Constraint solving over multiple similarity relations

Besik Dundua
FBT, International Black Sea University, Tbilisi, Georgia
VIAM, Ivane Javakhishvili Tbilisi State University, Georgia
bdundua@gmail.com

Temur Kutsia
Johannes Kepler University, Research Institute for Symbolic Computation, Linz, Austria
http://www.risc.jku.at/people/tkutsia/
kutsia@risc.jku.at

Mircea Marin
West University of Timișoara, Romania
http://staff.fmi.uvt.ro/~mircea.marin/
mircea.marin@e-uvt.ro

Cleopatra Pau
Johannes Kepler University, Research Institute for Symbolic Computation, Linz, Austria
ipau@risc.jku.at

Abstract
Similarity relations are reflexive, symmetric, and transitive fuzzy relations. They help to make approximate inferences, replacing the notion of equality. Similarity-based unification has been quite intensively investigated, as a core computational method for approximate reasoning and declarative programming. In this paper we consider solving constraints over several similarity relations, instead of a single one. Multiple similarities pose challenges to constraint solving, since we can not rely on the transitivity property anymore. Existing methods for unification with fuzzy proximity relations (reflexive, symmetric, non-transitive relations) do not provide a solution that would adequately reflect particularities of dealing with multiple similarities. To address this problem, we develop a constraint solving algorithm for multiple similarity relations, prove its termination, soundness, and completeness properties, and discuss applications.

2012 ACM Subject Classification Theory of computation → Logic; Computing methodologies → Symbolic and algebraic manipulation; Theory of computation → Semantics and reasoning.

Keywords and phrases Fuzzy relations, similarity, constraint solving.

Acknowledgements This research has been partially supported by the Austrian Science Fund (FWF) under the project 28789-N32 and the Shota Rustaveli National Science Foundation of Georgia under the grant YS-18-1480.

1 Introduction
Reasoning with incomplete, imperfect information is very common in human communication. Its modeling is a highly nontrivial task, and remains an important issue in applications of artificial intelligence. There are various notions associated to such information (e.g., uncertainty, imprecision, vagueness, fuzziness) and different methodologies have been proposed to deal with them (e.g., approaches based on default logic, probability, fuzzy sets, etc.)

For many problems in this area, exact equality is replaced by its approximation. Several approaches use similarity relations to express the approximation, modeling the corresponding imprecise information. Similarity relations are fuzzy binary relations, specifying to which degree two objects are similar to each other. They satisfy reflexivity, symmetry, and fuzzy min-transitivity properties, and can be characterized by “level-wise” equivalence. Once the
level $\lambda > 0$ of similarity is fixed, the set of all object-pairs, which have the similarity degree at least $\lambda$, form a classical equivalence relation.

Reasoning with similarity relations requires approximate inference techniques. Solving similarity-based constraints is the central computational mechanism for such inferences. Several approaches to unification modulo similarity have been proposed, see, e.g., [1, 2, 5–9, 14, 16, 17, 20, 21]. Recently, unification was studied also for proximity relations, which generalize similarity in the sense they are reflexive and symmetric but non-transitive fuzzy relations [10, 12, 15]. The techniques studied in these papers usually assume a single fuzzy (similarity or proximity) relation. However, in many practical situations, one needs to deal with several similarities between the objects from the same set, see, e.g. [18, 19], where examples about building online fashion compatibility representation and understanding visual similarities are considered in the context of learning image embeddings.

Multiple similarities pose challenges to constraint solving, since we can not rely on the transitivity property anymore. Note that proximity relations are not transitive either, but their unification methods have some limitations in dealing with multiple similarities simultaneously.

We address this problem, proposing an algorithm for solving constraints over multiple similarity relations. A simple example below illustrates the problem together with the results of different approaches, and motivates the development of a dedicated technique for it.

Example 1. Let white-circle, white-ellipse, gray-circle and gray-ellipse be four symbols and $R_1$ and $R_2$ be two similarity relations, where $R_1$ stands for “similar color, same shape” and $R_2$ denotes “same color, similar shape”. They are defined as

$R_1$ (white-circle, gray-circle) = $R_1$ (white-ellipse, gray-ellipse) = 0.5, and

$R_2$ (white-circle, white-ellipse) = $R_2$ (gray-circle, gray-ellipse) = 0.7.

Assume we want to find an object $X$ such that from the color point of view, it is at least 0.4-similar to white-circle and from the shape point of view, it is at least 0.5-similar to gray-ellipse. The expected answer is $X \simeq_{R_1, 0.4}$ white-circle and $X \simeq_{R_2, 0.5}$ gray-ellipse. The direct approach, trying to solve each equation separately by the weak unification algorithm from [17] leads to no solution in this case, because white-circle and gray-ellipse are not similar to each other by any of the given relations.

An alternative way could be to consider the constraint over the relation $R_1 \cup R_2$, which is a proximity, not a similarity, since transitivity is not satisfied. However, the proximity unification algorithm from [12] gives no solution. We can try to use the algorithm for solving proximity constraints from [15], but it would give two answers instead of one: $X = gray-circle$ and $X = white-ellipse$. On the other hand, the algorithm proposed in this paper computes the right solution $X = gray-circle$. Its similarity degrees are 0.5 for the relation $R_1$ and 0.7 for $R_2$.

It should be mentioned that the multi-adjoint framework [14, 16] is flexible enough to accommodate multiple similarities. It is a logic programming-based approach, where one needs to extend programs by fuzzy similarity axioms for each alphabet symbol and use classical unification. The authors show how to encode Sessa’s algorithm [17] in this framework.

Our approach is different. We develop the solving algorithm directly, without being dependent on the implementation or application preferences. It can be incorporated in a modular way in the constraint logic programming schema, can be used for constrained rewriting, querying, or similar purposes. It combines three parts: solving syntactic equations, solving similarity problems for one relation, and solving mixed problems. Except variables
for terms, we permit also variables for function symbols, since they are necessary in the
process of finding an “intermediate object” between terms in different similarity relations.
The paper is organized as follows: In Section 2 we introduce the basic notions, define
constraints and their solutions. Section 3 is the main section of the paper: it describes
all three parts of our algorithm and presents its termination, soundness, and completeness
results. In Section 4 we show how to include the computation of approximation degrees in
the algorithm. Concluding discussion can be found in Section 5.

2 Preliminaries

Similarity relations

We define basic notions about similarity relations following [9, 17]. A binary fuzzy relation
on a set $S$ is a mapping from $S \times S$ to the real interval $[0, 1]$. If $R$ is a fuzzy relation on $S$
and $\lambda$ is a number $0 < \lambda \leq 1$ (called cut value), then the $\lambda$-cut of $R$ on $S$, denoted $R_\lambda$, is an
ordinary (crisp) relation on $S$ defined as $R_\lambda := \{(s_1, s_2) \mid R(s_1, s_2) \geq \lambda\}$.

A fuzzy relation $R$ on a set $S$ is called a proximity relation, if it is reflexive and symmetric:

- Reflexivity: $R(s, s) = 1$ for all $s \in S$;
- Symmetry: $R(s_1, s_2) = R(s_2, s_1)$ for all $s_1, s_2 \in S$.

Let $\land$ be a T-norm: an associative, commutative, non-decreasing binary operation on
$[0, 1]$ with 1 as the unit element. A proximity relation (on $S$) is called a similarity relation
(on $S$) if it is transitive:

Transitivity $R(s_1, s_2) \geq R(s_1, s) \land R(s, s_2)$ for any $s_1, s_2, s \in S$.

In this paper, in the role of T-norm we take the minimum of two numbers, and write $\min$ instead of $\land$. In the role of $S$ we take a syntactic domain, defined in the next section.

Terms, atoms, substitutions

Our alphabet $A$ consists of the following pairwise disjoint sets of symbols:

- $\mathbf{V_T}$: term variables, denoted by $X, Y, Z$,
- $\mathbf{V_F}$: function variables, denoted by $F, G, H$,
- $\mathbf{C_F}$: function constants, denoted by $f, g, h$,

By $\mathbf{V}$ we denote the set of variables $\mathbf{V} = \mathbf{V_T} \cup \mathbf{V_F}$, and $\mathbf{V}$ is used for its elements.

A function symbol is a function variable or a function constant, i.e., an element of the set
$\mathbf{F} = \mathbf{C_F} \cup \mathbf{V_F}$. We use the letters $f, g, h$ to denote function symbols. Each function symbol
has a fixed arity.

Terms over $A$ are defined by the grammar $t := X \mid f(t_1, \ldots, t_n)$, where $f$ is an $n$-ary
function symbol. For terms we use the letters $t, s, r$. The set of terms over $A$ is denoted by
Terms($A$).

For a term $f(t_1, \ldots, t_n)$, if $n = 0$, we write just $f$ instead of $f()$. Usually, from the context
it is clear whether we are talking about a symbol or about a term.

A substitution $\sigma$ is a mapping from $\mathbf{V}$ to $\mathbf{F} \cup$ Terms($A$) such that

- $\sigma(X) \in$ Terms($A$) for all $X \in \mathbf{V_T}$,
- $\sigma(F) \in \mathbf{F}$ for all $F \in \mathbf{V_F}$,
- $\sigma(V) = V$ for all but finitely many variables $V \in \mathbf{V}$.
Substitutions are denoted by Greek letters $\sigma$, $\vartheta$, $\varphi$. The identity substitution is denoted by $\text{Id}$. The domain of a substitution $\sigma$ is the set $\text{dom}(\sigma) = \{ V \mid V \in \mathcal{V}, \sigma(V) \neq V \}$. The restriction of $\sigma$ to a set of variables $\mathcal{V}$ is the substitution $\sigma|_{\mathcal{V}}$ defined as $\sigma|_{\mathcal{V}}(V) = \sigma(V)$ if $V \in \mathcal{V}$ and $\sigma|_{\mathcal{V}}(V) = V$ otherwise. We will use the usual set representation of substitutions, writing $\sigma$ as $\{ V \mapsto \sigma(V) \mid V \in \text{dom}(\sigma) \}$.

Substitution application to variables, constants, and terms is defined as follows: $c \sigma = c$ for all $c \in \mathcal{C}_F$, $V \sigma = \sigma(V)$ for all $V \in \mathcal{V}$, and $f(t_1, \ldots, t_n) \sigma = (f\sigma)(t_1\sigma, \ldots, t_n\sigma)$.

### Similarity relations on syntactic domains

Our similarity relations are defined on the set of constants $\mathcal{C}_F$. Any such relation $\mathcal{R}$ should satisfy the restriction: $\mathcal{R}(f, g) = 0$, if $f$ and $g$ have different arity.

Given an $\mathcal{R}$ defined on $\mathcal{C}_F$, we extend it to $\mathcal{F} \cup \text{Terms}(\mathcal{A})$:

- For variables: $\mathcal{R}(V, V) = 1$.
- For nonvar. terms: $\mathcal{R}(f(t_1, \ldots, t_n), g(s_1, \ldots, s_n)) = \min(\mathcal{R}(f, g), \mathcal{R}(t_1, s_1), \ldots, \mathcal{R}(t_n, s_n))$, when $f$ and $g$ are both $n$-ary.
- In all other cases, $\mathcal{R}(\tau_1, \tau_2) = 0$ for $\tau_1, \tau_2 \in \mathcal{F} \cup \text{Terms}(\mathcal{A})$.

Given a similarity relation $\mathcal{R}$ and the cut value $\lambda \in (0, 1]$, we define $(\mathcal{R}, \lambda)$-neighborhood of $\tau$ as $\mathcal{N}(\tau, \mathcal{R}, \lambda) := \{ \tau' \mid \mathcal{R}(\tau, \tau') \geq \lambda \}$, where $\tau, \tau' \in \mathcal{F} \cup \text{Terms}(\mathcal{A})$. Based on the definition of similarity relations above, it is obvious that neighborhoods of function constants (resp. variables) contain only function constants (resp. variables) of the same arity. Neighborhoods of terms contain only terms. All terms in the same neighborhood have the same structure (same set of positions). We require for each $f \in \mathcal{C}_F$, $\mathcal{R}$, and $\lambda$, the set $\mathcal{N}(f, \mathcal{R}, \lambda)$ to be finite. It implies that term neighborhoods are finite as well.

### Constraints

In our constraint language, the elements of $\mathcal{F} \cup \text{Terms}(\mathcal{A})$ are the basic objects. In the rest of the paper, the letter $\tau$ is used to denote its elements. Besides, we have the equality predicate constant $\doteqdot$ (interpreted as syntactic equality), one or more similarity predicate constants $\simeq_1, \simeq_2, \ldots$ (interpreted as similarity relations on $\mathcal{F} \cup \text{Terms}(\mathcal{A})$), propositional constants $\text{true}$ and $\text{false}$, connectives $\land, \lor$, and the quantifier $\exists$.

Primitive constraints $\mathcal{P}$ are defined by the grammar

$$\mathcal{P} ::= \text{true} | \text{false} | t \doteqdot s | t \simeq s | f \doteqdot g | f \simeq g,$$

where $\simeq \in \{ \simeq_1, \simeq_2 \ldots \}$. Primitive $\doteqdot$- and $\simeq$-constraints are called primitive equality constraints and primitive similarity constraints, respectively. A literal $L$ is an atom or a primitive constraint. A (positive) constraint $\mathcal{C}$ over $\mathcal{A}$ is defined as $\mathcal{C} ::= \mathcal{P} | \mathcal{C} \land \mathcal{C} | \mathcal{C} \lor \mathcal{C} | \exists x. \mathcal{C}$.

In this paper we consider only positive constraints.

The domain of the intended interpretation of our constraint language is its Herbrand universe (the set of ground terms). The predicate constant $\doteqdot$ is interpreted as syntactic equality. Each similarity predicate constant $\simeq$ is interpreted as a similarity relation on the domain as defined in the previous section. When a predicate constant $\simeq$ is to be interpreted by a relation $\mathcal{R}$ with the cut value $\lambda \in (0, 1]$, we write $\simeq_{\mathcal{R}, \lambda}$ instead of $\simeq$.

A variable-predicate pair (VP-pair) is either $(V, \simeq_{\mathcal{R}, \lambda})$ or $(V, \doteqdot)$. We say that a substitution $\sigma$ is more general than $\vartheta$ on a set of VP-pairs $\mathcal{W}$ if there exists a substitution $\varphi$ such that $\mathcal{R}(V \sigma \varphi, V \vartheta) \geq \lambda$ for all $(V, \simeq_{\mathcal{R}, \lambda}) \in \mathcal{W}$ and $V \sigma \varphi = V \vartheta$ for all $(V, \doteqdot) \in \mathcal{W}$. In this case we write $\sigma \subseteq_{\mathcal{W}} \vartheta$. 
Example 2. Let $\mathcal{R}_1(a, b) = 0.7$, $\mathcal{R}_1(b, c) = 0.7$, $\mathcal{R}_1(a, c) = 0.8$, $\mathcal{R}_2(b, c) = 0.9$, and $\mathcal{W} = \{(X, \simeq_{\mathcal{R}_1, 0.5}), (Y, \simeq_{\mathcal{R}_2, 0.6}), (Y, \simeq_{\mathcal{R}_2, 0.7})\}$.

Let $\sigma = \{X \mapsto Y\}$ and $\vartheta = \{X \mapsto a, Y \mapsto b\}$. Then $\sigma \preceq_{\mathcal{W}} \vartheta$, because for $\varphi = \{X \mapsto b, Y \mapsto b\}$ we have $X \varphi = b \simeq_{\mathcal{R}_1, 0.5} a = X \vartheta$, $Y \varphi = b \simeq_{\mathcal{R}_1, 0.6} b = Y \vartheta$, and $Y \varphi = b \simeq_{\mathcal{R}_2, 0.7} b = Y \vartheta$.

Let $\sigma = \{X \mapsto Y\}$ and $\vartheta = \{X \mapsto a, Y \mapsto c\}$. Then $\sigma \preceq_{\mathcal{W}} \vartheta$, because for $\varphi = \{X \mapsto b, Y \mapsto b\}$ we have $X \varphi = b \simeq_{\mathcal{R}_1, 0.5} a = X \vartheta$, $Y \varphi = b \simeq_{\mathcal{R}_1, 0.6} c = Y \vartheta$, and $Y \varphi = b \simeq_{\mathcal{R}_2, 0.7} c = Y \vartheta$.

Let $\sigma = \{X \mapsto f(Y), Y \mapsto Z\}$ and $\vartheta = \{X \mapsto f(Z), Y \mapsto a, Z \mapsto X\}$. Then $\sigma \preceq_{\mathcal{W}} \vartheta$, because for $\varphi = \{X \mapsto Z, Z \mapsto a\}$ we have $X \varphi = f(Z) \simeq_{\mathcal{R}_1, 0.5} f(Z) = X \vartheta$, $Y \varphi = a \simeq_{\mathcal{R}_1, 0.6} a = Y \vartheta$, and $Y \varphi = a \simeq_{\mathcal{R}_2, 0.7} a = Y \vartheta$.

Theorem 3. $\preceq_{\mathcal{W}}$ is a quasi-ordering for all $\mathcal{W}$.

Proof. Reflexivity is obvious. For transitivity, assume $\sigma_1 \preceq_{\mathcal{W}} \sigma_2$ and $\sigma_2 \preceq_{\mathcal{W}} \sigma_3$. We will show $\sigma_1 \preceq_{\mathcal{W}} \sigma_3$. Take $(V, \simeq_{\mathcal{R}, \lambda}) \in \mathcal{W}$. Then for some $\varphi_1$ and $\varphi_2$ we have $R(V\varphi_1, V\varphi_2) \geq \lambda$ and $R(V\varphi_2, V\varphi_3) \geq \lambda$. Since similarity is stable for substitutions [17, Proposition 3.1], we have $R(V\varphi_1\varphi_2, V\varphi_2\varphi_3) \geq \lambda$. By transitivity of similarity, we get $R(V\varphi_1\varphi_2\varphi_3, V\varphi_3) \geq \min(R(V\varphi_1\varphi_2\varphi_3, V\varphi_2\varphi_3), R(V\varphi_2\varphi_3, V\varphi_3)) \geq \lambda$, which implies that $\sigma_1 \preceq_{\mathcal{W}} \sigma_3$.

Definition 4 (Solution). A substitution $\sigma$ is called a solution of a primitive constraint $\mathcal{P}$, if

$\mathcal{P} = \tau_1 \preceq_{\mathcal{R}, \lambda} \tau_2$ and $\tau_1 \sigma = \tau_2 \sigma$, or

$\mathcal{P} = \tau_1 \preceq_{\mathcal{R}, \lambda} \tau_2$ and $\mathcal{R}(\tau_1 \sigma, \tau_2 \sigma) \geq \lambda$.

Any substitution is a solution of true, while false has no solution.

A substitution $\sigma$ is a solution of a conjunction of primitive constraints $\mathcal{K}$ iff it solves each primitive constraint in $\mathcal{K}$. We denote the set of all solutions of $\mathcal{K}$ by $\text{Sol}(\mathcal{K})$. For a constraint $\mathcal{C} = \mathcal{K}_1 \land \cdots \land \mathcal{K}_n$ in disjunctive normal form (DNF), we define $\text{Sol}(\mathcal{C}) = \cup_{i=1}^n \text{Sol}(\mathcal{K}_i)$.

Given similarity relations $\mathcal{R}_1, \ldots, \mathcal{R}_n$, a conjunction of primitive constraints $\mathcal{K}$, and its solution $\sigma$, we say that $\sigma$ solves $\mathcal{K}$ with approximation degrees $\mathcal{D} = \{\langle \mathcal{R}_1, d_1 \rangle, \ldots, \langle \mathcal{R}_n, d_n \rangle\}$ if

$\mathcal{K} = \text{true}$ or $\mathcal{K} = \mathcal{K}_\bot$ and $d_1 = \cdots = d_n = 1$,

$\mathcal{K} = \tau_1 \preceq_{\mathcal{R}, \lambda_1} \tau_2$ for some $1 \leq j \leq n$, $d_j = \mathcal{R}_j(\tau_1 \sigma, \tau_2 \sigma) \geq \lambda_j$, and $d_i = 1$ for all

$1 \leq i \leq n$, $i \neq j$,

$\mathcal{K} = \mathcal{P} \land \mathcal{K}'$, $\sigma$ solves $\mathcal{P}$ and $\mathcal{K}'$ with approximation degrees $\{\langle \mathcal{R}_1, d_1' \rangle, \ldots, \langle \mathcal{R}_n, d_n' \rangle\}$ and

$\{\langle \mathcal{R}_1, d_1' \rangle, \ldots, \langle \mathcal{R}_n, d_n' \rangle\}$, respectively, and $d_i = \min(d_i', d_i')$ for all $1 \leq i \leq n$.

Such a definition of approximation degrees gives the flexibility to characterize approximations with respect to each involved relation independently from each other.

Theorem 5. Let $\mathcal{K} = \mathcal{K}_\bot \land \mathcal{K}_{\lambda_1} \land \cdots \land \mathcal{K}_{\lambda_m}$ be a constraint. If $\sigma$ is a solution of $\mathcal{K}$ and $\sigma \preceq_{\mathcal{W}(\mathcal{K})} \vartheta$, then $\vartheta$ is a solution of $\mathcal{K}$.
Proof. Let \( s_1 \simeq_{\mathcal{R}, \lambda_i} s_2 \in \mathcal{K}_{\mathcal{R}, \lambda_i} \). From \( \sigma \succeq_{W(\mathcal{K})} \vartheta \), by definition of \( \succeq_{W(\mathcal{K})} \), there exists a \( \varphi \) such that \( \mathcal{R}(V \sigma \varphi, V \vartheta) \geq \lambda_i \) for each \( V \in \text{var}(\mathcal{K}_{\mathcal{R}, \lambda_i}) \). It implies that

\[
\mathcal{R}(s_j \sigma \varphi, s_j \vartheta) \geq \lambda_i, \quad j = 1, 2. \tag{1}
\]

On the other hand, for similarity relations \( \mathcal{R}(s_1 \sigma \varphi, s_2 \sigma \varphi) = \mathcal{R}(s_1 \sigma, s_2 \sigma) \) (see [17]). Since \( \sigma \) is a solution of \( \mathcal{K} \), \( \mathcal{R}(s_1 \sigma, s_2 \sigma) \geq \lambda_i \). Hence, we have \( \mathcal{R}(s_1 \sigma \varphi, s_2 \sigma \varphi) \geq \lambda_i \). From this inequality and (1), by symmetry and transitivity of \( \mathcal{R} \), we get \( \mathcal{R}(s_1 \vartheta, s_2 \vartheta) \geq \lambda_i \). Hence, \( \vartheta \) is a solution of \( s_1 \simeq_{\mathcal{R}, \lambda_i}, s_2 \).

It is straightforward that \( \vartheta \) is a solution of any equation from \( \mathcal{K}_{\mathcal{R}} \). Hence, \( \vartheta \) is a solution of \( \mathcal{K} \).

**Definition 6** (Solved form, approximately solved form). A conjunction of primitive constraints \( \mathcal{K} \) is in solved form, if \( \mathcal{K} \) is either true or each primitive constraint in \( \mathcal{K} \) has a form \( V \equiv \tau \) or \( V \preceq_{\mathcal{R}, \lambda} \tau \), where \( V \) appears only once in \( \mathcal{K} \). A constraint in DNF \( \mathcal{K}_1 \lor \cdots \lor \mathcal{K}_n \) is in solved form, if each \( \mathcal{K}_i \) is in solved form.

A conjunction of primitive constraints \( \mathcal{K}_{\text{sol}} \land \mathcal{K}_{\text{var}} \) is in solved form (approx-solved form) if \( \mathcal{K}_{\text{sol}} \) is in solved form, and \( \mathcal{K}_{\text{var}} \) is a conjunction of primitive similarity constraints between variables \( V_1 \preceq_{\mathcal{R}, \lambda} V_2 \) such that neither \( V_1 \) nor \( V_2 \) appear in the left hand side of any primitive constraint in \( \mathcal{K}_{\text{sol}} \). A constraint in DNF \( \mathcal{K}_1 \lor \cdots \lor \mathcal{K}_n \) is in appr-solved form, if each \( \mathcal{K}_i \) is in appr-solved form.

Appr-solved forms are also solved forms, but not vice versa. Each solved form \( \mathcal{K} \) induces a substitution, denoted by \( \sigma_{\mathcal{K}} \): if \( \mathcal{K} = \text{true} \), then \( \sigma_{\mathcal{K}} = \text{Id} \), otherwise \( \sigma_{\mathcal{K}} = \{ V \mapsto \tau \mid V \in \mathcal{K} \} \). Obviously, \( \sigma_{\mathcal{K}} \) is a solution of \( \mathcal{K} \). A constraint \( \mathcal{K}_{\text{sol}} \land \mathcal{K}_{\text{var}} \) in appr-solved form is also solvable, because \( \sigma_{\mathcal{K}_{\text{sol}}} \) solves \( \mathcal{K}_{\text{sol}} \) and \( \mathcal{K}_{\text{var}} \) always has at least a trivial solution mapping all terms variables to the same term variable and all function variables to the same function variable.

**Example 7.** Let \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) be defined as in Example 1 and \( \mathcal{K} = \mathcal{K}_{\mathcal{R}_1,0.4} \land \mathcal{K}_{\mathcal{R}_2,0.5} \) be a constraint, where \( \mathcal{K}_{\mathcal{R}_1,0.4} = X \preceq_{\mathcal{R}_1,0.4} \text{white-circle} \land X \preceq_{\mathcal{R}_1,0.4} Y \) and \( \mathcal{K}_{\mathcal{R}_2,0.5} = X \preceq_{\mathcal{R}_2,0.5} \text{gray-ellipse} \land Y \preceq_{\mathcal{R}_2,0.5} \text{white-ellipse} \). One can bring \( \mathcal{K}_{\mathcal{R}_1,0.4} \) to its equivalent solved form (e.g., by an algorithm along the lines of the weak unification algorithm in [17]). \( \mathcal{K}_{\mathcal{R}_2,0.5} \) is already in the solved form. Hence, \( \mathcal{K} \) is equivalent to the constraint

\[
X \preceq_{\mathcal{R}_1,0.4} \text{white-circle} \land Y \preceq_{\mathcal{R}_1,0.4} \text{white-circle} \land
X \preceq_{\mathcal{R}_2,0.5} \text{gray-ellipse} \land Y \preceq_{\mathcal{R}_2,0.5} \text{white-ellipse},
\]

which is not yet in a solved form. A solved form, equivalent to \( \mathcal{K} \), would be \( X = \text{gray-circle} \land Y = \text{white-circle} \). It induces the substitution \( \sigma = \{ X \mapsto \text{gray-circle}, Y \mapsto \text{white-circle} \} \).

**Example 8.** Let \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) be two similarity relations defined as

\[
\mathcal{R}_1 : \quad \mathcal{R}_1(a_1, c_1) = \mathcal{R}_1(b_1, c_1) = 0.7, \quad \mathcal{R}_1(a_1, b_1) = 0.8,
\]

\[
\mathcal{R}_1(a_2, c_2) = \mathcal{R}_1(b_2, c_2) = 0.6, \quad \mathcal{R}_1(a_2, b_2) = 0.7,
\]

\[
\mathcal{R}_2 : \quad \mathcal{R}_2(b_1, b_3) = \mathcal{R}_2(b_2, b_3) = 0.6, \quad \mathcal{R}_2(b_1, b_3) = 0.7, \quad \mathcal{R}_2(c_1, c_2) = 0.8.
\]

Visually:
They are applied modulo associativity, commutativity, and idempotence of the constraint solving algorithm.

In this subsection we describe the rules that solve equality constraints. Essentially, these are

\[ K \vee C \]

its rules are divided into three groups: for equalities, for similarities, and for mixed problems.

The solved forms induce two substitutions: \( \sigma_1 = \{ X \mapsto f(b_1,b_2), Y \mapsto b_1 \} \) and \( \sigma_2 = \{ X \mapsto f(c_1,c_2), Y \mapsto c_1 \} \). They are solutions of \( K \). There are other solutions of \( K \) that are \( \cong \)-equivalent to \( \sigma_1 \) or \( \sigma_2 \): \( \vartheta_1 = \{ X \mapsto f(b_1,b_2), Y \mapsto b_2 \} \cong \sigma_1 \), \( \vartheta_2 = \{ X \mapsto f(b_1,b_2), Y \mapsto b_3 \} \cong \sigma_1 \), and \( \vartheta_3 = \{ X \mapsto f(c_1,c_2), Y \mapsto c_2 \} \cong \sigma_2 \).

Now let \( K = X \cong_{\mathbb{R}_1,0.5} g(Y) \land X \cong_{\mathbb{R}_2,0.6} g(Z) \). A solved form \( K_s = X \equiv g(Z) \land Y \equiv Z \) implies \( K \), but is not equivalent to it, because \( K \) has solutions \( \{ X \mapsto g(b_1), Y \mapsto a_1, Z \mapsto b_2 \} \) and \( \{ X \mapsto g(b_1), Y \mapsto a_1, Z \mapsto b_3 \} \), which do not solve \( K_s \). On the other hand, if we take the approximate solved form \( K_{as} = X \equiv g(X_1) \land X_1 \equiv_{\mathbb{R}_1,0.8} Y \land X_1 \equiv_{\mathbb{R}_2,0.6} Z \), then every solution of \( K_{as} \) solves \( K \), and \( \exists X_1, K_{as} \) holds for any solution \( \sigma \) of \( K \). (Substitution application to a quantified constraint avoids variable capture.) Alternatively, we could have taken another solved form \( K'_s = X \equiv g(X_1) \land Y \equiv_{\mathbb{R}_1,0.8} X_1 \land Z \equiv_{\mathbb{R}_2,0.6} X_1 \) which has the same properties as \( K_{as} \).

**Example 9.** Let \( R_1(a,b_1) = R_1(b_1,b_2) = R_1(a,b_2) = 0.8, R_2(c,b_1) = R_2(b_1,b_2) = R_2(b_2,c) = 0.7 \) and consider a constraint \( K = X \cong_{\mathbb{R}_1,0.6} f(Y,Y) \land X \cong_{\mathbb{R}_2,0.5} f(Z,Z) \). The straightforward solved form \( X \equiv f(Z,Z) \land Y \equiv Z \), as in the previous example, has fewer solutions than \( K \), e.g., \( \{ X \mapsto f(b_1,b_2), Y \mapsto a, Z \mapsto c \} \) would be lost. If we take an appr-solved form \( K_{as} = X \equiv f(X_1,X_2) \land X_1 \equiv_{\mathbb{R}_1,0.8} Y \land X_1 \equiv_{\mathbb{R}_2,0.5} Z \land X_2 \equiv_{\mathbb{R}_1,0.6} Y \land X_2 \equiv_{\mathbb{R}_2,0.5} Z \), then all solutions of \( K_{as} \) solve \( K \) and for each solution \( \sigma \) of \( K \), we have \( \exists X_1, X_2, K_{as} \)\( \sigma \).

Unlike the previous example, we can not turn this appr-solved form into a solved form by swapping sides of variables-only equations.

### 3 Constraint solving

The constraint solving algorithm Solve presented in this section works on constraints in DNF.

Its rules are divided into three groups: for equalities, for similarities, and for mixed problems.

They are applied modulo associativity, commutativity, and idempotence of \( \land \) and \( \lor \), treating false as the unit element of \( \lor \). We first introduce the rules and then show how Solve is defined using them.

In the rules, the superscript ? indicates that the constraints are supposed to be solved.

The sides of an equation \( V \equiv^? \tau \) belong to the same syntactic category, i.e., it stands either for \( X \equiv^? t \) or for \( F \equiv^? f \). The same holds for \( V \equiv^? \tau \).

#### Equality rules

In this subsection we describe the rules that solve equality constraints. Essentially, these are

first-order unification rules with a slight modification, which concerns dealing with function and predicate variables. The rules have the form \( K \hookrightarrow K' \), which defines the transformation \( K \lor C \hookrightarrow K' \lor C \). Note that \( C \) does not change.

The rules are Del-eq (deletion), Dec-eq (decomposition), Ori-eq (orientation), Elim-eq (variable elimination), Confl-eq (conflict), Mism-eq (arity mismatch), Occ-eq (occurrence...
Constraint solving over multiple similarity relations

The rules in this section are designed for similarity relations. They resemble weak unification rules [9,17], with the difference that function variables and multiple similarity relations are permitted, and aims at computing the answer in solved form, instead of a substitution.

In the Elim-sim rule, the variable \( V \) is replaced by \( \tau \) only in the constraints for the same similarity relation. This is justified by the fact that although a \( \lambda \)-cut of each similarity relation is transitive, from \( t \simeq_{R_1,\lambda_1} s, s \simeq_{R_2,\lambda_2} r \) we cannot conclude anything about similarity between \( t \) and \( r \).
The similarity rules have the same form as the equality rules: $K \rightsquigarrow K'$, which defines the transformation $K \lor C \rightsquigarrow K' \lor C$. The names are also similar to those for equalities, using $\text{sim}$ instead of $\text{eq}$.

**Del-sim**: $\tau_1 \sim_{R,\lambda}^2 \tau_2 \land K \rightsquigarrow K$, where $\tau_1, \tau_2 \in C_F \cup V_F \cup V_T$ and $R(\tau_1, \tau_2) \geq \lambda$.

**Dec-sim**: $f(t_1, \ldots, t_n) \equiv_{R,\lambda}^2 g(s_1, \ldots, s_n) \land K \rightsquigarrow$

$\neg \equiv_{R,\lambda}^2 g \land l_1 \equiv_{R,\lambda}^2 s_1 \land \ldots \land l_n \equiv_{R,\lambda}^2 s_n \land K$, where $n > 0$.

**Ori-sim**: $\tau \equiv_{R,\lambda}^2 V \land K \rightsquigarrow V \equiv_{R,\lambda}^2 \tau \land K$, where $\tau \notin V$.

**Elim-sim**: $V \equiv_{R,\lambda}^2 \tau \land \mathcal{K} \rightsquigarrow V \equiv_{R,\lambda}^2 \tau \land \mathcal{K}\{\tau \mapsto \tau\} \land \mathcal{K}$

where $\mathcal{K}$ does not contain primitive $\equiv_{R,\lambda}$-constraints, $\mathcal{V} \notin \text{var}(\tau)$, and $V \in \mathcal{K}_{R,\lambda}$.

**Confl-sim**: $\tau_1 \sim_{R,\lambda}^2 \tau_2 \land K \rightsquigarrow \text{false}$, where $\tau_1, \tau_2 \in C_F \cup V_F \cup V_T$, and $R(\tau_1, \tau_2) < \lambda$.

**Mism-sim**: $f(t_1, \ldots, t_n) \equiv_{R,\lambda}^2 g(s_1, \ldots, s_m) \land K \rightsquigarrow \text{false}$, if $n \neq m$.

**Occ-sim**: $X \equiv_{R,\lambda}^2 t \land K \rightsquigarrow \text{false}$, if $X \in \text{var}(t)$ and $X \neq t$.

The algorithm Sim applies the similarity rules as long as possible. When there are more than one applicable rule, the algorithm may choose one nondeterministically.

Termination of Sim can be proved as termination of Unif:

**Theorem 12.** Sim is terminating.

**Lemma 13 (Soundness lemma for Sim).** If $K \rightsquigarrow K'$ is a step performed by a rule in Sim, then $\text{Sol}(K) = \text{Sol}(K')$.

**Proof.** When we have only one similarity relation, soundness follows from soundness of the original unification algorithm [17]. For the extension to multiple similarity relations, the only nontrivial rule is Elim simulations. (For the others, $\text{Sol}(K) = \text{Sol}(K')$ holds directly.) It is important to notice that in this rule, $\{V \mapsto \tau\}$ applies only to $\mathcal{K}_{R,\lambda}$. Then for $V \equiv_{R,\lambda}^2 \tau \land \mathcal{K}_{R,\lambda}$ we have $\text{Sol}(V \equiv_{R,\lambda}^2 \tau \land \mathcal{K}_{R,\lambda}) = \text{Sol}(V \equiv_{R,\lambda}^2 \tau \land \mathcal{K}_{R,\lambda}\{V \mapsto \tau\})$. (It follows from soundness of the original unification algorithm [17], since the constraint is over a single similarity relation.) Constraints for all other relations remain unchanged. It implies that the solution sets for constraints in both sides of the Elim-sim rule are the same.

**Example 14.** Let $\mathcal{K}_{R,0.4} = X \equiv_{R,0.4} \text{white-circle} \land X \equiv_{R,0.4} Y$ and $\mathcal{K}_{R,0.5} = X \equiv_{R,0.5} \text{gray-ellipse} \land Y \equiv_{R,0.5} \text{white-ellipse}$ be the constraints from Example 7. The reduction mentioned in that example is modeled by performing the Elim-sim step and replacing $X$ by $\text{white-circle}$ in $\mathcal{K}_{R,0.4}$:

$X \equiv_{R,0.4} \text{white-circle} \land X \equiv_{R,0.4} Y \land$

$X \equiv_{R,0.5} \text{gray-ellipse} \land Y \equiv_{R,0.5} \text{white-ellipse} \rightsquigarrow$ Elim-sim

$X \equiv_{R,0.4} \text{white-circle} \land \text{white-circle} \equiv_{R,0.4} Y \land$

$X \equiv_{R,0.5} \text{gray-ellipse} \land Y \equiv_{R,0.5} \text{white-ellipse}$.

If Elim-sim permitted to replace $X$ not only in $\mathcal{K}_{R,0.4}$, but also in $\mathcal{K}_{R,0.5}$, we would get

$X \equiv_{R,0.4} \text{white-circle} \land \text{white-circle} \equiv_{R,0.4} Y \land$

$\text{white-circle} \equiv_{R,0.5} \text{gray-ellipse} \land Y \equiv_{R,0.5} \text{white-ellipse}$,

but $\text{white-circle} \equiv_{R,0.5} \text{gray-ellipse}$ is unsolvable. Hence, we would lose a solution.
Mixed rules

The rules in this section apply when there are at least two primitive constraints over different similarity relations. The notation \( t[X] \) below means that the variable \( X \) occurs in the term \( t \).

**Definition 15 (Occurrence cycle).** An occurrence cycle for a variable \( X_1 \) is called the conjunction of primitive constraints \( X_1 \; \simeq_{R_1,\lambda_1} \; t_1 \; [X_2] \; \land \; X_2 \; \simeq_{R_2,\lambda_2} \; t_2 \; [X_3] \; \land \; \cdots \; \land \; X_n \; \simeq_{R_n,\lambda_n} \; t_n \; [X_1] \), where \( n > 1 \), \( R_i \neq R_{i+1} \) for all \( 1 \leq i \leq n-1 \), \( R_n \neq R_1 \), and at least one \( t \) is not a variable.

**Remark 16.** Note that in the definition of occurrence cycle, if two neighboring primitive similarity constraints use the same relation, they can be contracted into one constraint by transitivity, i.e., instead of \( X_i \; \simeq_{R_i,\lambda_i} \; t_i \; [X_{i+1}] \; \land \; X_{i+1} \; \simeq_{R_{i+1},\lambda_{i+1}} \; t_{i+1} \; [X_{i+2}] \) we can have \( X_i \; \simeq_{R_i,\lambda_i} \; t_i \; [t_{i+1} \; [X_{i+2}]] \), getting rid of consecutive identical similarity relations. The same is true for the last and the first constraints.

**Theorem 17.** If a conjunction of primitive constraints contains an occurrence cycle modulo symmetry of \( \simeq_{R,\lambda} \), then it has no solution.

**Proof.** In similarity relations, symbols of different arities can not be similar. Therefore, similar terms have the same set of positions, i.e., as trees they are the same up to renaming of nodes.

Assume by contradiction that the given occurrence cycle has a solution \( \vartheta \). It means that the following term pairs have the same structure: \( X_1 \vartheta \) and \( t_1 \; [X_2] \; \vartheta \), \( X_2 \vartheta \) and \( t_2 \; [X_3] \; \vartheta \), \ldots, \( X_n \vartheta \) and \( t_n \; [X_1] \; \vartheta \). Then \( X_1 \vartheta \) and \( t_1 \; [t_2 \cdots [t_n \; [X_1] \cdots \cdots ] \vartheta \) have the same structure. Since at least one of \( t_i 's \) is not a variable, \( X_1 \vartheta \) is a proper subterm of \( t_1 \; [t_2 \cdots [t_n \; [X_1] \cdots \cdots ] \vartheta \). But a term and its proper subterm can not have the same structure. A contradiction.

The phrase “modulo symmetry of \( \simeq_{R,\lambda} \)” in the theorem means that the sides of primitive constraints can be swapped, in order to detect an occurrence cycle. Since side swapping does not affect solvability of constraints, the theorem remains true if an occurrence cycle is not in the explicit form in the constraint.

Below, when we talk about existence of an occurrence cycle in a constraint, we mean existence modulo symmetry of the similarity predicate.

The rules in the mixed group are rules for occurrence check (Occ-mix), mismatch (Mism-mix), and elimination of term and function variables (TVE-mix and FVE-mix, respectively). All of them except FVE-mix have the form \( K \; \leadsto \; K' \). As usual, they define the constraint transformation \( K \; \lor \; C \; \leadsto \; K' \; \lor \; C' \). As for FVE-mix, its form is \( K \; \leadsto \; K' \; \lor \; \cdots \; \lor \; K'_n \), defining a transformation \( K \; \lor \; C \; \leadsto \; K'_1 \; \lor \; \cdots \; \lor \; K'_n \; \lor \; C \). Note that in any of these rules, \( C \) does not change.

In all the rules it is assumed that the constraint to be transformed (i.e., the constraint in the left hand side of \( \leadsto \)) has the form \( K_1 \land K_{R_1,\lambda_1} \land \cdots \land K_{R_m,\lambda_m} \), where \( K_1 \) and each \( K_{R_i,\lambda_i} \), \( 1 \leq i \leq m \), are in solved form.

The TVE-mix rule uses the renaming function \( \rho \). Applied to a term, \( \rho \) gives its fresh copy, obtained by replacing each occurrence of a constant from \( F_T \) by a new function variable, each occurrence of a term variable by a fresh term variable, and each occurrence of a function variable by a fresh function variable. For instance, if the term is \( f(\tilde{F}(a, X, X, f(a))) \), we have \( \rho(\tilde{F}(a, X, X, f(a))) = G_1(G_2(G_3(\tilde{Y}_1, \tilde{Y}_2, G_4(G_5(\tilde{Y}_2))))), \) where \( G_1, G_2, G_3, G_4, G_5 \in \nu_T \) are new function variables and \( \tilde{Y}_1, \tilde{Y}_2 \in V_T \) are new term variables.

**Occ-mix:** \( X \; \simeq_{R,\lambda} \; t \; \land \; K \; \leadsto \; \text{false} \), if \( X \; \simeq_{R,\lambda} \; t \; \land \; K \) contains an occurrence cycle for \( X \).
Mism-mix: $X \simeq_{\mathcal{R}_1,\lambda_1} f(t_1, \ldots, t_n) \wedge X \simeq_{\mathcal{R}_2,\lambda_2} g(s_1, \ldots, s_m) \wedge \mathcal{K} \not\rightarrow \text{false}$, if $\mathcal{R}_1 \neq \mathcal{R}_2$ and $m \neq n$.

TVE-mix: $X \simeq_{\mathcal{R}_A} f(t_1, \ldots, t_n) \wedge \mathcal{K} \not\rightarrow$

$X \Rightarrow F(t_1', \ldots, t_n') \wedge F \simeq_{\mathcal{R}_A} f \wedge t_1' \simeq_{\mathcal{R}_A} t_1 \wedge \cdots \wedge t_n' \simeq_{\mathcal{R}_A} t_n \wedge \mathcal{K} \vartheta,$

where $X \in \text{var}(\mathcal{K})$, $X \simeq_{\mathcal{R}_1,\lambda_1} f(t_1, \ldots, t_n) \wedge \mathcal{K}$ does not contain an occurrence cycle for $X$, $F(t_1', \ldots, t_n') = \rho(f(t_1, \ldots, t_n))$, and $\vartheta = \{X \Rightarrow F(t_1', \ldots, t_n')\}$.

FVE-mix: $F \simeq_{\mathcal{R}_A} f \wedge \mathcal{K} \not\rightarrow \lor_{g \in \mathbb{N}(f,\mathcal{R}_A)}(F \not\rightarrow g)$, where $F \in \text{var}(\mathcal{K})$.

By Mix we denote one application of any of the mixed rules.

**Lemma 18** (Soundness lemma for Mix). If $\mathcal{K} \not\rightarrow \mathcal{C}$ is a step performed by a rule in Mix, and $\vartheta \in \text{Sol}(\mathcal{C})$, then $\vartheta \in \text{Sol}(\mathcal{K})$.

**Proof.** For failing rules it is trivial as false has no solution. For FVE-mix, the definition of neighborhood implies it. For TVE-mix we reason as follows: Let $\mathcal{K} = \{X \simeq_{\mathcal{R}_A} f(t_1, \ldots, t_n)\} \wedge K_1$ and $\vartheta$ be a solution of the right hand side of this rule. Then $X \vartheta = F(t_1', \ldots, t_n') \vartheta \simeq_{\mathcal{R}_A} f(t_1, \ldots, t_n) \vartheta$ and $\vartheta$ solves $X \simeq_{\mathcal{R}_A} f(t_1, \ldots, t_n)$. As for any other equation $eq \in K_1$, we have $eq \vartheta$ in the right hand side, where $\vartheta = \{X \Rightarrow F(t_1', \ldots, t_n')\}$. Moreover, $\vartheta$ is a solution of $eq \vartheta$ iff $\vartheta \vartheta$ is a solution of $eq$. The equality $X \vartheta = F(t_1', \ldots, t_n') \vartheta$ implies $\vartheta \vartheta = \vartheta$. Hence, $\vartheta$ is a solution of $eq$. ▸

Our constraint solving algorithm Solve is designed as a strategy of applying Unif, Sim, and Mix. To solve a conjunction of primitive equality and similarity constraints $\mathcal{K} = K_{\Delta} \wedge \mathcal{K}_{\mathcal{R}_1,\lambda_1} \wedge \cdots \wedge \mathcal{K}_{\mathcal{R}_m,\lambda_m}$, it performs the following steps:

\[
\begin{align*}
\mathcal{C} & := \mathcal{K} \\
\text{while } \mathcal{C} \text{ is not in the appr-solved form do} & \\
\quad \mathcal{C} := \text{Unif}(\mathcal{C}), \text{ if } \mathcal{C} = \text{false}, \text{ return } \text{false} \\
\quad \mathcal{C} := \text{Sim}(\mathcal{C}), \text{ if } \mathcal{C} = \text{false}, \text{ return } \text{false} \\
\quad \mathcal{C} := \text{Mix}(\mathcal{C}), \text{ if } \mathcal{C} = \text{false}, \text{ return } \text{false} \\
\text{return } \mathcal{C}
\end{align*}
\]

We write $\text{Solve}(\mathcal{K}) = \mathcal{C}$, if the algorithm returns $\mathcal{C}$ for the input $\mathcal{K}$. Respectively, $\text{Solve}(\mathcal{K}) = \text{false}$ if false is returned.

**Example 19.** Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the relations defined in Example 8 and illustrate the steps Solve would make to solve $X \simeq_{\mathcal{R}_{1,0.5}} f(a_1, a_2) \wedge X \simeq_{\mathcal{R}_{2,0.6}} f(Y, Y)$. We will explicitly distinguish between function variables and terms made of a function variable only, i.e., between $F$ and $f()$. For the same reason, we write constant-terms $a_1$ and $a_2$ in their full form $a_1()$ and $a_2()$. Primitive constraints selected to perform a particular step are underlined.

\[
\begin{align*}
X \simeq_{\mathcal{R}_{1,0.5}} f(a_1(), a_2()) & \wedge X \simeq_{\mathcal{R}_{2,0.6}} f(Y, Y) \quad \text{\simeq_{TVE-mix}} \\
X \Rightarrow F(G_1(), G_2()) & \wedge F \simeq_{\mathcal{R}_{1,0.5}} f \wedge G_1() \simeq_{\mathcal{R}_{1,0.5}} a_1() \wedge G_2() \simeq_{\mathcal{R}_{1,0.5}} a_2() \wedge \\
F(G_1(), G_2()) & \simeq_{\mathcal{R}_{2,0.6}} f(Y, Y) \quad \text{\simeq_{Dec-sim} x 2} \\
X \Rightarrow F(G_1(), G_2()) & \wedge F \simeq_{\mathcal{R}_{1,0.5}} f \wedge G_1() \simeq_{\mathcal{R}_{1,0.5}} a_1() \wedge G_2() \simeq_{\mathcal{R}_{1,0.5}} a_2() \wedge \\
F(G_1(), G_2()) & \simeq_{\mathcal{R}_{2,0.6}} f(Y, Y) \quad \text{\simeq_{Dec-sim, Ori-sim}} \\
X \Rightarrow F(G_1(), G_2()) & \wedge F \simeq_{\mathcal{R}_{1,0.5}} f \wedge G_1() \simeq_{\mathcal{R}_{1,0.5}} a_1() \wedge G_2() \simeq_{\mathcal{R}_{1,0.5}} a_2() \wedge 
\end{align*}
\]
Constraint solving over multiple similarity relations

Let \( G_1 \) be two dags with the same set of vertices \( V \). The relation \(<_d\) is defined on such graphs, having the same set of vertices.

Let \( G_1 = (\text{Vert}, E_1) \) and \( G_2 = (\text{Vert}, E_2) \) be two dags with the same set of vertices \( \text{Vert} \).

The vertices are marked in \( G_1 \) and \( G_2 \). Then \( G_1 <_d G_2 \) iff \( E_1 \supseteq E_2 \) and the following condition holds:
Let $\emptyset \neq D \subseteq \text{Vert}$ be the set of all vertices, for which the marks in the graphs differ (i.e., $D := \{ v \in \text{Vert} \mid \text{mark}(v, G_1) \neq \text{mark}(v, G_2) \}$: same vertex, different markings), and $\emptyset \neq M \subseteq D$ be the set of those elements of $D$, which are not reachable from any of the elements in $D$ in $G_1$. (Such a subset of $D$ exists, because the graphs are acyclic.) Then $\text{mark}(v, G_1) \subset \text{mark}(v, G_2)$ for all $v \in M$.

We write $G >_{\text{dag}} G'$ if $G <_{\text{dag}} G$.

\textbf{Theorem 22.} The relation $>_{\text{dag}}$ is a well-founded ordering on dags.

\textbf{Proof.} First, we show that $>_{\text{dag}}$ is a strict partial order (irreflexive and transitive relation). Irreflexivity is obvious. For transitivity, assume $G_1 >_{\text{dag}} G_2$, $G_2 >_{\text{dag}} G_3$ and show $G_1 >_{\text{dag}} G_3$. Let $G_i = (\text{Vert}, E_i)$ for $i = 1, 2, 3$. By transitivity of set inclusion, we have $E_1 \subseteq E_3$.

Let $D(G_i, G_j) := \{ v \in \text{Vert} \mid \text{mark}(v, G_i) \neq \text{mark}(v, G_j) \}$ and $M(G_i, G_j)$ be the set of all those elements in $D(G_i, G_j)$ that are not reachable in $G_j$ from $D(G_i, G_j)$, $1 \leq i < j \leq 3$. Assume first that $D(G_1, G_3) \neq \emptyset$ and take $v \in M(G_1, G_3)$. We want to show that $\text{mark}(v, G_1) \supset \text{mark}(v, G_3)$. The possible cases are

i. $\text{mark}(v, G_1) \neq \text{mark}(v, G_2)$ and $\text{mark}(v, G_2) = \text{mark}(v, G_3)$.

ii. $\text{mark}(v, G_1) = \text{mark}(v, G_2)$ and $\text{mark}(v, G_2) \neq \text{mark}(v, G_3)$.

iii. $\text{mark}(v, G_1) \neq \text{mark}(v, G_2)$ and $\text{mark}(v, G_2) \neq \text{mark}(v, G_3)$.

In case i, $v \in D(G_1, G_2)$. If $v \in M(G_1, G_2)$, then $\text{mark}(v, G_1) \supset \text{mark}(v, G_2) = \text{mark}(v, G_3)$. Now we show that the case $v \notin M(G_1, G_2)$ is impossible. Assume by contradiction that $v \notin M(G_1, G_2)$. It means that there exists $v' \in M(G_1, G_2)$ such that $v' \rightarrow^+ v$ in $G_2$. But then, since $E_2 \subseteq E_3$, we have $v' \rightarrow^+ v$ in $G_3$. Since $v \notin M(G_1, G_3)$, we should have $v' \notin D(G_1, G_3)$, i.e., $\text{mark}(v', G_1) = \text{mark}(v', G_3)$. On the other hand, from $G_1 >_{\text{dag}} G_2$ and $v' \in M(G_1, G_2)$, we have $\text{mark}(v', G_1) \supset \text{mark}(v', G_2)$, which implies $\text{mark}(v', G_2) \subseteq \text{mark}(v', G_3)$. Hence, we get $v'' \rightarrow^+ v' \rightarrow^+ v$ in $G_3$. Therefore, we can not have $v'' \notin D(G_1, G_3)$, because there would be a contradiction: $v \in M(G_1, G_3)$ and $v$ is reachable in $G_3$ from $v'' \in D(G_1, G_3)$. Hence, we get $\text{mark}(v'', G_1) = \text{mark}(v'', G_3)$, which, together with $v'' \in M(G_2, G_3)$ implies that $\text{mark}(v'', G_1) \subset \text{mark}(v'', G_2)$. Hence, $v'' \in D(G_1, G_2)$ and since $G_1 >_{\text{dag}} G_2$, there should exist $v''' \in M(G_1, G_2)$ such that $\text{mark}(v'''', G_1) \supset \text{mark}(v'''', G_2)$ and $v''' \rightarrow^+ v''$ in $G_2$. Then we have also $v''' \rightarrow^+ v'$ in $G_3$, since $E_2 \subseteq E_3$. Moreover, $\text{mark}(v'''', G_2) = \text{mark}(v'''', G_3)$, because otherwise we would have a contradiction with $v'' \in M(G_2, G_3)$ (there would be $v''' \in D(G_2, G_3)$ with $v''' \rightarrow^+ v''$ in $G_3$). Hence, $\text{mark}(v'''', G_1) \supset \text{mark}(v'''', G_2) = \text{mark}(v'''', G_3)$ and we get $v''' \notin D(G_1, G_3)$. But it contradicts our assumption that $v \in M(G_1, G_3)$, because we got $v''' \in D(G_1, G_3)$ with $v''' \rightarrow^+ v$ in $G_3$. The obtained contradiction shows that the case $v \notin M(G_1, G_2)$ is impossible. (See also the diagram in Appendix A.)

The case ii can be proved analogously. In case iii, if we have $\text{mark}(v, G_1) \supset \text{mark}(v, G_2)$ and $\text{mark}(v, G_2) \supset \text{mark}(v, G_3)$, we immediately get $\text{mark}(v, G_1) \supset \text{mark}(v, G_3)$. The other cases are not possible. For instance, assuming $\text{mark}(v, G_1) \supset \text{mark}(v, G_2)$ and $\text{mark}(v, G_2) \subset \text{mark}(v, G_3)$ will lead to contradiction by the same reasoning as in the proof of case i, where we reached $\text{mark}(v', G_1) \supset \text{mark}(v', G_2)$ and $\text{mark}(v', G_2) \subset \text{mark}(v', G_3)$.

For the assumption $D(G_1, G_3) = \emptyset$ we get a contradiction analogously.

Well-foundedness follows from well-foundedness of $\supset$, from the facts that the set of vertices is fixed and the set of edges can not be infinitely increased, and from boundedness of the length of paths due to acyclicity.
Theorem 23 (Termination of Solve). The algorithm Solve terminates and gives either false or a constraint in appr-solved form.

Proof. Let \( K_0 \) be a given conjunction of primitive constraints. We build a term variable dependency graph \( G = (\text{Vert}, E) \) from \( K_0 \) and maintain it during the process of solving. The vertices of \( G \) correspond to term variables in \( K_0 \) so that to each variable a single vertex is assigned. For instance, if \( K_0 \) contains three term variables \( X_1, X_2, \) and \( Y_3 \), then
\[
V = \{v_{X_1}, v_{X_2}, v_{X_3}\}.
\]
The initial marking is defined as \( \text{mark}(v_X, G) = \{X\} \) for each \( v_X \in \text{Vert}. \) Next, to define \( E \), we do the following: If \( K_0 \) contains a solved primitive constraint \( X \leq^\gamma t \) or \( X \simeq_{K, \lambda} t \) (i.e., if \( X \) occurs in \( K_0 \) once), then \( \{v_X, v_Y\} \in E \) for all \( Y \in \text{var}(t) \) and \( \text{mark}(v_Y, G) \) is updated as \( \text{mark}(v_Y, G) \setminus \{X\} \). This is how the initial version of the graph is created. It is denoted by \( G_{K_0} \).

In the process of the application of Solve, the graph gets modified as follows:

a) The applied rule is of the form \( K \hookrightarrow K' \). Then from the graph \( G_K \) we obtain the graph \( G_K' \) depending on the rule:
   - Elim-\( \text{eq} \) with \( X \leq^\gamma t \) adds edges and removes \( X \) from the marking set of the vertex \( v_X \) exactly as described above.
   - Elim-\( \text{sim} \) with \( X \simeq_{K, \lambda} t \) either keeps the graph unchanged (when \( X \) occurs more than once in the resulting constraint after the application of Elim-\( \text{sim} \)), or modifies it as described above (when \( X \) occurs exactly once in the resulting constraint after the application of Elim-\( \text{sim} \)).
   - \( \text{TVE-mix} \) with \( X \simeq_{K, \lambda} t \) does the same modification as Elim-\( \text{eq} \) and, in addition, modifies marking: Let \( Y_1, \ldots, Y_m, m \geq 0 \), be all the copies of a term variable \( Y \in \text{var}(t) \) created by the renaming function \( \rho \) in the TVE-mix step. Then each \( Y_i \) is associated with the vertex \( v_Y \) (i.e., \( v_{Y_1} = v_Y \)) and \( \text{mark}(v_Y, G) \) gets updated as \( \text{mark}(v_Y, G) \cup \{Y_1, \ldots, Y_m\} \).
   - No other rule of the form \( K \hookrightarrow K' \) modifies the graph.

b) The applied rule is of the form \( K \hookrightarrow K'_1 \lor \cdots \lor K'_n, n > 1 \). Then \( G_K = G_{K'_1} = \cdots = G_{K'_n} \).

One can see that the set \( \text{Vert} \) remains unchanged during the process.

Let \( G_1 = (\text{Vert}, E_1) \) be the graph before applying a rule, and \( G_2 = (\text{Vert}, E_2) \) be the one after a rule application so that \( G_1 \neq G_2 \), i.e., Elim-\( \text{eq} \), Elim-\( \text{sim} \), or TVE-mix is applied. Let the chosen primitive constraint be \( X \leq t \) (for Elim-\( \text{eq} \)) or \( X \simeq_{K, \lambda} t \) (for Elim-\( \text{sim} \) and TVE-mix). Then \( E_1 \subseteq E_2 \), because new edges from \( v_X \) to \( v_Y \) for each \( Y \in \text{var}(t) \), are not changed for any of the vertices which is not reachable from \( X \) in \( G_2 \).

Hence, \( G_1 \succ_{\text{dag}} G_2 \).

Let us consider the pair \( (G_K, V_F(K)) \) of such a term variable dependency graph \( G_K \) associated to a constraint \( K \) and a set of function variables \( V_F(K) \) occurring in \( K \). These pairs are ordered lexicographically by \( \succ_{\text{dag}} \) and \( \succ \). By Theorem 22, \( \succ_{\text{dag}} \) is well-founded. The relation \( \succ \) on natural numbers is well-founded. Therefore, their lexicographic combination is well-founded. Since Unif and Sim are terminating (Theorems 10 and 12), each iteration of the while loop in the definition of Solve either stops with false, or reaches the application of Mix. In this process, measure of the pair \( (G_K, V_F(K)) \) does not increase, because the Unif and Sim rules, as we have already seen, either decrease or keep unchanged \( G_K \), and do not add new function variables. The application of Mix either fails, or strictly reduces \( (G_K, V_F(K)) \): TVE-mix strictly decreases \( G_K \), and FVE-mix does not change \( G_K \) but strictly decreases \( V_F(K) \). Hence, the while loop in Solve can be executed only finitely many times.

If Solve does not stop with false, the only possible non-solved primitive constraints are those between variables, whose left hand side has occurrences in at least two different kind of
Theorem 24 (Soundness of Solve). Let $\mathcal{K}$ be a conjunction of primitive constraints. Then every solution of the constraint $\text{Solve}(\mathcal{K})$ is a solution of $\mathcal{K}$.

Proof. By induction on the length of a rule application sequence leading from $\mathcal{K}$ to $\text{Solve}(\mathcal{K})$, using the soundness lemmas for equality, similarity, and mixed rules (Lemmas 11, 13, 18).

Theorem 25 (Completeness of Solve). Let $\mathcal{K}$ be a conjunction of primitive constraints, and $\vartheta$ be its solution. Then $\text{Solve}(\mathcal{K})$ is a constraint $(\mathcal{K}_{\text{sol}} \land \mathcal{K}_{\text{var}}) \lor \mathcal{C}$, where $\mathcal{K}_{\text{sol}} \land \mathcal{K}_{\text{var}}$ is in appr-solved form, and $\mathcal{K}_{\text{var}} \sigma_{\mathcal{K}_{\text{var}}} \preceq_{W(\mathcal{K})} \vartheta$, where $\sigma_{\mathcal{K}_{\text{sol}}}$ is the substitution induced by $\mathcal{K}_{\text{sol}}$, and $\mathcal{K}_{\text{var}}$ is a solution of $\mathcal{K}_{\text{var}}$.

Proof. In the proof we use completeness of unification and weak unification algorithms [4,9,17].

First, note that if one of the failure rules is applicable to a constraint, then it has no solution.

For Occ-mix it follows from Theorem 17. For Mism-mix, it is guaranteed by the fact that symbols with different arities are not similar. For failure rules in Unif and Sim it is known from their completeness results.

Application of Unif to $\mathcal{K}$ leads to a new constraint $\mathcal{C}_{\text{un}}$, which contains a solved from $\mathcal{K}_{\text{un-sol}}$ such that $\sigma_{\mathcal{K}_{\text{un-sol}}} \preceq_{W(\mathcal{K})} \vartheta$. Application of Sim to $\mathcal{C}_{\text{un}}$ gives $\mathcal{C}_{\text{sim}}$, which contains a solved form $\mathcal{K}_{\text{sim-sol}}$ (an extension of $\mathcal{K}_{\text{un-sol}}$) such that $\sigma_{\mathcal{K}_{\text{sim-sol}}} \preceq_{W(\mathcal{K})} \vartheta$. After that, if TVE-mix is applicable, we have an equation $X \preceq_{\mathcal{K}_{\text{var}}} f(t_1, \ldots , t_n)$. TVE-mix extends the solved form by a new equation $X \equiv^{\tau} \rho(f(t_1, \ldots , t_n))$, obtaining $\mathcal{K}_{\text{tve-sol}}$. By definition of $\rho$, the term $\rho(f(t_1, \ldots , t_n))$ contains fresh variables for each symbol in $f(t_1, \ldots , t_n)$ and, hence, $\sigma_{\mathcal{K}_{\text{tve-sol}}} \preceq_{W(\mathcal{K})} \vartheta$. It is important at this step to record which fresh variable is a copy of which original variable, maintaining an function $\text{original-of}(V') = V$, where $V \in \text{var}(\mathcal{K})$ and $V'$ is zero or more applications of $\rho$ to it (i.e., $V'$ is $V$, or its copy, or a copy of its copy etc.).

If the rule FVE-mix is applicable, we have an equation $F \preceq_{\mathcal{K}_{\text{var}}} f$. We make a step by this rule, adding a new equation $F \equiv^{\tau}_{\text{original-of}(F)} \vartheta$ and obtaining a new solved form $\mathcal{K}_{\text{fve-sol}}$. Let $\varphi$ be the substitution $\{\text{original-of}(F) \mapsto \text{original-of}(F) \vartheta\}$. Then we have $\sigma_{\mathcal{K}_{\text{fve-sol}}} \varphi \preceq_{W(\mathcal{K})} \vartheta$. Iterating this process, we do not get false, since $\mathcal{K}$ was solvable. By Theorem 23, the process terminates with an appr-solved form $\mathcal{K}_{\text{sol}} \land \mathcal{K}_{\text{var}}$ such that $\sigma_{\mathcal{K}_{\text{var}}} \varphi_1 \cdots \varphi_k \preceq_{W(\mathcal{K})} \vartheta$, where the $\varphi$’s are substitutions of the form $\{\text{original-of}(V) \mapsto \text{original-of}(V) \vartheta\}$. Let $\mathcal{K}_{\text{var}}$ be the restriction of $\vartheta$ to the variables of $\mathcal{K}_{\text{var}}$. Then $\sigma_{\mathcal{K}_{\text{var}}}$ is a solution of $\mathcal{K}_{\text{var}}$. And we have $\sigma_{\mathcal{K}_{\text{sol}}} \varphi_1 \cdots \varphi_k \sigma_{\mathcal{K}_{\text{var}}} \preceq_{W(\mathcal{K})} \vartheta$.

For the $\varphi$’s, composition is commutative, because they are ground substitutions with disjoint domains. For some of them we have $\sigma_{\mathcal{K}_{\text{sol}}} \varphi_1 = \sigma_{\mathcal{K}_{\text{sol}}}$, because at some step we might have solved an equation with $\text{original-of}(V)$ variable in its left for $\text{original-of}(V) \in \text{dom}(\varphi_1)$. We assume that the $\varphi$’s in the composition are rearranged so that $\sigma_{\mathcal{K}_{\text{sol}}} \varphi_1 \cdots \varphi_i = \sigma_{\mathcal{K}_{\text{sol}}} \varphi_{i+1} \cdots \varphi_k$. These remaining $\varphi$’s are those for which the algorithm reached a variables-only equation containing $\text{original-of}(V)$, which occurs in the domain of one of the $\varphi$’s. But then $\varphi_{i+1} \cdots \varphi_k$ is a part of $\sigma_{\mathcal{K}_{\text{var}}}$. Hence, we can get rid of them, obtaining $\sigma_{\mathcal{K}_{\text{sol}}} \sigma_{\mathcal{K}_{\text{var}}}$.

Hence, we get $\sigma_{\mathcal{K}_{\text{sol}}} \sigma_{\mathcal{K}_{\text{var}}} \preceq_{W(\mathcal{K})} \vartheta$.

4 Computing approximation degrees

In the algorithm, we have not included the computation of approximation degrees, but it can be done easily. Instead of constraints in DNF of the form $K_1 \lor \cdots \lor K_n$, we will be working with expressions (we call them extended constraints) $(K_1, D_1) \lor \cdots \lor (K_n, D_n)$,
where $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are approximation degrees. The rules will carry the degree ("computed so far") as an additional parameter, but only two rules would change them: Del-sim and FVE-mix. Their variants with degree modification would work on constraint-degree pairs ($\psi$ stands for disjoint union):

$$\text{Del-sim-deg} : (\tau_1 \simeq_{\mathcal{R}, \lambda} \tau_2 \land \mathcal{K}, \{(\mathcal{R}, \emptyset) \cup \mathcal{D}\}) \leadsto (\mathcal{K}, \{\langle \mathcal{R}, \min\{\Delta, \mathcal{R}(\tau_1, \tau_2)\}\rangle \cup \mathcal{D}\})$$

where $\tau_1, \tau_2 \in C_T \cup V_F \cup V_T$ and $\mathcal{R}(\tau_1, \tau_2) \geq \lambda$.

$$\text{FVE-mix-deg} : (F \simeq_{\mathcal{R}, \lambda} f \land \mathcal{K}, \{(\mathcal{R}, \emptyset) \cup \mathcal{D}\}) \leadsto \forall g \in N(f, \mathcal{R}, \lambda) \left( F \simeq g \land \mathcal{K}\{F \mapsto g\}, \{(\mathcal{R}, \min\{\Delta, \mathcal{R}(f, g)\}) \cup \mathcal{D}\}, \right)$$

where $F \in \text{var}(\mathcal{K})$.

For any other rule $\mathcal{R}$ of the form $\mathcal{K} \leadsto \mathcal{K}_1 \lor \cdots \lor \mathcal{K}_n$, its degree variant $\mathcal{R}$-deg will have the form $(\mathcal{K}, \emptyset) \leadsto (\mathcal{K}_1, \emptyset) \lor \cdots \lor (\mathcal{K}_n, \emptyset)$, i.e., $\emptyset$ will not change. Let us denote the corresponding versions of Unif, Sim, and Mix by Unif-deg, Sim-deg, and Mix-deg. The notions of solved and approx-solved forms generalize directly to extended constraints. Then we can define $\text{Solve-deg}$ along the lines of $\text{Solve}$: To solve a conjunction of primitive equality and similarity constraints $\mathcal{K}$ with respect to similarity relations $\mathcal{R}_1, \ldots, \mathcal{R}_m$, it performs the following steps:

$$\mathcal{C} := (\mathcal{K}, \{(\mathcal{R}_1, 1), \ldots, (\mathcal{R}_m, 1)\})$$

while $\mathcal{C}$ is not in the appr-solved form do

- $\mathcal{C} := \text{Unif-deg}(\mathcal{C})$, if $\mathcal{C} = \text{false}$, return false
- $\mathcal{C} := \text{Sim-deg}(\mathcal{C})$, if $\mathcal{C} = \text{false}$, return false
- $\mathcal{C} := \text{Mix-deg}(\mathcal{C})$, if $\mathcal{C} = \text{false}$, return false

return $\mathcal{C}$

5 Discussion and summary

The proposed solver can be used in constraint-based formalisms such as, for instance, constraint logic programming [11] or term rewriting with constraints [13]. We can envisage an instance of the CLP schema with constraints over multiple similarities. Without going into much details, a simple constraint logic program below can illustrate this possibility:

Example 26. The letter $P$ in the program stands for a predicate variable. In constraints, it is treated as a function variable.

$$P(X, Y) \leftarrow P \simeq_{\mathcal{R}_1, \lambda_1} p, X \simeq_{\mathcal{R}_2, \lambda_2} F(Y), Y \simeq_{\mathcal{R}_3, \lambda_3} c, r(X), r(F(Y)).$$

$$r(F(X)) \leftarrow F \simeq_{\mathcal{R}_2, \lambda_2} h, X \simeq_{\mathcal{R}_1, \lambda_1} a.$$  
Assume $\mathcal{R}_1(p, q) = 0.9$, $\mathcal{R}_2(a, b) = 0.8$, $\mathcal{R}_1(f, g) = 0.6$, $\mathcal{R}_2(g, h) = 0.5$, $\mathcal{R}_2(b, c) = 0.7$, $\lambda_1 = \lambda_2 = 0.4$. Then by performing the usual CLP inference (i.e., syntactically unifying the selected query and the head of the corresponding clause) for the query $\leftarrow g(X, Y)$, the resulting constraint (together with the approximation degrees) will be $(X \simeq g(b) \land Y \simeq b; \{(\mathcal{R}_1, 0.8), (\mathcal{R}_2, 0.5)\}) \lor (X \simeq h(b) \land Y \simeq b; \{(\mathcal{R}_1, 0.8), (\mathcal{R}_2, 0.7)\})$.

To summarize, the algorithm $\text{Solve}$ presented in the paper solves positive equational and similarity constraints, where multiple similarity relations are permitted. Given such a constraint in DNF, it computes disjunction of approximately solved forms, from which solution substitutions can be read off. It can be easily extended to include the computation of approximation degrees of the solutions. The algorithm is terminating, sound, and complete.
References


Constraint solving over multiple similarity relations


A Intuitive explanation of a part of the proof of Theorem 3

In the proof of Theorem 3, case i, we assumed $v \notin M(G_1, G_2)$ and reached a contradiction. The reasoning line there can be illustrated by the diagram below, where we step by step construct the vertices $v', v'', v'''$ until we reach a contradiction with another assumption $v \in M(G_1, G_3)$. The symbols $\subset$, $\supset$, $=$ between the vertices show the relation between the marked sets of those vertices. The leftmost graph is a part of $G_1$, the middle one of $G_2$, and the rightmost one of $G_3$. 

$$
\begin{align*}
G_1 & \quad G_2 & \quad G_3 \\
\text{mark}(v, G_1) \subset \text{mark}(v, G_2) = \text{mark}(v, G_3) \\
\text{mark}(v, G_1) \supset \text{mark}(v, G_2) \subset \text{mark}(v, G_3) \\
\text{mark}(v, G_1) \subset \text{mark}(v, G_2) \supset \text{mark}(v, G_3) \\
\text{mark}(v, G_1) \supset \text{mark}(v, G_2) = \text{mark}(v, G_3)
\end{align*}
$$