + \cdots + x_n a(e_n\!\downarrow).$$

Single Equation: Idea

Case $m = 1$: Single homogeneous LDE $a_1 x_1 + \cdots + a_n x_n = 0$.
Fortenbacher's idea:

- Search minimal solutions starting from the elements in the canonical basis of $\mathbb{N}^n$.
- Suppose the current vector $v\!\downarrow$ is not a solution.
- It can be nondeterministically increased, component by component, until it becomes a solution or greater than a solution.
- To decrease the search space, the following restrictions can be imposed:
  - If $a(v\!\downarrow) > 0$, then increase by one some v_j with $a_j < 0$.
  - If $a(v\!\downarrow) < 0$, then increase by one some v_j with $a_j > 0$.

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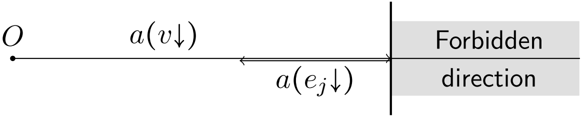
- To decrease the search space, the following restrictions can be imposed:
  - If $a(v{\downarrow}) > 0$, then increase by one some v_j with $a_j < 0$.
  - If $a(v{\downarrow}) < 0$, then increase by one some v_j with $a_j > 0$.
  - (If $a(v{\downarrow})a(e_j{\downarrow}) < 0$ for some j, increase v_j by one.)

Single Equation: Geometric Interpretation of the Idea

- Fortenbacher's condition
  If $a(v\downarrow)a(e_j\downarrow) < 0$ for some j, increase v_j by one.
- Increasing v_j by one: $a(v\downarrow + e_j\downarrow) = a(v\downarrow) + a(e_j\downarrow)$.
- Going to the "right direction", towards the origin.



Single Equation: Algorithm

Case $m = 1$: Single homogeneous LDE $a_1 x_1 + \cdots + a_n x_n = 0$.
Fortenbacher's algorithm:

- Start with the pair P, M of the set of potential solutions $P = \{e_1\!\downarrow, \ldots, e_n\!\downarrow\}$ and the set of minimal nontrivial solutions $M = \emptyset$.
- Apply repeatedly the rules:

1. $\{v\!\downarrow\} \cup P', M \Longrightarrow P', M$,
   if $v\!\downarrow \gg u\!\downarrow$ for some $u\!\downarrow \in M$.
2. $\{v\!\downarrow\} \cup P', M \Longrightarrow P', \{v\!\downarrow\} \cup M$,
   if $a(v\!\downarrow) = 0$ and rule 1 is not applicable.
3. $P, M \Longrightarrow \{v\!\downarrow + e_j\!\downarrow \mid v\!\downarrow \in P,\ a(v\!\downarrow)a(e_j\!\downarrow) < 0,\ j \in 1..n\}, M$,
   if rules 1 and 2 are not applicable.

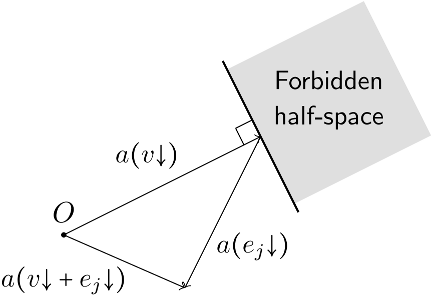
- If $\emptyset, M$ is reached, return M.

System of Equations: Idea

- General case: System of homogeneous LDEs.
- $a(x\downarrow) = 0\downarrow$.
- Generalizing Fortenbacher's idea:
  - Search minimal solutions starting from the elements in the canonical basis of $\mathbb{N}^n$.
  - Suppose the current vector $v\downarrow$ is not a solution.
  - It can be nondeterministically increased, component by component, until it becomes a solution or greater than a solution.
  - To decrease the search space, increase only those components that lead to the "right direction".

System of Equations: How to Restrict

- "Right direction": Towards the origin.
- If $a(v{\downarrow}) \neq 0{\downarrow}$, then do $a(v{\downarrow} + e_j{\downarrow}) = a(v{\downarrow}) + a(e_j{\downarrow})$.
- $a(v{\downarrow}) + a(e_j{\downarrow})$ should lie in the half-space containing O.
- Contejean-Devie condition: If $a(v{\downarrow}) \cdot a(e_j{\downarrow}) < 0$ for some j, increase v_j by one. ($\cdot$ is the scalar product.)

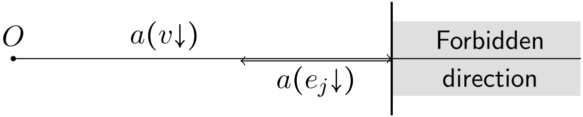


How to Restrict: Comparison

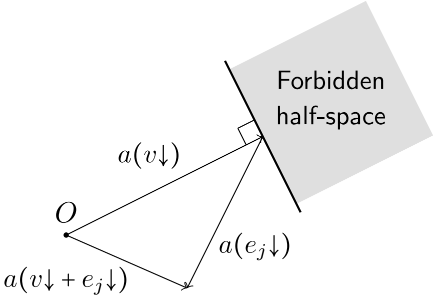
- Fortenbacher's condition
  If $a(v{\downarrow})a(e_j{\downarrow}) < 0$ for some j, increase v_j by one.
- Contejean-Devie condition
  If $a(v{\downarrow}) \cdot a(e_j{\downarrow}) < 0$ for some j, increase v_j by one.

How to Restrict: Comparison

Fortenbacher's condition



Contejean-Devie condition



System of Equations: Algorithm

System of homogeneous LDEs: $a(x\downarrow) = 0\downarrow$.

Contejean-Devie algorithm:

- Start with the pair P, M where
  - $P = \{e_1\downarrow, \dots, e_n\downarrow\}$ is the set of potential solutions,
  - $M = \varnothing$ is the set of minimal nontrivial solutions.
- Apply repeatedly the rules:

  1. $\{v\downarrow\} \cup P', M \Longrightarrow P', M$,
     if $v\downarrow \gg u\downarrow$ for some $u\downarrow \in M$.
  2. $\{v\downarrow\} \cup P', M \Longrightarrow P', \{v\downarrow\} \cup M$,
     if $a(v\downarrow) = 0\downarrow$ and rule 1 is not applicable.
  3. $P, M \Longrightarrow \{v\downarrow + e_j\downarrow \mid v\downarrow \in P,\ a(v\downarrow) \cdot a(e_j\downarrow) < 0,\ j \in 1..n\}, M$,
     if rules 1 and 2 are not applicable.

- If $\varnothing, M$ is reached, return M.

Contejean-Devie Algorithm on an Example

$$\begin{cases} -x_1 + x_2 + 2x_3 - 3x_4 = 0 \\ -x_1 + 3x_2 - 2x_3 - x_4 = 0 \end{cases}$$

$$e_1{\downarrow} = (1,0,0,0)^T \quad e_2{\downarrow} = (0,1,0,0)^T$$
$$e_3{\downarrow} = (0,0,1,0)^T \quad e_4{\downarrow} = (0,0,0,1)^T$$

Start: $\{e_1{\downarrow}, \dots, e_4{\downarrow}\}, \varnothing$.

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Contejean-Devie Algorithm on an Example

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$\boxed{\begin{matrix}-1\\-1\end{matrix}\;\Big|\;1000}$ $\boxed{\begin{matrix}1\\3\end{matrix}\;\Big|\;0100}$ $\boxed{\begin{matrix}2\\-2\end{matrix}\;\Big|\;0010}$ $\boxed{\begin{matrix}-3\\-1\end{matrix}\;\Big|\;0001}$

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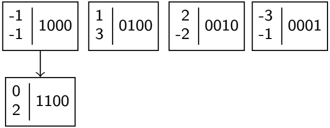

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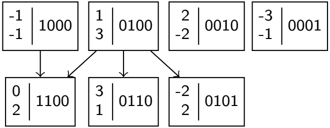

$\begin{smallmatrix}-1\\-1\end{smallmatrix}$ | 1000
$\begin{smallmatrix}1\\3\end{smallmatrix}$ | 0100
$\begin{smallmatrix}2\\-2\end{smallmatrix}$ | 0010
$\begin{smallmatrix}-3\\-1\end{smallmatrix}$ | 0001
$\begin{smallmatrix}0\\2\end{smallmatrix}$ | 1100
$\begin{smallmatrix}3\\1\end{smallmatrix}$ | 0110
$\begin{smallmatrix}-2\\2\end{smallmatrix}$ | 0101

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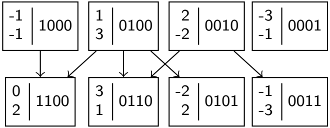

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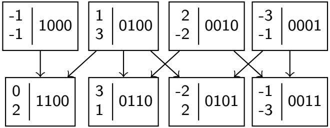

-1 -1 | 1000

1 3 | 0100

2 -2 | 0010

-3 -1 | 0001

0 2 | 1100

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-1 -3 | 0011

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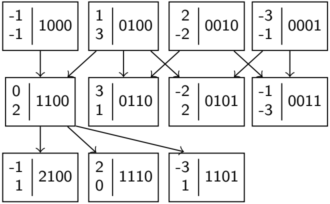

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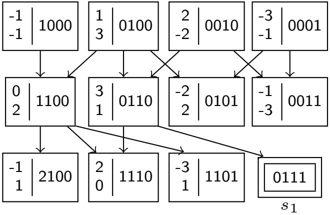

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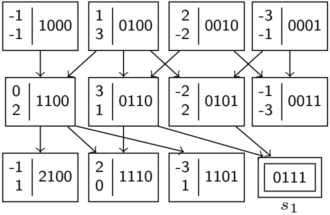

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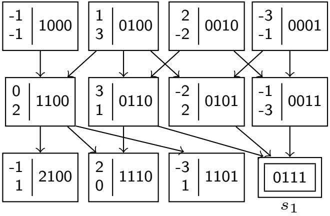

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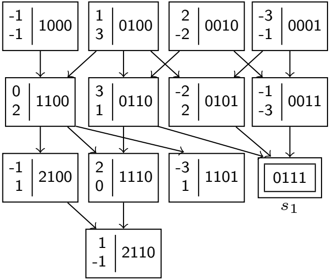

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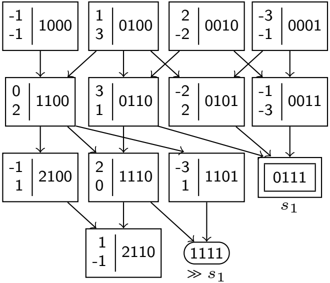

$\begin{array}{c|c} -1 & 1000 \\ -1 & \end{array}$ $\begin{array}{c|c} 1 & 0100 \\ 3 & \end{array}$ $\begin{array}{c|c} 2 & 0010 \\ -2 & \end{array}$ $\begin{array}{c|c} -3 & 0001 \\ -1 & \end{array}$

$\begin{array}{c|c} 0 & 1100 \\ 2 & \end{array}$ $\begin{array}{c|c} 3 & 0110 \\ 1 & \end{array}$ $\begin{array}{c|c} -2 & 0101 \\ 2 & \end{array}$ $\begin{array}{c|c} -1 & 0011 \\ -3 & \end{array}$

$\begin{array}{c|c} -1 & 2100 \\ 1 & \end{array}$ $\begin{array}{c|c} 2 & 1110 \\ 0 & \end{array}$ $\begin{array}{c|c} -3 & 1101 \\ 1 & \end{array}$ $\boxed{\boxed{0111}}$ s_1

$\begin{array}{c|c} 1 & 2110 \\ -1 & \end{array}$ 1111 $\gg s_1$

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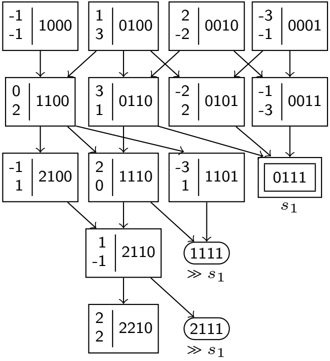

$\begin{smallmatrix}-1\\-1\end{smallmatrix}$ 1000 $\begin{smallmatrix}1\\3\end{smallmatrix}$ 0100 $\begin{smallmatrix}2\\-2\end{smallmatrix}$ 0010 $\begin{smallmatrix}-3\\-1\end{smallmatrix}$ 0001

$\begin{smallmatrix}0\\2\end{smallmatrix}$ 1100 $\begin{smallmatrix}3\\1\end{smallmatrix}$ 0110 $\begin{smallmatrix}-2\\2\end{smallmatrix}$ 0101 $\begin{smallmatrix}-1\\-3\end{smallmatrix}$ 0011

$\begin{smallmatrix}-1\\1\end{smallmatrix}$ 2100 $\begin{smallmatrix}2\\0\end{smallmatrix}$ 1110 $\begin{smallmatrix}-3\\1\end{smallmatrix}$ 1101 0111 s_1

$\begin{smallmatrix}1\\-1\end{smallmatrix}$ 2110 1111 $\gg s_1$

$\begin{smallmatrix}2\\2\end{smallmatrix}$ 2210 2111 $\gg s_1$

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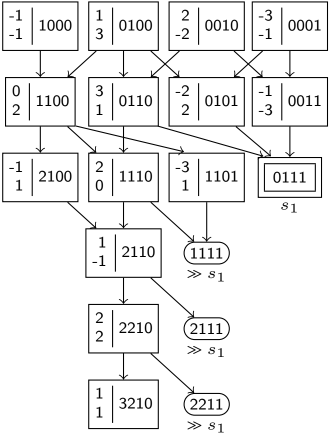


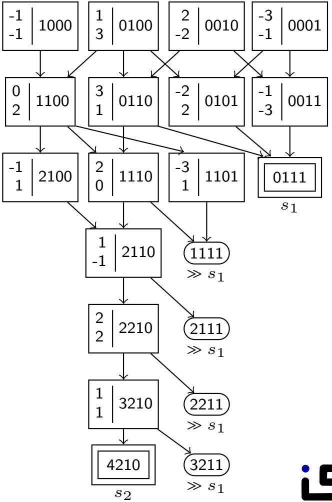
Contejean-Devie Algorithm on an Example

$$\begin{cases} -x_1 + x_2 + 2x_3 - 3x_4 = 0 \\ -x_1 + 3x_2 - 2x_3 - x_4 = 0 \end{cases}$$

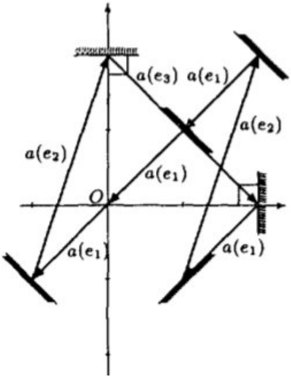
$$e_1{\downarrow} = (1,0,0,0)^T \quad e_2{\downarrow} = (0,1,0,0)^T$$
$$e_3{\downarrow} = (0,0,1,0)^T \quad e_4{\downarrow} = (0,0,0,1)^T$$

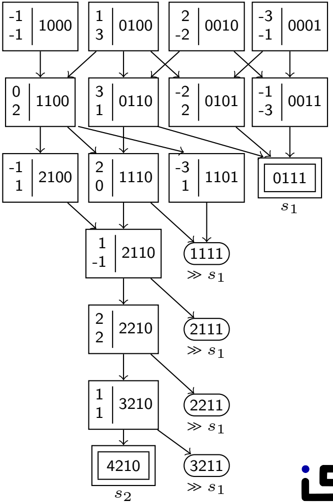
Start: $\{e_1{\downarrow}, \dots, e_4{\downarrow}\}, \emptyset$.

1. $\{v{\downarrow}\} \cup P', M \Longrightarrow P', M$,
   if $v{\downarrow} \gg u{\downarrow}$ for some $u{\downarrow} \in M$.
2. $\{v{\downarrow}\} \cup P', M \Longrightarrow P', \{v{\downarrow}\} \cup M$,
   if $a(v{\downarrow}) = 0{\downarrow}$ and rule 1 is not applicable.
3. $P, M \Longrightarrow \{v{\downarrow} + e_j{\downarrow} \mid v{\downarrow} \in P,$
   $a(v{\downarrow}) \cdot a(e_j{\downarrow}) < 0,\ j \in 1..n\}, M$,
   if rules 1 and 2 are not applicable.



Contejean-Devie Algorithm on an Example





-1 -1 | 1000
1 3 | 0100
2 -2 | 0010
-3 -1 | 0001

0 2 | 1100
3 1 | 0110
-2 2 | 0101
-1 -3 | 0011

-1 1 | 2100
2 0 | 1110
-3 1 | 1101
0111
s_1

1 -1 | 2110
1111
$\gg s_1$

2 2 | 2210
2111
$\gg s_1$

1 1 | 3210
2211
$\gg s_1$

4210
s_2
3211
$\gg s_1$

Properties of the Algorithm

$a(x\!\downarrow) = 0\!\downarrow$: An n-variate system of homogeneous LDEs.

$(e_1\!\downarrow, \dots, e_n\!\downarrow)$: The canonical basis of $\mathbb{N}^n$.

$\mathcal{B}(a(x\!\downarrow) = 0\!\downarrow)$: Basis in the set of nontrivial natural solutions of $a(x\!\downarrow) = 0\!\downarrow$.

Theorem

- *The Contejean-Devie algorithm terminates on any input.*
- *Let* $(e_1\!\downarrow, \dots, e_n\!\downarrow), \varnothing \Longrightarrow^* \varnothing, M$ *be the sequence of transformations performed by the Contejean-Devie algorithm for* $a(x\!\downarrow) = 0\!\downarrow$*. Then*

$$\mathcal{B}(a(x\!\downarrow) = 0\!\downarrow) = M.$$

Notation

- $\|x{\downarrow}\| = \sqrt{x_1^2 + \cdots + x_n^2}$.
- $|(s_1, \ldots, s_n)| = s_1 + \cdots + s_n$.

Completeness

Theorem

Let $P_0, M_0 \Longrightarrow^* \varnothing, M$ *be the sequence of transformations performed by the Contejean-Devie algorithm for* $a(x{\downarrow}) = 0{\downarrow}$ *with* $P_0 = (e_1{\downarrow}, \dots, e_n{\downarrow})$ *and* $M_0 = \varnothing$*. Then* $\mathcal{B}(a(x{\downarrow}) = 0{\downarrow}) \subseteq M$*.*

Proof.

Assume $s{\downarrow} \in \mathcal{B}(a(x{\downarrow}) = 0{\downarrow})$ and show that there exists a sequence of vectors

$$v_1{\downarrow} = e_{j_0}{\downarrow} \ll \cdots \ll v_k{\downarrow} \ll v_{k+1}{\downarrow} = v_k{\downarrow} + e_{j_k}{\downarrow} \ll \cdots \ll v_{|s{\downarrow}|}{\downarrow} = s{\downarrow}$$

such that $v_i{\downarrow} \in P_{l_i}$, where P_{l_i} is from the given sequence of transformations and $l_i < l_j$ for $i < j$.

Completeness

Theorem

Let $P_0, M_0 \Longrightarrow^* \varnothing, M$ *be the sequence of transformations performed by the Contejean-Devie algorithm for* $a(x\downarrow) = 0\downarrow$ *with* $P_0 = (e_1\downarrow, \dots, e_n\downarrow)$ *and* $M_0 = \varnothing$*. Then* $\mathcal{B}(a(x\downarrow) = 0\downarrow) \subseteq M$.

Proof (cont.)

For $e_{j0}\downarrow$, any basic vector $\ll s\downarrow$ can be chosen. Such basic vectors do exist (since $s\downarrow \neq 0\downarrow$) and are in P_0. Assume now we have $v_1\downarrow \ll \dots \ll v_k\downarrow \ll s\downarrow$ with $v_k\downarrow \in P_{l_k}$. Then there exists $s_k\downarrow$ with $s\downarrow = v_k\downarrow + s_k\downarrow$ and $0 = \|a(s\downarrow)\|^2 = \|a(v_k\downarrow)\|^2 + \|a(s_k\downarrow)\|^2 + 2a(v_k\downarrow) \cdot a(s_k\downarrow)$, which implies $a(v_k\downarrow) \cdot a(s_k\downarrow) < 0$.

Completeness

Theorem

Let $P_0, M_0 \Longrightarrow^* \emptyset, M$ *be the sequence of transformations performed by the Contejean-Devie algorithm for* $a(x\!\downarrow) = 0\!\downarrow$ *with* $P_0 = (e_1\!\downarrow, \dots, e_n\!\downarrow)$ *and* $M_0 = \emptyset$*. Then* $\mathcal{B}(a(x\!\downarrow) = 0\!\downarrow) \subseteq M$.

Proof (cont.)

Hence, there exists $e_{j_k}\!\downarrow$ with $s_k\!\downarrow \gg e_{j_k}\!\downarrow$ such that $a(v_k\!\downarrow) \cdot a(e_{j_k}\!\downarrow) < 0$. We take $v_{k+1}\!\downarrow = v_k\!\downarrow + e_{j_k}\!\downarrow$. Then $s\!\downarrow \gg v_{k+1}\!\downarrow$ and by rule 3, $v_{k+1}\!\downarrow \in P_{l_{k+1}}$. After $|s\!\downarrow|$ steps, we reach s. Hence, $s\!\downarrow \in P_{l_{|s|}}$. Since $a(s\!\downarrow) = 0$, application of rule 2 moves $s\!\downarrow$ to M. □

Soundness

Theorem

Let $P_0, M_0 \Longrightarrow^* \varnothing, M$ *be the sequence of transformations performed by the Contejean-Devie algorithm for* $a(x{\downarrow}) = 0{\downarrow}$ *with* $P_0 = (e_1{\downarrow}, \dots, e_n{\downarrow})$ *and* $M_0 = \varnothing$*. Then* $M \subseteq \mathcal{B}(a(x{\downarrow}) = 0{\downarrow})$.

Proof.

Any $s{\downarrow} \in M$ is a solution. Show that it is minimal. Assume it is not: $s{\downarrow} = s_1{\downarrow} + s_2{\downarrow}$, where $s_1{\downarrow}$ and $s_2{\downarrow}$ are non-null solutions smaller than s. Assume $s{\downarrow}$ was obtained during the transformations as $s{\downarrow} = v_i{\downarrow} + e_{j_i}{\downarrow}$, where $v_i{\downarrow} \in P_i$. But then $v_i{\downarrow} \gg s_1{\downarrow}$ or $v_i{\downarrow} = s_1{\downarrow}$ or $v_i{\downarrow} \gg s_2{\downarrow}$ or $v_i{\downarrow} = s_1{\downarrow}$ and $v_i{\downarrow}$ is greater than an already computed minimal solution. Therefore, it should have been removed from P_i. A contradiction. □

Termination

Theorem

Let $v_1{\downarrow}, v_2{\downarrow}, \ldots$ *be an infinite sequence satisfying the Contejean-Devie condition for* $a(x{\downarrow}) = 0{\downarrow}$*:*

- u_1 *is a basic vector and for each* $i \geq 1$ *there exists* $1 \leq j \leq n$ *such that* $a(v_i{\downarrow}) \cdot a(e_j{\downarrow}) < 0$ *and* $v_{i+1}{\downarrow} = v_i{\downarrow} + e_j{\downarrow}$*.*

Then there exist $v{\downarrow}$ *and* k *such that*

- $v{\downarrow}$ *is a solution of* $a(x{\downarrow}) = 0{\downarrow}$*, and*
- $v{\downarrow} \ll v_k{\downarrow}$*.*

Non-Homogeneous Case

Non-homogeneous linear Diophantine system with m equations and n variables:

$$\begin{cases} a_{11}x_1 & +\cdots+ & a_{1n}x_n & = & b_1 \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & +\cdots+ & a_{mn}x_n & = & b_m \end{cases}$$

- a's and b's are integers.
- Matrix form: $a(x\downarrow) = b\downarrow$.

Non-Homogeneous Case. Solving Idea

Turn the system into a homogeneous one, denoted S_0:

$$\begin{cases} -b_1 x_0 & + & a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ -b_m x_0 & + & a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & 0 \end{cases}$$

- Solve S_0 and keep only the solutions with $x_0 \leq 1$.
- $x_0 = 1$: a minimal solution for $a(x\downarrow) = b\downarrow$.
- $x_0 = 0$: a minimal solution for $a(x\downarrow) = 0\downarrow$.
- Any solution of the non-homogeneous system $a(x\downarrow) = b\downarrow$ has the form $x\downarrow + y\downarrow$ where:
  - $x\downarrow$ is a minimal solution of $a(x\downarrow) = b\downarrow$.
  - $y\downarrow$ is a linear combination (with natural coefficients) of minimal solutions of $a(x\downarrow) = 0\downarrow$.

Back to ACU-Unification

Theorem

The decision problem for ACU-Matching and ACU-unification is NP-complete.

Specific vs General Results

For each specific equational theory separately studying

- decidability,
- unification type,
- unification algorithm/procedure.

Can one study these problems for bigger classes of equational theories?

General Results

In general, unification modulo equational theories

- is undecidable,
- unification type of a given theory is undecidable,
- admits a complete unification procedure
  (Gallier & Snyder, called an universal E-unification procedure).

General Results

Universal E-unification procedure $\mathcal{U}_E$.

Rules:

- **Trivial**, **Orient**, **Decomposition**, **Variable Elimination** from $\mathcal{U}$, plus
- **Lazy Paramodulation:**

$$\{e[u]\} \cup P'; S \Longrightarrow \{l \doteq^? u, e[r]\} \cup P'; S,$$

  for a fresh variant of the identity $l \approx r$ from $E \cup E^{-1}$, where
  - $e[u]$ is an equation where the term u occurs,
  - u is not a variable,
  - if l is not a variable, then the top symbol of l and u are the same.

General Results

Universal E-unification procedure. Control.

In order to solve a unification problem Γ modulo a given E:

- Create an initial system $\Gamma; \varnothing$.
- Apply successively rules from $\mathcal{U}_E$, building a complete tree of derivations.
- No other inference rule may be applied to the equation $l \doteq^? u$ that is generated by the Lazy Paramodulation rule before it is subjected to a Decomposition step.

General Results

Example

$E = \{f(a,b) \approx a, a \approx b\}$.

Unification problem: $\{f(x,x) \doteq^?_E x\}$.

Computing a unifier $\{x \mapsto a\}$ by the universal procedure:

$$\begin{aligned}
\{f(x,x) \doteq^?_E x\};\varnothing &\Longrightarrow_{LP} \{f(a,b) \doteq^?_E f(x,x), a \doteq^?_E x\};\varnothing \\
&\Longrightarrow_{D} \{a \doteq^?_E x, b \doteq^?_E x, a \doteq^?_E x\};\varnothing \\
&\Longrightarrow_{O} \{x \doteq^?_E a, b \doteq^?_E x, a \doteq^?_E x\};\varnothing \\
&\Longrightarrow_{S} \{b \doteq^?_E a, a \doteq^?_E a\};\{x \doteq a\} \\
&\Longrightarrow_{LP} \{a \doteq^?_E a, b \doteq^?_E b, a \doteq^?_E a\};\{x \doteq a\} \\
&\Longrightarrow^+_{T} \varnothing;\{x \doteq a\}
\end{aligned}$$

General Results

Pros and cons of the universal procedure:

- Pros: Is sound and complete. Can be used for any E.
- Cons: Very inefficient. Usually does not yield a decision procedure or a (minimal) E-unification algorithm even for unitary or finitary theories with decidable unification.

General Results

More useful results can be obtained by imposing additional restrictions on equational theories:

- Syntactic approaches: Restricting syntactic form of the identities defining equational theories.
- Semantic approaches: Depend on properties of the free algebras defined by the equational theory.

Summary

- Syntactic unification and matching.
  - Unification and matching algorithms.
  - Unification on term graphs, algorithms with improved complexity.
- Equational unification and matching
  - Classification with respect to unification type.
  - Algorithms for commutative and ACU-unification, including solving systems of linear Diophantine equations.
  - Universal E-unification procedure.

Summary

- Syntactic unification and matching.
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