MUG: Matching, Unification, Generalizations.
ISR 2012

Exercises

For the definitions and notation see the course materials. Some of the exarcises are taken or adapted from F. Baader and T. Nipkow, Term Rewriting and All That. Cambridge University Press, 1998.

1. Prove the elementary properties of substitutions:
   (a) Composition of substitutions is associative.
   (b) For all $\mathcal{X} \subseteq \mathcal{V}$, t and σ, if $vars(t) \subseteq \mathcal{X}$ then $t\sigma = t\sigma|_{\mathcal{X}}$.
   (c) For all σ, ϑ, and t, if $t\sigma = t\vartheta$ then $t\sigma|_{vars(t)} = t\vartheta|_{vars(t)}$
2. Prove that for any σ and ϑ, $\sigma \mathrel{\dot{=}} \vartheta$ iff there exists a variable renaming substitution η such that $\sigma\eta = \vartheta$.
3. Prove the following statements about idempotent substitutions:
   (a) A substitution σ is idempotent iff $dom(\sigma) \cap vran(\sigma) = \emptyset$.
   (b) If σ is an idempotent substitution and ϑ is an arbitrary substitution with the property $\vartheta \subseteq \sigma$, then $\vartheta \leq \sigma$. Find a counterexample for the case when σ is not idempotent.
   (c) Let σ be an idempotent mgu of a unification problem P. Prove that $\vartheta = \vartheta\sigma$ for any unifier ϑ of P.
4. Formulate a sufficient condition for the equality $\sigma\vartheta = \sigma \cup \vartheta$ to hold for any σ and ϑ.
5. Let σ_1 and σ_2 be two substitutions such that $\sigma_1 \leq \sigma_2$. Prove or find a counterexample to the following statements:
   (a) $\sigma_1\vartheta \leq \sigma_2\vartheta$ for any ϑ.
   (b) $\vartheta\sigma_1 \leq \vartheta\sigma_2$ for any ϑ.
6. How many unifiers do the following unification problems have (restricted to the set of variables of the problem)?
   (a) $\{x \doteq^? a\}$.
   (b) $\{x \doteq^? y\}$.
   (c) $\{x \doteq^? f(y)\}$.
   (d) $\{x \doteq^? f(x)\}$.
   (e) $\{f(x, x) \doteq^? f(a, b)\}$.
   (f) $\{f(x) \doteq^? g(x)\}$.
   (g) $\{x \doteq^? x\}$.

7. Let $\{x \doteq^? y\}$ be a unification problem. Which of the following substitutions are its mgu? Justify your answer.
   (a) $\{x \mapsto y\}$
   (b) $\{y \mapsto x\}$
   (c) $\{x \mapsto z, y \mapsto z\}$
   (d) $\{x \mapsto a, y \mapsto a\}$
   (e) $\{x \mapsto y, y \mapsto z, z \mapsto x\}$
   (f) $\{x \mapsto y, z \mapsto u, u \mapsto z\}$
   (g) $\{x \mapsto y, z \mapsto u, u \mapsto v, v \mapsto z\}$
8. Implement the inference system $\mathcal{U}$ in your favorite programming language.
9. Perform the steps the inference system $\mathcal{U}$ to the unification problems below:
   (a) $\{f(a, x, g(y, z)) \doteq^? f(z, y, x)$.
   (b) $\{f(x, y, z) \doteq^? f(h(u, u), h(x, x), h(y, y))\}$.
   (c) $\{f(x, f(y, f(a, a))) \doteq^? f(f(y, y), f(f(z, z), z))\}$.
10. Let P_1 and P_2 be unification problems. Show that if σ_1 is an mgu of P_1 and σ_2 is an mgu of $P_2\sigma_1$, then $\sigma_1\sigma_2$ is an mgu of $P_1 \cup P_2$.
11. Check if the following unification / matching problems are solvable:
   (a) $f(x, y) \doteq^? / \ll^? f(h(a), x)$.
   (b) $f(x, y) \doteq^? / \ll^? f(h(x), a)$.
   (c) $f(x, b) \doteq^? / \ll^? f(h(y), x)$.
   (d) $f(x, x) \doteq^? / \ll^? f(h(y), y)$.
12. Find a sufficient condition for the existence of an idempotent solution of a matching problem.
13. What is the space and time complexity om the matching algorithm $\mathcal{M}$.
14. What happens if you try to unify $f(x, f(x, x))$ and $f(f(x, f(x, x)), f(x, f(x, x)))$ without occurrence check?
15. Implement almost linear unification.
16. Solve the C-unification problem $f(x, y) \doteq^?_{\mathsf{C}} f(z, f(z, z))$. What is the cardinality of the minimal complete set of unifiers?
17. Complete all the steps omitted in the slides to solve the ACU-unification problem with constants $\{f(x, f(x, y)) \doteq^?_{\mathsf{ACU}} f(a, f(z, f(z, z)))\}$.
18. Solve the elementary ACU-unification problem $\{f(x, y) \doteq^?_{\mathsf{ACU}} f(z, u)\}$.
19. Prove NP-completeness of the decision problem of C-unification with constants.
20. Prove NP-hardness of the decision problem of ACU-unification with constants.