

# Unification

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- ▶ Method: Replace certain subexpressions (variables) by other expressions.



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(Syntax: variables, function symbols, terms, etc.)
- ▶ The *substitution*  $\{x \mapsto b, y \mapsto a\}$  *unifies* the *terms*  $f(x, a)$  and  $f(b, y)$ .
- ▶ Solving the equation  $f(x, a) = f(b, y)$  for  $x$  and  $y$ .



# What is Unification

- ▶ Goal of unification: Identify two symbolic expressions.
- ▶ Method: Replace certain subexpressions (variables) by other expressions.

Depending what is meant under "identify" (syntactic identity or equality modulo some equations) one speaks about *syntactic unification* or *equational unification*.

## Example (Equational Unification)

- ▶ The terms  $f(x, a)$  and  $g(a, x)$  are not syntactically unifiable.
- ▶ However, they are unifiable modulo the equation  $f(a, a) = g(a, a)$  with the substitution  $\{x \mapsto a\}$ .



# What is Unification

- ▶ Goal of unification: Identify two symbolic expressions.
- ▶ Method: Replace certain subexpressions (variables) by other expressions.

Depending at which positions the variables are allowed to occur, and which kind of expressions they are allowed to be replaced by, one speaks about *first-order unification* or *higher-order unification*.

## Example (Higher-Order Unification)

- ▶ If  $G$  and  $x$  are variables, the terms  $f(x, a)$  and  $G(a, x)$  can not be subjected to first-order unification.
- ▶  $G(a, x)$  is not a first-order term:  $G$  occurs in the top position.
- ▶ However,  $f(x, a)$  and  $G(a, x)$  can be unified by higher-order unification with the substitution  $\{x \mapsto a, G \mapsto f\}$ .





# In this Lecture

First-order syntactic unification:

- ▶ Recursive Descent Algorithm
- ▶ Improved algorithms

# What is Unification Good For?

- ▶ To make an inference step in theorem proving.
- ▶ To perform an inference in logic programming.
- ▶ To make a rewriting step in term rewriting.
- ▶ To generate a critical pair in completion.
- ▶ To extract a part from structured or semistructured data (e.g. from an XML document).
- ▶ For type inference in programming languages.
- ▶ For matching in pattern-based languages.
- ▶ For program schemas manipulation and software engineering.
- ▶ For various formalisms in computational linguistics.
- ▶ etc.



# Literature

## Survey papers:



[F. Baader and W. Snyder. Unification Theory.](#)

In A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*, pages 447–533. Elsevier, 2001.



[F. Baader and J. Siekmann. Unification Theory.](#)

In D. Gabbay, C. Hogger and A. Robinson, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming*, Oxford University Press, 1994.



[J.-P. Jouannaud, C. Kirchner. Solving Equations in Abstract Algebras. A Rule-Based Survey on Unification.](#)

In J.-L. Lassez and G. Plotkin, editors. *Computational Logic. Essays in Honor of A. Robinson*. MIT Press, 1991.







[K. Knight. Unification: A Multidisciplinary Survey.](#)

ACM Computing Surveys, 21(1), 1989.



# Literature

Some other useful sources:

-  F. Baader and T. Nipkow. *Term Rewriting and All That*. Cambridge University Press, 1998.
-  C. Kirchner (editor). *Unification*. Academic Press, London, 1990.
-  C. Kirchner and H. Kirchner. *Rewriting, Solving, Proving*. Online version from 2006.
-  J.-L. Lassez, M. Maher, and K. Marriott. *Unification revisited*.  
In J. Minker, editor, *Foundations of Deductive Databases and Logic Programming*, pages 587-625. Morgan Kaufman, 1987.

Slides are available from

<http://www.risc.jku.at/~tkutsia/tbilisi2011.html>



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- 1965:** Alan Robinson introduced unification as the basic operation of his resolution principle, and gave a formal account of an algorithm that computes a most general unifier for first-order terms.
- 1972:** Gordon Plotkin introduced equational unification.
- 1972:** Gerard Huet and Claudio Lucchesi showed undecidability of higher-order unification.



# Terms

## Alphabet:

- ▶ A set of fixed arity function symbols  $\mathcal{F}$ .
- ▶ A countable set of variables  $\mathcal{V}$ .
- ▶  $\mathcal{F}$  and  $\mathcal{V}$  are disjoint.

## Terms over $\mathcal{F}$ and $\mathcal{V}$ :

$$t ::= x \mid f(t_1, \dots, t_n),$$

where

- ▶  $n \geq 0$ ,
- ▶  $x$  is a variable,
- ▶  $f$  is an  $n$ -ary function symbol.



# Terms

## Conventions, notation:

- ▶ Constants: 0-ary function symbols.
- ▶  $x, y, z$  denote variables.
- ▶  $a, b, c$  denote constants.
- ▶  $f, g, h$  denote arbitrary function symbols.
- ▶  $s, t, r$  denote terms.
- ▶ Parentheses omitted in terms with the empty list of arguments:  $a$  instead of  $a()$ .



# Terms

## Conventions, notation:

- ▶  $\mathcal{T}(\mathcal{F}, \mathcal{V})$ : the set of terms over  $\mathcal{F}$  and  $\mathcal{V}$ .
- ▶ Ground terms: terms without variables.
- ▶  $\mathcal{T}(\mathcal{F})$ : the set of ground terms over  $\mathcal{F}$ .
- ▶ Equation: a pair of terms, written  $s \doteq t$ .
- ▶  $\text{vars}(t)$ : the set of variables in  $t$ . This notation will be used also for sets of terms, equations, and sets of equations.



# Terms

## Example

- ▶  $f(x, g(x, a), y)$  is a term, where  $f$  is ternary,  $g$  is binary,  $a$  is a constant.
- ▶  $\text{vars}(f(x, g(x, a), y)) = \{x, y\}$ .
- ▶  $f(b, g(b, a), c)$  is a ground term.
- ▶  $\text{vars}(f(b, g(b, a), c)) = \emptyset$ .



# Substitutions

## Substitution

- ▶ A mapping from variables to terms, where all but finitely many variables are mapped to themselves.

## Example

A substitution is represented as a set of *bindings*:

- ▶  $\{x \mapsto f(a, b), y \mapsto z\}$ .
- ▶  $\{x \mapsto f(x, y), y \mapsto f(x, y)\}$ .

All variables except  $x$  and  $y$  are mapped to themselves by these substitutions.

## Notation

- ▶  $\sigma, \vartheta, \eta, \rho$  denote arbitrary substitutions.
- ▶  $\varepsilon$  denotes the identity substitution.



# Substitutions

## Substitution Application

Applying a substitution  $\sigma$  to a term  $t$ :

$$t\sigma = \begin{cases} \sigma(x) & \text{if } t = x \\ f(t_1\sigma, \dots, t_n\sigma) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

## Example

- ▶  $\sigma = \{x \mapsto f(x, y), y \mapsto g(a)\}$ .
- ▶  $t = f(x, g(f(x, f(y, z))))$ .
- ▶  $t\sigma = f(f(x, y), g(f(f(x, y), f(g(a), z))))$ .



# Substitutions

## Domain, Range, Variable Range

For a substitution  $\sigma$ :

- ▶ The *domain* is the set of variables:

$$\text{dom}(\sigma) = \{x \mid x\sigma \neq x\}.$$

- ▶ The *range* is the set of terms:

$$\text{ran}(\sigma) = \bigcup_{x \in \text{dom}(\sigma)} \{x\sigma\}.$$

- ▶ The *variable range* is the set of variables:

$$\text{vran}(\sigma) = \text{vars}(\text{ran}(\sigma)).$$





# Substitutions

## Example (Domain, Range, Variable Range)

$$\text{dom}(\{x \mapsto f(a, y), y \mapsto g(z)\}) = \{x, y\}$$

$$\text{ran}(\{x \mapsto f(a, y), y \mapsto g(z)\}) = \{f(a, y), g(z)\}$$

$$\text{vran}(\{x \mapsto f(a, y), y \mapsto g(z)\}) = \{y, z\}$$

$$\text{dom}(\{x \mapsto f(a, b), y \mapsto g(c)\}) = \{x, y\}$$

$$\text{ran}(\{x \mapsto f(a, b), y \mapsto g(c)\}) = \{f(a, b), g(c)\}$$

$$\text{vran}(\{x \mapsto f(a, b), y \mapsto g(c)\}) = \emptyset \text{ (ground substitution)}$$

$$\text{dom}(\varepsilon) = \emptyset$$

$$\text{ran}(\varepsilon) = \emptyset$$

$$\text{vran}(\varepsilon) = \emptyset$$





# Substitutions

## Composition of Substitutions

- ▶ Written:  $\sigma\vartheta$ .
- ▶  $t(\sigma\vartheta) = (t\sigma)\vartheta$ .
- ▶ Informal algorithm for constructing the representation of the composition  $\sigma\vartheta$ :
  1.  $\sigma$  and  $\vartheta$  are given by their representation.
  2. Apply  $\vartheta$  to every term in  $\text{ran}(\sigma)$  to obtain  $\sigma_1$ .
  3. Remove from  $\vartheta$  any binding  $x \mapsto t$  with  $x \in \text{dom}(\sigma)$  to obtain  $\vartheta_1$ .
  4. Remove from  $\sigma_1$  any trivial binding  $x \mapsto x$  to obtain  $\sigma_2$ .
  5. Take the union of the sets of bindings  $\sigma_2$  and  $\vartheta_1$ .

▶ Jump to RDA



# Substitutions

## Example (Composition)

1.  $\sigma = \{x \mapsto f(y), y \mapsto z\}$   
 $\vartheta = \{x \mapsto a, y \mapsto b, z \mapsto y\}$
2.  $\sigma_1 = \{x \mapsto f(y)\vartheta, y \mapsto z\vartheta\} = \{x \mapsto f(b), y \mapsto y\}$
3.  $\vartheta_1 = \{z \mapsto y\}$
4.  $\sigma_2 = \{x \mapsto f(b)\}$
5.  $\sigma\vartheta = \{x \mapsto f(b), z \mapsto y\}$

Composition is not commutative:

$$\vartheta\sigma = \{x \mapsto a, y \mapsto b\} \neq \sigma\vartheta.$$



# Substitutions

## Elementary Properties of Substitutions

### Theorem

- ▶ *Composition of substitutions is associative.*
- ▶ *For all  $\mathcal{X} \subseteq \mathcal{V}$ ,  $t$  and  $\sigma$ , if  $\text{vars}(t) \subseteq \mathcal{X}$  then  $t\sigma = t\sigma|_{\mathcal{X}}$ .*
- ▶ *For all  $\sigma$ ,  $\vartheta$ , and  $t$ , if  $t\sigma = t\vartheta$  then  $t\sigma|_{\text{vars}(t)} = t\vartheta|_{\text{vars}(t)}$*

Proof.

Exercise.



# Substitutions

## Triangular Form

Sequential list of bindings:

$$[x_1 \mapsto t_1; x_2 \mapsto t_2; \dots; x_n \mapsto t_n],$$

represents composition of  $n$  substitutions each consisting of a single binding:

$$\{x_1 \mapsto t_1\} \{x_2 \mapsto t_2\} \dots \{x_n \mapsto t_n\}.$$



# Substitutions

## Variable Renaming, Inverse

A substitution  $\sigma = \{x_1 \mapsto y_1, x_2 \mapsto y_2, \dots, x_n \mapsto y_n\}$  is called *variable renaming* iff

- ▶  $y$ 's are distinct variables, and
- ▶  $\{x_1, \dots, x_n\} = \{y_1, \dots, y_n\}$ .

The *inverse* of  $\sigma$ , denoted  $\sigma^{-1}$ , is the substitution

$$\sigma^{-1} = \{y_1 \mapsto x_1, y_2 \mapsto x_2, \dots, y_n \mapsto x_n\}$$

## Example

- ▶  $\{x \mapsto y, y \mapsto z, z \mapsto x\}$  is a variable renaming.
- ▶  $\{x \mapsto a\}$ ,  $\{x \mapsto y\}$ , and  $\{x \mapsto z, y \mapsto z\}$  are not.



# Substitutions

## Idempotent Substitution

A substitution  $\sigma$  is *idempotent* iff  $\sigma\sigma = \sigma$ .

### Example

Let  $\sigma = \{x \mapsto f(z), y \mapsto z\}$ ,  $\vartheta = \{x \mapsto f(y), y \mapsto z\}$ .

- ▶  $\sigma$  is idempotent.
- ▶  $\vartheta$  is not:  $\vartheta\vartheta = \sigma \neq \vartheta$ .

## Theorem

$\sigma$  is idempotent iff  $\text{dom}(\sigma) \cap \text{vran}(\sigma) = \emptyset$ .

## Proof.

Exercise. □





# Substitutions

## Instantiation Quasi-Ordering

- ▶ A substitution  $\sigma$  is *more general* than  $\vartheta$ , written  $\sigma \leq \vartheta$ , if there exists  $\eta$  such that  $\sigma\eta = \vartheta$ .
- ▶ The relation  $\leq$  is quasi-ordering (reflexive and transitive binary relation), called *instantiation quasi-ordering*.
- ▶  $\equiv$  is the equivalence relation corresponding to  $\leq$ .

## Example

Let  $\sigma = \{x \mapsto y\}$ ,  $\rho = \{x \mapsto a, y \mapsto a\}$ ,  $\vartheta = \{y \mapsto x\}$ .

- ▶  $\sigma \leq \rho$ , because  $\sigma\{y \mapsto a\} = \rho$ .
- ▶  $\sigma \leq \vartheta$ , because  $\sigma\{y \mapsto x\} = \vartheta$ .
- ▶  $\vartheta \leq \sigma$ , because  $\vartheta\{x \mapsto y\} = \sigma$ .
- ▶  $\sigma \equiv \vartheta$ .



# Substitutions

## Theorem

*For any  $\sigma$  and  $\vartheta$ ,  $\sigma = \vartheta$  iff there exists a variable renaming substitution  $\eta$  such that  $\sigma\eta = \vartheta$ .*

## Proof.

Exercise. □

## Example

$\sigma, \vartheta$  from the previous example:

- ▶  $\sigma = \{x \mapsto y\}$ .
- ▶  $\vartheta = \{y \mapsto x\}$ .
- ▶  $\sigma = \vartheta$ .
- ▶  $\sigma\{x \mapsto y, y \mapsto x\} = \vartheta$ .



# Substitutions

## Unifier, Most General Unifier

- ▶ A substitution  $\sigma$  is a *unifier* of the terms  $s$  and  $t$  if  $s\sigma = t\sigma$ .
- ▶ A unifier  $\sigma$  of  $s$  and  $t$  is a *most general unifier (mgu)* if  $\sigma \leq \vartheta$  for every unifier  $\vartheta$  of  $s$  and  $t$ .
- ▶ A unification problem for  $s$  and  $t$  is represented as  $s \doteq? t$ .



# Substitutions

## Example (Unifier, Most General Unifier)

Unification problem:  $f(x, z) \doteq? f(y, g(a))$ .

- Some of the unifiers:

$$\{x \mapsto y, z \mapsto g(a)\}$$

$$\{y \mapsto x, z \mapsto g(a)\}$$

$$\{x \mapsto a, y \mapsto a, z \mapsto g(a)\}$$

$$\{x \mapsto g(a), y \mapsto g(a), z \mapsto g(a)\}$$

$$\{x \mapsto f(x, y), y \mapsto f(x, y), z \mapsto g(a)\}$$

...

- mgu's:  $\{x \mapsto y, z \mapsto g(a)\}$ ,  $\{y \mapsto x, z \mapsto g(a)\}$ .
- mgu is unique up to a variable renaming:

$$\{x \mapsto y, z \mapsto g(a)\} \doteq \{y \mapsto x, z \mapsto g(a)\}$$



# Unification Algorithm

- ▶ Goal: Design an algorithm that for a given unification problem  $s \doteq? t$ 
  - ▶ returns an mgu of  $s$  and  $t$  if they are unifiable,
  - ▶ reports failure otherwise.

# Naive Algorithm

Write down two terms and set markers at the beginning of the terms. Then:

1. Move the markers simultaneously, one symbol at a time, until both move off the end of the term (**success**), or until they point to two different symbols;
2. If the two symbols are both non-variables, then **fail**;  
otherwise, one is a variable (call it  $x$ ) and the other one is the first symbol of a subterm (call it  $t$ ):
  - ▶ If  $x$  occurs in  $t$ , then **fail**;
  - ▶ Otherwise, replace  $x$  everywhere by  $t$  (including in the solution), write down " $x \mapsto t$ " as a part of the solution, and return to 1.



# Naive Algorithm

- ▶ Finds disagreements in the two terms to be unified.
- ▶ Attempts to repair the disagreements by binding variables to terms.
- ▶ Fails when function symbols clash, or when an attempt is made to unify a variable with a term containing that variable.



# Example

$$f(x, g(a), g(z))$$



$$f(g(y), g(y), g(g(x)))$$





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$$f(g(y), g(a), g(z))$$



$$f(g(y), g(y), g(g(g(y))))$$



# Example

$f(g(y), g(a), g(z))$



$f(g(y), g(y), g(g(g(y))))$



$\{x \mapsto g(y)\}$



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$$\{x \mapsto g(y)\}$$

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$$f(g(a), g(a), g(z))$$

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$$\{x \mapsto g(a)\}$$




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$$f(g(a), g(a), g(z))$$



$$f(g(a), g(a), g(g(a)))$$



$$\{x \mapsto g(a), y \mapsto a\}$$



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$f(g(a), g(a), g(z))$



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# Example

$$f(g(a), g(a), g(g(g(a))))$$

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# Interesting Questions

## Implementation:

- ▶ What data structures should be used for terms and substitutions?
- ▶ How should application of a substitution be implemented?
- ▶ What order should the operations be performed in?

## Correctness:

- ▶ Does the algorithm always terminate?
- ▶ Does it always produce an mgu for two unifiable terms, and fail for non-unifiable terms?
- ▶ Do these answers depend on the order of operations?

## Complexity:

- ▶ How much space does this take, and how much time?





# Answers

On the coming slides, for various unification algorithms.

# Implementation: Unification by Recursive Descent

Implementation of the naive algorithm:

- ▶ Term representation: either by explicit pointer structures or by built-in recursive data types (depending on the implementation language).
- ▶ Substitution representation: a list of pairs of terms.
- ▶ Application of a substitution: constructing a new term or replacing a variable with a new term.
- ▶ The left-to-right search for disagreements: implemented by recursive descent through the terms.



## Example

The Recursive Descent Algorithm we are going to describe will correspond to a slightly modified version of the naive algorithm:

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$$f(g(y), g(y), g(g(x)))$$



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## Example

The Recursive Descent Algorithm we are going to describe will correspond to a slightly modified version of the naive algorithm:

$$f(g(y), g(a), g(g(a)))$$



$$f(g(y), g(a), g(g(a)))$$



$$\{x \mapsto g(a), y \mapsto a, z \mapsto g(g(a))\}$$



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# Unification by Recursive Descent

**Input:** Terms  $s$  and  $t$

**Output:** An mgu of  $s$  and  $t$

**Global:** Substitution  $\sigma$ . Initialized to  $\varepsilon$

Unify ( $s,t$ )

**begin**

**if**  $s$  is a variable **then**  $s := s\sigma$ ;  $t := t\sigma$ ;

Print( $s$ , '  $\doteq$  ',  $t$ , '  $\sigma =$  ',  $\sigma$ );

**if**  $s$  is a variable **and**  $s = t$  **then** Do nothing;

**else if**  $s = f(s_1, \dots, s_n)$  **and**  $t = g(t_1, \dots, t_m)$ ,  $n, m \geq 0$  **then**

**if**  $f = g$  **then for**  $i := 1$  **to**  $n$  **do** Unify ( $s_i, t_i$ );

**else** Exit with failure;

**else if**  $s$  is not a variable **then** Unify ( $t, s$ );

**else if**  $s$  occurs in  $t$  **then** Exit with failure;

**else**  $\sigma := \sigma\{s \mapsto t\}$ ;

**end**

**Algorithm 1:** Recursive descent algorithm



# Recursive Descent Algorithm

- ▶ Implementation of substitution composition: Without the steps 3 and 4 of the composition algorithm. [▶ Jump to composition](#)
- ▶ Reason: When a binding  $x \mapsto t$  created and applied,  $x$  does not appear in the terms anymore.

The Recursive Descent Algorithm is essentially the Robinson's Unification Algorithm.



## Example

$$s = f(x, g(a), g(z)), t = f(g(y), g(y), g(g(x))), \sigma = \varepsilon.$$

Printing outputs are given in *blue*.

---

$$\text{Unify}(f(x, g(a), g(z)), f(g(y), g(y), g(g(x))))$$

$$f(x, g(a), g(z)) \doteq? f(g(y), g(y), g(g(x))), \sigma = \varepsilon$$

$$\text{Unify}(x, g(y))$$

$$x \doteq? g(y), \sigma = \varepsilon$$

$$\text{Unify}(g(a), g(y))$$

$$g(a) \doteq? g(y), \sigma = \{x \mapsto g(y)\}$$

Continues on the next slide.



## Example (Cont.)

$Unify(a, y)$

$$a \doteq? y, \sigma = \{x \mapsto g(y)\}$$

$Unify(y, a)$

$$y \doteq? a, \sigma = \{x \mapsto g(y)\}$$

$Unify(g(z), g(g(x)))$

$$g(z) \doteq? g(g(x)), \sigma = \{x \mapsto g(a), y \mapsto a\}$$

$Unify(z, g(x))$

$$z \doteq? g(g(a)), \sigma = \{x \mapsto g(a), y \mapsto a\}$$

---

**Result:**  $\sigma = \{x \mapsto g(a), y \mapsto a, z \mapsto g(g(a))\}$



# Properties of Recursive Descent Algorithm

- ▶ Goal: Prove logical properties of the Recursive Descent Algorithm.
- ▶ Method (rule-based approach):
  1. Describe an inference system for deriving solutions for unification problems.
  2. Show that the inference system simulates the actions of the Recursive Descent Algorithm.
  3. Prove logical properties of the inference system.



# The Inference System $\mathcal{U}$

- ▶ A set of equations in *solved form*:

$$\{x_1 \doteq t_1, \dots, x_n \doteq t_n\}$$

where each  $x_i$  occurs exactly once.

- ▶ For each idempotent substitution there exists exactly one set of equations in solved form.
- ▶ Notation:
  - ▶  $[\sigma]$  for the solved form set for an idempotent substitution  $\sigma$ .
  - ▶  $\sigma_S$  for the idempotent substitution corresponding to a solved form set  $S$ .





# The Inference System $\mathcal{U}$

- ▶ *System*: The symbol  $\perp$  or a pair  $P; S$  where
  - ▶  $P$  is a multiset of unification problems,
  - ▶  $S$  is a set of equations in solved form.
- ▶  $\perp$  represents failure.
- ▶ A unifier (or a solution) of a system  $P; S$ : A substitution that unifies each of the equations in  $P$  and  $S$ .
- ▶  $\perp$  has no unifiers.





# The Inference System $\mathcal{U}$

Six transformation rules on systems:<sup>1</sup>

**Trivial:**  $\{s \doteq^? s\} \cup P'; S \Longrightarrow P'; S.$

**Decomposition:**  $\{f(s_1, \dots, s_n) \doteq^? f(t_1, \dots, t_n)\} \cup P'; S \Longrightarrow$   
 $\{s_1 \doteq^? t_1, \dots, s_n \doteq^? t_n\} \cup P'; S,$   
where  $n \geq 0$ .

**Symbol Clash:**  $\{f(s_1, \dots, s_n) \doteq^? g(t_1, \dots, t_m)\} \cup P'; S \Longrightarrow \perp$   
if  $f \neq g$ .

---

<sup>1</sup> $\cup$  when applied to  $P$  is a multiset union.



# The Inference System $\mathcal{U}$

**Orient:**  $\{t \doteq^? x\} \cup P'; S \implies \{x \doteq^? t\} \cup P'; S,$   
if  $t$  is not a variable.

**Occurs Check:**  $\{x \doteq^? t\} \cup P'; S \implies \perp$   
if  $x \in \text{vars}(t)$  but  $x \neq t$ .

**Variable Elimination:**  $\{x \doteq^? t\} \cup P'; S \implies$   
 $P'\{x \mapsto t\}; S\{x \mapsto t\} \cup \{x \doteq t\},$   
if  $x \notin \text{vars}(t)$ .



# Unification with $\mathcal{U}$

In order to unify  $s$  and  $t$ :

1. Create an initial system  $\{s \stackrel{?}{=} t\}; \emptyset$ .
2. Apply successively rules from  $\mathcal{U}$ .

The system  $\mathcal{U}$  is essentially the Herbrand's Unification Algorithm.



# Simulating the Recursive Descent Algorithm by $\mathcal{U}$

$s$ ,  $t$ ,  $\sigma$  when printed in the Recursive Descent Algorithm:

$$\begin{array}{l} s_1 \quad t_1 \quad \varepsilon \\ s_2 \quad t_2 \quad \sigma_2 \\ s_3 \quad t_3 \quad \sigma_3 \\ \dots \end{array}$$

Can be simulated by the sequence of transformations:

$$\begin{array}{l} \{s_1 \stackrel{?}{=} t_1\}; \emptyset \\ \implies \{s_2 \stackrel{?}{=} t_2\} \cup P_2; S_2 \\ \implies \{s_3 \stackrel{?}{=} t_3\} \cup P_3; S_3 \\ \dots \end{array}$$

where  $s_i \stackrel{?}{=} t_i$  is the equation acted on by a rule, and  $\sigma_i$  is  $\sigma_{S_i}$ .



# Simulating the Recursive Descent Algorithm by $\mathcal{U}$

Furthermore:

- ▶ If the call to Unify in RDA ends in failure, then the transformation sequence ends in  $\perp$ .
- ▶ If the call to Unify in RDA terminates with success, with a global substitution  $\sigma_n$ , then the transformation sequence ends in  $\emptyset; S$  where  $\sigma_S = \sigma_n$ .



# Simulating the Recursive Descent Algorithm by $\mathcal{U}$

Furthermore:

- ▶ If the call to Unify in RDA ends in failure, then the transformation sequence ends in  $\perp$ .
- ▶ If the call to Unify in RDA terminates with success, with a global substitution  $\sigma_n$ , then the transformation sequence ends in  $\emptyset; S$  where  $\sigma_S = \sigma_n$ .

This simulation can be achieved by

- ▶ treating  $P$  as a stack,
- ▶ always applying the rule to the top equation,
- ▶ only using **Trivial** when  $s$  is a variable.

There is only one rule applicable at each step under this control.

$\mathcal{U}$  — an abstract version of RDA.





# Properties of $\mathcal{U}$ : Termination

## Lemma

*For any finite multiset of equations  $P$ , every sequence of transformations in  $\mathcal{U}$*

$$P; \emptyset \Longrightarrow P_1; \sigma_1 \Longrightarrow P_2; \sigma_2 \Longrightarrow \dots$$

*terminates either with  $\perp$  or with  $\emptyset; S$ , with  $S$  in solved form.*



# Properties of $\mathcal{U}$ : Termination

## Proof.

Complexity measure on the multisets of equations:  $\langle n_1, n_2, n_3 \rangle$ , ordered lexicographically on triples of naturals, where

$n_1$  = The number of distinct variables in  $P$ .

$n_2$  = The number of symbols in  $P$ .

$n_3$  = The number of equations in  $P$  of the form  $t \doteq^? x$  where  $t$  is not a variable.

Each rule in  $\mathcal{U}$  reduces the complexity measure.



# Properties of $\mathcal{U}$ : Termination

## Proof [Cont.]

- ▶ A rule can always be applied to a system with non-empty  $P$ .
- ▶ The only systems to which no rule can be applied are  $\perp$  and  $\emptyset; S$ .
- ▶ Whenever an equation is added to  $S$ , the variable on the left-hand side is eliminated from the rest of the system, i.e.  $S_1, S_2, \dots$  are in solved form.



## Corollary

If  $P; \emptyset \Longrightarrow^+ \emptyset; S$  then  $\sigma_S$  is idempotent.



# Properties of $\mathcal{U}$ : Correctness

Notation:  $\Gamma$  for systems.

## Lemma

*For any transformation  $P; S \implies \Gamma$ , a substitution  $\vartheta$  unifies  $P; S$  iff it unifies  $\Gamma$ .*



# Properties of $\mathcal{U}$ : Correctness

Proof.

**Occurs Check:** If  $x \in \text{vars}(t)$  and  $x \neq t$ , then

- ▶  $x$  contains fewer symbols than  $t$ ,
- ▶  $x\vartheta$  contains fewer symbols than  $t\vartheta$  (for any  $\vartheta$ ).

Therefore,  $x\vartheta$  and  $t\vartheta$  can not be unified.

**Variable Elimination:** From  $x\vartheta = t\vartheta$ , by structural induction on  $u$ :

$$u\vartheta = u\{x \mapsto t\}\vartheta$$

for any term, equation, or multiset of equations  $u$ . Then

$$P'\vartheta = P'\{x \mapsto t\}\vartheta, \quad S\vartheta = S\{x \mapsto t\}\vartheta.$$



# Properties of $\mathcal{U}$ : Correctness

## Theorem (Soundness)

*If  $P; \emptyset \Longrightarrow^+ \emptyset; S$ , the  $\sigma_S$  unifies any equation in  $P$ .*

## Proof.

$\sigma_S$  unifies  $S$ . Induction using the previous lemma finishes the proof. □



# Properties of $\mathcal{U}$ : Correctness

## Theorem (Completeness)

*If  $\vartheta$  unifies every equation in  $P$ , then any maximal sequence of transformations  $P; \emptyset \Longrightarrow \dots$  ends in a system  $\emptyset; S$  such that  $\sigma_S \leq \vartheta$ .*

### Proof.

Such a sequence must end in  $\emptyset; S$  where  $\vartheta$  unifies  $S$  (why?). For every binding  $x \mapsto t$  in  $\sigma_S$ ,  $x\sigma_S\vartheta = t\vartheta = x\vartheta$  and for every  $x \notin \text{dom}(\sigma_S)$ ,  $x\sigma_S\vartheta = x\vartheta$ . Hence,  $\vartheta = \sigma_S\vartheta$ . □

### Corollary

*If  $P$  has no unifiers, then any maximal sequence of transformations from  $P; \emptyset$  must have the form  $P; \emptyset \Longrightarrow \dots \Longrightarrow \perp$ .*



# Observations

- ▶ The choice of rules in computations via  $\mathcal{U}$  is “don’t care” nondeterminism (the word “any” in Completeness Theorem).
- ▶ Any control strategy will result to an mgu for unifiable terms, and failure for non-unifiable terms.
- ▶ Any practical algorithm that proceeds by performing transformations of  $\mathcal{U}$  in any order is
  - ▶ sound and complete,
  - ▶ generates mgus for unifiable terms.
- ▶ Not all transformation sequences have the same length.
- ▶ Not all transformation sequences end in exactly the same mgu.





# Observations

- ▶ Any substitution generated by  $\mathcal{U}$  is a compact representation of the (infinite) set of all unifiers.
- ▶ The unifiers can be generated by composing all the possible substitutions with the mgu.
- ▶ Any two mgu's of a given pair of terms are instances of each other.
- ▶ The mgu's can be obtained from a single mgu by composition with variable renaming.
- ▶ By this operation it is possible to create an infinite number of mgu's.
- ▶ The finite search tree for  $\mathcal{U}$  is not able to produce every idempotent mgu.



# Complexity of Recursive Descent

Can take exponential time and space.

## Example

Let

$$s = h(x_1, x_2, \dots, x_n, f(y_0, y_0), f(y_1, y_1), \dots, f(y_{n-1}, y_{n-1}), y_n)$$

$$t = h(f(x_0, x_0), f(x_1, x_1), \dots, f(x_{n-1}, x_{n-1}), y_1, y_2, \dots, y_n, x_n)$$

Unifying  $s$  and  $t$  will create an mgu where each  $x_i$  and each  $y_i$  is bound to a term with  $2^{i+1} - 1$  symbols:

$$\{x_1 \mapsto f(x_0, x_0), x_2 \mapsto f(f(x_0, x_0), f(x_0, x_0)), \dots, \\ y_0 \mapsto x_0, y_1 \mapsto f(x_0, x_0), y_2 \mapsto f(f(x_0, x_0), f(x_0, x_0)), \dots\}$$

Can we do better?



# Complexity of Recursive Descent

First idea: Use triangular substitutions.

## Example

Triangular unifier of  $s$  and  $t$  from the previous example:

$$[y_0 \mapsto x_0; y_n \mapsto f(y_{n-1}, y_{n-1}); y_{n-1} \mapsto f(y_{n-2}, y_{n-2}); \dots]$$

- ▶ Triangular unifiers are not larger than the original problem.
- ▶ However, it is not enough to rescue the algorithm:
  - ▶ Substitutions have to be applied to terms in the problem, that leads to duplication of subterms.
  - ▶ In the example, calling Unify on  $x_n$  and  $y_n$ , which by then are bound to terms with  $2^{n+1} - 1$  symbols, will lead to exponential number of recursive calls.



# How to Speed up Unification?

Develop

- (a) more subtle data structures for terms.
- (b) a different method for applying substitutions.

Details to come.