McCarthy-Kleene fuzzy automata and MSO logics*

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Abstract

We introduce McCarthy-Kleene fuzzy automata (MK-fuzzy automata) over a bimonoid K which is related to the fuzzification of the McCarthy-Kleene logic. Our automata are inspired by, and intend to contribute to, practical applications being in development in a project on runtime network monitoring based on predicate logic. We investigate closure properties of the class of recognizable MK-fuzzy languages accepted by MK-fuzzy automata as well as of deterministically recognizable MK-fuzzy languages accepted by their deterministic counterparts. Moreover, we establish a Nivat-like result for recognizable MK-fuzzy languages. We introduce an MK-fuzzy MSO logic and show the expressive equivalence of a fragment of this logic with MK-fuzzy automata, i.e., a Büchi type theorem.

Keywords: Bimonoids, McCarthy-Kleene logic, MK-fuzzy automata, MK-fuzzy MSO logics

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1. Introduction

Fuzzy automata constitute a special model of weighted automata but historically have been defined and studied separately, mostly inspired by fuzzy logic theory. The original fuzzy automaton model assigned to words values from the lattice $[0, 1]$ with the usual max and min operations. Later on, fuzzy automata were investigated also over more general structures like for instance lattices, residuated lattices, and l-monoids. Several real world applications are modelled by fuzzy automata. We refer the reader to [20] for fuzzy automata theory and applications, to [22] for a generalization of them and their connection to weighted automata, and to [1] for fuzzy semirings related to automata. For weighted automata theory, the interested reader should consult for instance [7, 8, 10].

On the other hand, McCarthy-Kleene logic (MK-logic for short), a combination of three-valued logics of McCarthy [19] and Kleene [13], has been introduced in [14, 2] to reason about computation errors. The original idea, according to [2], was to distinguish between two types of errors: critical ones, which make the whole computation stop and cause a total failure of the program, and non-critical ones, which stop only part of the computation and can be fixed or circumvented by a success in some other part. MK-logic is a four-valued logic, where alongside the truth values t (true) and f (false) there are also u (undefined, which originates from Kleene's logic) and e (error, which comes from McCarthy's logic). In this combination, 'undefined' is intended to represent non-critical errors, while 'error' is reserved for critical ones. As in McCarthy's logic, interpretation of binary connectives is asymmetric, which means, for instance, that the disjunction of t and e is t, while the disjunction of e and t gives e. In the combination it is assumed that e prevails u in whatever order they appear.

MK-logic has found an application in the LogicGuard project [17, 18, 15, 4] which pursues research on network security, developing a specification and verification formalism and tool for runtime network monitoring based on predicate logic. A monitor, which is a logical formula (usually with quantifiers), is interpreted over a network (an infinite stream of messages). The goal is to check whether the property specified in the monitor is satisfied by the stream, and report violating messages, if any. For instance, the following monitoring formula

$$\textbf{monitor } x : p(x) \Rightarrow \textbf{exists } y \textbf{ with } x \leq y \leq x + T : q(x, y)$$

investigates for every stream position x that satisfies $p(x)$ whether there exists some position y in range $[x, x + T]$ such that property $q(x, y)$ holds. Operationally, the monitor formula is translated into a program, which accepts stream messages one after the other, keeps evaluating the monitored property on the known part of the stream, and if it is violated (i.e., its truth value becomes f), reports the message that caused the violation. At each moment, the monitor observes only a finite initial part of the stream. Hence, it is not always possible to decide whether the property holds or not ('not enough' messages have arrived). In this case, a new copy of the current instance of the monitoring formula is created. Its truth value is u: undefinedness here really corresponds to 'unknown', not to a non-critical error. The copy is added to the pile of copies of some previous instances, which also wait to be decided. Each of these copies will be evaluated for the incoming messages and will be removed from consideration if its truth value becomes t or f. In the latter case, the violated message is reported. If something causes an error (i.e., if the truth value e is generated for some reason), monitoring stops. The LogicGuard framework has met the expectations of the developers, being successfully used for runtime network monitoring. As the next step, it is planned to deploy it for new application scenarios such as, for instance, "Internet of Things". Such applications pose new challenges, related to the difficulties with quantification of decisions, or to the fact that it is not a priori clear what the expectations of a correct execution of a system are. To deal with such problems, reasoning with some kind of probabilistic or fuzzy knowledge is required. As the first step towards this direction, we envisage the extension of the LogicGuard specification language to a fuzzy quantified logic that is able to handle specifications including uncertainty and vagueness. On this strand, and for the development of the fuzzification of the MK-logic and relative models, we introduce MK-fuzzy automata, and this paper is a first attempt to study these models. Our MK-fuzzy automata assign, to words, values from the bimonoid

$$K = \{(t, f, u, e) \in [0, 1]^4 \mid t + f + u + e = 1\}$$

where its operations, called MK-disjunction and MK-conjunction, are inspired by the fuzzification of the MK-logic. Formal series with values in K are called MK-fuzzy languages.

Classical operations in formal series over semirings cannot be defined in the usual way over bimonoids due to the lack of commutativity and distributivity properties. Notable examples are the Cauchy product and the star operation. If the weight structure is weaker than a semiring, for instance a bimonoid like in our case, then the lack of commutativity, distributivity, and multiplicative zero properties has a serious impact on the automata models considered over such a weight structure. For instance the value assigned by the automaton to a word cannot be defined in the usual way. Due to these difficulties, and since no interesting bimonoid structures have been considered so far, there is a lack of work on weighted automata over bimonoids. According to our best knowledge, the most relative works deal with automata and transducers over strong bimonoids where the first operation is commutative and there is a multiplicative zero [5, 11, 16]. For our MK-fuzzy automata, where a multiplicative zero is missing from the bimonoid K, we consider a set of initial states, a set of transitions, and a set of final states and define on these sets the initial distribution, the mapping assigning truth values to the transitions of the automaton, and the terminal distribution, respectively. Our model is nondeterministic. Since the MK-disjunction is not commutative, we require the state set of the MK-fuzzy automaton to be linearly ordered. Then the paths of the automaton over any word can be ordered according to lexicographic order, and hence we can define the value of K assigned by the MK-fuzzy automaton to the given word.

We show that the class of recognizable MK-fuzzy languages accepted by MK-fuzzy automata is closed under MK-disjunction, strict alphabetic homomorphisms and inverse strict alphabetic homomorphisms. Moreover, we establish a Nivat-like decomposition result [21] showing that recognizable MK-fuzzy languages can be obtained from very particular MK-fuzzy automata (in fact, with only one state), restriction to recognizable languages and strict alphabetic homomorphisms. We introduce also the deterministic counterpart of our model and show that the class of MK-fuzzy languages accepted by these automata, called deterministically recognizable, is closed under MK-disjunction with scalars from the right. MK-disjunction with scalars form the left results to recognizable MK-fuzzy languages. The Cauchy product of two deterministically recognizable MK-fuzzy languages is a recognizable MK-fuzzy language. Due to the structure of the bimonoid K, we can define several notions of supports of MK-fuzzy languages. We show that the strong support, related to the first component of the elements in K, of a deterministically recognizable MK-fuzzy language is a recognizable language. Furthermore, we introduce an MK-fuzzy MSO logic and determine a fragment of sentences which is expressively equivalent to the class of MK-fuzzy automata, i.e., a Büchi type theorem.

A preliminary version of this paper appeared in [9]. The present version contains detailed proofs and further explanations. In addition we proved a normalization result (Proposition 16) for nondeterministic MK-fuzzy automata, needed for the proofs and being of independent value due to the differences of the notions of normalization for weighted automata and our models.

2. Preliminaries

Let A be an alphabet, i.e., a finite nonempty set. As usually, we denote by A^* the set of all finite words over A and define $A^+ = A^* \setminus \{\varepsilon\}$, where ε is the empty word. The length of a word w, i.e., the number of the letters of w is denoted as usual by $|w|$. A word $w = a_0 \dots a_{n-1}$ over A, with $a_0, \dots, a_{n-1} \in A$, is written also as $w = w(0) \dots w(n-1)$ with $w(i) = a_i$ for every $0 \leq i \leq n-1$. Assume now that $\leq$ is a linear order on A. The lexicographic order $\leq_{lex}$ on A^* is defined as follows:

$$w \leq_{lex} w' \quad \text{iff} \quad ((w' = wv \text{ with } v \in A^*) \text{ or } (w = v_1 a v_2,\ w' = v_1 b v_2',\ v_1 \in A^*,\ a, b \in A \text{ with } a < b))$$

for every $w, w' \in A^*$. Let now A and B be linearly ordered sets, respectively by $\leq_A$ and $\leq_B$. Then, the Cartesian product $A \times B$ is linearly ordered by $\leq$ which is defined, as usual, in the following way:

$$(a, b) \leq (a', b') \quad \text{iff} \quad ((a <_A a') \text{ or } (a = a' \text{ and } b \leq_B b'))$$

for every $(a, b), (a', b') \in A \times B$. In a similar way, the linear orders of three sets induce a linear order on their Cartesian product. If no confusion arises, we shall use the same symbol $\leq$ to denote every linear order considered in the sequel.

Throughout the paper A will denote an alphabet.

A *bimonoid* $(K, +, \cdot, 0, 1)$ (cf. [11]) consists of a set K, two binary operations $+$ and $\cdot$ and two constant elements 0 and 1 such that $(K, +, 0)$ and $(K, \cdot, 1)$ are monoids. If the monoid $(K, +, 0)$ is commutative and 0 acts as a multiplicative zero, i.e., $k \cdot 0 = 0 \cdot k = 0$ for every $k \in K$, then the bimonoid is called *strong*. The bimonoid is denoted simply by K if the operations and the constant elements are understood. A semiring is a strong bimonoid where multiplication distributes over addition. A bimonoid K is called zero-sum free if $k + k' = 0$ implies $k = k' = 0$, and it is called zero-divisor free if $k \cdot k' = 0$ implies $k = 0$ or $k' = 0$, for every $k, k' \in K$.

In this paper we deal with a new type of fuzzy sets with values in the Cartesian product $[0,1]^4 = [0,1] \times [0,1] \times [0,1] \times [0,1]$, such that their components are summing up to 1. This type of fuzzy sets is inspired by McCarthy-Kleene logic (MK-logic for short). MK-logic which is a combination of three-valued logics of McCarthy [19] and Kleene [13], has been introduced in [14, 2] to reason about computation errors. It is a four-valued logic, where alongside the truth values t (true) and f (false) there are also u (undefined, which originates from Kleene's logic) and e (error, which comes from McCarthy's logic). In this combination, 'undefined' is intended to represent non-critical errors, while ‘error’ is reserved for critical ones. For the reader’s convenience we recall the truth tables of MK-logic:

or	t	f	u	e
t	t	t	t	t
f	t	f	u	e
u	t	u	u	e
e	e	e	e	e

not	t	f	u	e
	f	t	u	e

and	t	f	u	e
t	t	f	u	e
f	f	f	f	f
u	u	f	u	e
e	e	e	e	e

implies	t	f	u	e
t	t	f	u	e
f	t	t	t	t
u	t	u	u	e
e	e	e	e	e

For the fuzzification of the MK-logic we assign to t, f, u, e values from the interval $[0, 1]$ with the restriction that they are summing up to 1. Therefore, our fuzzy sets get their values in the subset K of the Cartesian product $[0, 1]^4$ which is defined as follows:

$$K = \{(t, f, u, e) \in [0, 1]^4 \mid t + f + u + e = 1\}.$$

Due to practical applications, by which our theory is motivated (cf. [15]), we refer to the four components of the elements of K to as the *true*, *false*, *unknown*, and *error* value, respectively. We shall denote the elements of K with bold symbols and we shall call them the *truth values* of our fuzzy sets. For $\mathbf{k} = (t, f, u, e) \in K$ we shall write sometimes $x(\mathbf{k})$ for $x \in \{t, f, u, e\}$, to denote the x value of $\mathbf{k}$. For every $\mathbf{k_1} = (t_1, f_1, u_1, e_1), \mathbf{k_2} = (t_2, f_2, u_2, e_2) \in K$ we let $\mathbf{k_3} = \mathbf{k_1} \sqcup \mathbf{k_2}$ and $\mathbf{k_4} = \mathbf{k_1} \sqcap \mathbf{k_2}$ where $\mathbf{k_3} = (t_3, f_3, u_3, e_3)$ and $\mathbf{k_4} = (t_4, f_4, u_4, e_4)$ are defined by the relations

$$\begin{aligned}
&t_3 = t_1 + (f_1 + u_1)t_2 && t_4 = t_1 t_2 \\
&f_3 = f_1 f_2 && f_4 = f_1 + (t_1 + u_1)f_2 \\
&u_3 = f_1 u_2 + u_1(f_2 + u_2) && u_4 = t_1 u_2 + u_1(t_2 + u_2) \\
&e_3 = e_1 + (f_1 + u_1)e_2 && e_4 = e_1 + (t_1 + u_1)e_2.
\end{aligned}$$

It is not difficult to see that $\mathbf{k_3}, \mathbf{k_4} \in K$, therefore $\sqcup$ and $\sqcap$ are well-defined operations on K. Indeed, let us present the proof for $\mathbf{k_4}$; the proof for $\mathbf{k_3}$ is similar. By standard computations we get $0 \leq t_4, f_4, u_4, e_4 \leq 1$. Furthermore, we calculate

$$\begin{aligned} t_4 + f_4 + u_4 + e_4 &= t_1t_2 + f_1 + (t_1 + u_1)f_2 + t_1u_2 + u_1(t_2 + u_2) + e_1 + (t_1 + u_1)e_2 \\ &= t_1t_2 + f_1 + t_1f_2 + u_1f_2 + t_1u_2 + u_1t_2 + u_1u_2 + e_1 + t_1e_2 + u_1e_2 \\ &= t_1(t_2 + f_2 + u_2 + e_2) + f_1 + u_1(f_2 + t_2 + u_2 + e_2) + e_1 \\ &= t_1 + f_1 + u_1 + e_2 = 1 \end{aligned}$$

as wanted. We call $\sqcup$ the *MK-disjunction* (*disjunction* for simplicity) and $\sqcap$ the *MK-conjunction* (*conjunction* for simplicity). The result of the empty MK-conjunction equals $\mathbf{1}$. MK-disjunction and MK-conjunction correspond to the fuzzification of the connectives 'or', 'and' of the MK-logic, respectively. To clarify this, we preserve the above notations for $\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}$, and $\mathbf{k_4}$ and construct the following multiplication table:

	t_2	f_2	u_2	e_2
t_1	t_1t_2	t_1f_2	t_1u_2	t_1e_2
f_1	f_1t_2	f_1f_2	f_1u_2	f_1e_2
u_1	u_1t_2	u_1f_2	u_1u_2	t_1e_2
e_1	e_1t_2	e_1f_2	e_1u_2	e_1e_2

(1)

For $y = t, f, u, e$, we compute every component $y_3 \in \{t_3, f_3, u_3, e_3\}$ of $\mathbf{k_3}$ by summing up the values of the cells in table (1) above, such that the corresponding cells in the truth table of 'or' contain the value y. Similarly, for $\mathbf{k_4}$ we compute every component $y_4 \in \{t_4, f_4, u_4, e_4\}$ of $\mathbf{k_4}$ by summing up the values of the cells in table (1) above, such that the corresponding cells in the truth table of 'and' contain the value y. For instance $t_3 = t_1t_2 + t_1f_2 + t_1u_2 + t_1e_2 + f_1t_2 + u_1t_2 = t_1(t_2 + f_2 + u_2 + e_2) + (f_1 + u_1)t_2 = t_1 + (f_1 + u_1)t_2$ and $t_4 = t_1t_2$.

Proposition 1. *The disjunction and conjunction operations on K are associative with unit elements $\mathbf{0} = (0, 1, 0, 0)$ and $\mathbf{1} = (1, 0, 0, 0)$, respectively.*

Proof. Let $\mathbf{k_1} = (t_1, f_1, u_1, e_1), \mathbf{k_2} = (t_2, f_2, u_2, e_2), \mathbf{k_3} = (t_3, f_3, u_3, e_3) \in K$. We show firstly, the associativity property for the disjunction operation. For this we let $(\mathbf{k_1} \sqcup \mathbf{k_2}) \sqcup \mathbf{k_3} = (t, f, u, e)$ and $\mathbf{k_1} \sqcup (\mathbf{k_2} \sqcup \mathbf{k_3}) = (t', f', u', e')$. By definition, we have

$$\mathbf{k_1} \sqcup \mathbf{k_2} = (t_1 + (f_1 + u_1)t_2,\ f_1f_2,\ f_1u_2 + u_1(f_2 + u_2),\ e_1 + (f_1 + u_1)e_2)$$

and

$\mathbf{k_2} \sqcup \mathbf{k_3} = (t_2 + (f_2 + u_2)t_3,\ f_2 f_3,\ f_2 u_3 + u_2(f_3 + u_3),\ e_2 + (f_2 + u_2)e_3).$

Furthermore, we get

$$
\begin{aligned}
t &= t_1 + (f_1 + u_1)t_2 + (f_1 f_2 + f_1 u_2 + u_1(f_2 + u_2))t_3 \\
&= t_1 + (f_1 + u_1)t_2 + f_1 f_2 t_3 + f_1 u_2 t_3 + u_1 f_2 t_3 + u_1 u_2 t_3 \\
&= t_1 + (f_1 + u_1)t_2 + (f_1 + u_1) f_2 t_3 + (f_1 + u_1) u_2 t_3 \\
&= t_1 + (f_1 + u_1)(t_2 + (f_2 + u_2)t_3) \\
&= t',
\end{aligned}
$$

$$
\begin{aligned}
f &= (f_1 f_2) f_3 \\
&= f_2(f_2 f_3) \\
&= f',
\end{aligned}
$$

$$
\begin{aligned}
u &= f_1 f_2 u_3 + (f_1 u_2 + u_1(f_2 + u_2))(f_3 + u_3) \\
&= f_1 f_2 u_3 + f_1 u_2 f_3 + f_1 u_2 u_3 + u_1 f_2 f_3 + u_1 f_2 u_3 + u_1 u_2 f_3 + u_1 u_2 u_3 \\
&= f_1(f_2 u_3 + u_2(f_3 + u_3)) + u_1(f_2 f_3 + f_2 u_3 + u_2(f_3 + u_3)) \\
&= u',
\end{aligned}
$$

$$
\begin{aligned}
e &= e_1 + (f_1 + u_1)e_2 + (f_1 f_2 + f_1 u_2 + u_1(f_2 + u_2))e_3 \\
&= e_1 + (f_1 + u_1)e_2 + f_1 f_2 e_3 + f_1 u_2 e_3 + u_1 f_2 e_3 + u_1 u_2 e_3 \\
&= e_1 + (f_1 + u_1)e_2 + (f_1 + u_1) f_2 e_3 + (f_1 + u_1) u_2 e_3 \\
&= e_1 + (f_1 + u_1)(e_2 + (f_2 + u_2)e_3) \\
&= e'
\end{aligned}
$$

which implies that $(\mathbf{k_1} \sqcup \mathbf{k_2}) \sqcup \mathbf{k_3} = \mathbf{k_1} \sqcup (\mathbf{k_2} \sqcup \mathbf{k_3})$.

Next we proceed with the associativity of conjunction. For this we let $(\mathbf{k_1} \sqcap \mathbf{k_2}) \sqcap \mathbf{k_3} = (\tilde{t}, \tilde{f}, \tilde{u}, \tilde{e})$ and $\mathbf{k_1} \sqcap (\mathbf{k_2} \sqcap \mathbf{k_3}) = (\tilde{t}', \tilde{f}', \tilde{u}', \tilde{e}')$. By definition, we have

$\mathbf{k_1} \sqcap \mathbf{k_2} = (t_1 t_2,\ f_1 + (t_1 + u_1) f_2,\ t_1 u_2 + u_1(t_2 + u_2),\ e_1 + (t_1 + u_1)e_2)$

and

$\mathbf{k_2} \sqcap \mathbf{k_3} = (t_2t_3,\ f_2 + (t_2 + u_2)f_3,\ t_2u_3 + u_2(t_3 + u_3),\ e_2 + (t_2 + u_2)e_3).$

Then we get

$$\begin{aligned}
\tilde{t} &= (t_1t_2)t_3 \\
&= t_1(t_2t_3) \\
&= \tilde{t}',
\end{aligned}$$

$$\begin{aligned}
\tilde{f} &= f_1 + (t_1 + u_1)f_2 + (t_1t_2 + t_1u_2 + u_1(t_2 + u_2))f_3 \\
&= f_1 + (t_1 + u_1)f_2 + t_1t_2f_3 + t_1u_2f_3 + u_1t_2f_3 + u_1u_2f_3 \\
&= f_1 + (t_1 + u_1)f_2 + (t_1 + u_1)t_2f_3 + (t_1 + u_1)u_2f_3 \\
&= f_1 + (t_1 + u_1)f_2 + (t_1 + u_1)(t_2 + u_2)f_3 \\
&= f_1 + (t_1 + u_1)(f_2 + (t_2 + u_2)f_3) \\
&= \tilde{f}',
\end{aligned}$$

$$\begin{aligned}
\tilde{u} &= t_1t_2u_3 + (t_1u_2 + u_1(t_2 + u_2))(t_3 + u_3) \\
&= t_1t_2u_3 + t_1u_2t_3 + t_1u_2u_3 + u_1t_2t_3 + u_1t_2u_3 + u_1u_2t_3 + u_1u_2u_3 \\
&= t_1(t_2u_3 + u_2t_3 + u_2u_3) + u_1(t_2t_3 + t_2u_3 + u_2t_3 + u_2u_3) \\
&= t_1(t_2u_3 + u_2(t_3 + u_3)) + u_1(t_2t_3 + t_2u_3 + u_2(t_3 + u_3)) \\
&= \tilde{u}',
\end{aligned}$$

$$\begin{aligned}
\tilde{e} &= e_1 + (t_1 + u_1)e_2 + (t_1t_2 + t_1u_2 + u_1(t_2 + u_2))e_3 \\
&= e_1 + t_1e_2 + u_1e_2 + t_1t_2e_3 + t_1u_2e_3 + u_1t_2e_3 + u_1u_2e_3 \\
&= e_1 + (t_1 + u_1)e_2 + (t_1 + u_1)t_2e_3 + (t_1 + u_1)u_2e_3 \\
&= e_1 + (t_1 + u_1)(e_2 + t_2e_3 + u_2e_3) \\
&= e_1 + (t_1 + u_1)(e_2 + (t_2 + u_2)e_3) \\
&= \tilde{e}'
\end{aligned}$$

and hence $(\mathbf{k_1} \sqcap \mathbf{k_2}) \sqcap \mathbf{k_3} = \mathbf{k_1} \sqcap (\mathbf{k_2} \sqcap \mathbf{k_3})$, as required.

We show now that $\mathbf{0}, \mathbf{1}$ are the unit elements of disjunction and conjunction, respectively. Indeed, we have $\mathbf{k_1} \sqcup \mathbf{0} = (t_1 + (f_1 + u_1)0, f_1 1, f_1 0 + u_1(1+0), e_1 + (f_1 + u_1)0) = (t_1, f_1, u_1, e_1)$ and $\mathbf{0} \sqcup \mathbf{k_1} = (0 + (1+0)t_1, 1f_1, 1u_1 + 0(f_1 + u_1), 0 + (1+0)e_1) = (t_1, f_1, u_1, e_1)$. Finally, $\mathbf{k_1} \sqcap \mathbf{1} = (t_1 1, f_1 + (t_1 + u_1)0, t_1 0 + u_1(1+0), e_1 + (t_1 + u_1)0) = (t_1, f_1, u_1, e_1)$ and $\mathbf{1} \sqcap \mathbf{k_1} = (1t_1, 0 + (1 + 0)f_1, 1u_1 + 0(t_1 + u_1), 0 + (1+0)e_1) = (t_1, f_1, u_1, e_1)$, and we are done. ■

The result of the empty MK-conjunction equals $\mathbf{1}$. By Proposition 1, we immediately get the next corollary.

Corollary 2. *The structure $(K, \sqcup, \sqcap, \mathbf{0}, \mathbf{1})$ is a bimonoid.*

Nevertheless, by the following proposition we conclude that the bimonoid $(K, \sqcup, \sqcap, \mathbf{0}, \mathbf{1})$ is not strong.

Proposition 3. *Both the disjunction and conjunction operations on K are not commutative and not idempotent. Furthermore, for every $\mathbf{k} = (t, f, u, e) \in K$ we get $\mathbf{0} \sqcap \mathbf{k} = \mathbf{0}$ and $\mathbf{k} \sqcap \mathbf{0} = (0, t + f + u, 0, e)$.*

Proof. Consider the elements $\mathbf{k} = (0.3, 0.2, 0.4, 0.1), \mathbf{k}' = (0.9, 0.05, 0.03, 0.02)$ of K. Then we get $\mathbf{k} \sqcup \mathbf{k}' = (0.84, 0.01, 0.038, 0.112)$, $\mathbf{k}' \sqcup \mathbf{k} = (0.924, 0.01, 0.038, 0.028)$, $\mathbf{k} \sqcap \mathbf{k}' = (0.27, 0.235, 0.381, 0.114)$, $\mathbf{k}' \sqcap \mathbf{k} = (0.27, 0.236, 0.381, 0.113)$, $\mathbf{k} \sqcup \mathbf{k} = (0.48, 0.04, 0.32, 0.16)$, and $\mathbf{k} \sqcap \mathbf{k} = (0.09, 0.34, 0.4, 0.17)$. The remaining part of our proposition is proved by a standard calculation. ■

Proposition 4. *Both the disjunction and conjunction on K do not distribute over each other.*

Proof. Let $\mathbf{k} = (t, f, u, e)$ be an arbitrary element in K. Then we can easily show that $\mathbf{k} \sqcap (\mathbf{0} \sqcup \mathbf{1}) \neq (\mathbf{k} \sqcap \mathbf{0}) \sqcup (\mathbf{k} \sqcap \mathbf{1})$ and $(\mathbf{1} \sqcup \mathbf{k}) \sqcap \mathbf{0} \neq (\mathbf{1} \sqcap \mathbf{0}) \sqcup (\mathbf{k} \sqcap \mathbf{0})$, which imply that conjunction is neither left nor right distributive over disjunction. Similarly, we get $\mathbf{k} \sqcup (\mathbf{0} \sqcap \mathbf{1}) \neq (\mathbf{k} \sqcup \mathbf{0}) \sqcap (\mathbf{k} \sqcup \mathbf{1})$ and $(\mathbf{0} \sqcap \mathbf{k}) \sqcup \mathbf{1} \neq (\mathbf{0} \sqcup \mathbf{1}) \sqcap (\mathbf{k} \sqcup \mathbf{1})$, i.e., disjunction is neither left nor right distributive over conjunction. ■

Proposition 5. *The bimonoid K is zero-sum free and zero-divisor free.*

Proof. We show firstly that K is zero-sum free. For this let $\mathbf{k_1} = (t_1, f_1, u_1, e_1), \mathbf{k_2} = (t_2, f_2, u_2, e_2) \in K$ and assume that $\mathbf{k}_1 \sqcup \mathbf{k}_2 = \mathbf{0}$. Hence, we get

$$(t_1 + (f_1 + u_1)t_2,\ f_1 f_2,\ f_1 u_2 + u_1(f_2 + u_2),\ e_1 + (f_1 + u_1)e_2) = (0, 1, 0, 0),$$

i.e.,

- $t_1 + (f_1 + u_1)t_2 = 0$,
- $f_1 f_2 = 1$,
- $f_1 u_2 + u_1(f_2 + u_2) = 0$,
- $e_1 + (f_1 + u_1)e_2 = 0$.

Since $0 \leq f_1, f_2 \leq 1$ the second equality implies that $f_1 = f_2 = 1$, which in turn means that $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{0}$, as required.

Next assume that $\mathbf{k}_1 \sqcap \mathbf{k}_2 = \mathbf{0}$, i.e.,

$(t_1 t_2,\ f_1 + (t_1 + u_1)f_2,\ t_1 u_2 + u_1(t_2 + u_2),\ e_1 + (t_1 + u_1)e_2) = (0, 1, 0, 0)$

and hence,

- $t_1 t_2 = 0$,
- $f_1 + (t_1 + u_1)f_2 = 1$,
- $t_1 u_2 + u_1(t_2 + u_2) = 0$,
- $e_1 + (t_1 + u_1)e_2 = 0$.

The first equality implies that $t_1 = 0$ or $t_2 = 0$.

i) Let $t_1 = 0$. Then, we get

   - $f_1 + u_1 f_2 = 1$,
   - $u_1 t_2 = u_1 u_2 = 0$,
   - $e_1 = u_1 e_2 = 0$.

   By the second relation we get $u_1 = 0$ or $u_2 = 0$. If $u_1 = 0$ since also $e_1 = 0$, by our assumption we get $f_1 = 1$, i.e., $\mathbf{k}_1 = \mathbf{0}$.

   Assume that $u_1 \neq 0$. Then by the second and third relations we get respectively, $t_2 = u_2 = 0$ and $e_2 = 0$, which implies $f_2 = 1$, i.e., $\mathbf{k}_2 = \mathbf{0}$.

ii) Let $t_1 \neq 0$ and $t_2 = 0$. Then we have

   - $f_1 + (t_1 + u_1)f_2 = 1$,
   - $t_1 u_2 = u_1 u_2 = 0$,

- $e_1 = t_1e_2 = u_1e_2 = 0$.

By the second equality we get $u_2 = 0$ and by the third one $e_2 = 0$. Taking into account our assumption, we conclude that $f_2 = 1$, and hence $\mathbf{k}_2 = \mathbf{0}$.

Therefore, in every case $\mathbf{k}_1 = \mathbf{0}$ or $\mathbf{k}_2 = \mathbf{0}$, and our proof is completed. ■

An *MK-fuzzy language over* A *and* K is a mapping $s : A^* \to K$. The *strong support of* s is the language $\mathrm{stgsupp}(s) = \{w \in A^* \mid t(s(w)) \neq 0\}$. For every $w \in A^*$ the MK-fuzzy language $\overline{w}$ is determined by $\overline{w}(v) = \mathbf{1}$ if $v = w$, and $\overline{w}(v) = \mathbf{0}$ otherwise. The *constant* MK-fuzzy language $\widetilde{\mathbf{k}}$ ($\mathbf{k} \in K$) is defined, for every $w \in A^*$, by $\widetilde{\mathbf{k}}(w) = \mathbf{k}$. We shall denote by $K\left\langle\left\langle A^*\right\rangle\right\rangle$ the class of all MK-fuzzy languages over A and K. The *characteristic MK-fuzzy language* $\mathbf{1}_L \in K\left\langle\left\langle A^*\right\rangle\right\rangle$ *of* a language $L \subseteq A^*$ is defined by $\mathbf{1}_L(w) = \mathbf{1}$ if $w \in L$ and $\mathbf{1}_L(w) = \mathbf{0}$ otherwise. Let $s, r \in K\left\langle\left\langle A^*\right\rangle\right\rangle$ and $\mathbf{k} \in K$. The *MK-disjunction* (or simply *disjunction*) $s \sqcup r$, the *MK-conjunction* (or simply *conjunction*) $s \sqcap r$, and the *MK-conjunctions with scalars* (simply *scalar conjunctions*) $\mathbf{k} \sqcap s$ and $s \sqcap \mathbf{k}$ are defined as follows: $(s \sqcup r)(w) = s(w) \sqcup r(w)$, $(s \sqcap r)(w) = s(w) \sqcap r(w)$, and $(\mathbf{k} \sqcap s)(w) = \mathbf{k} \sqcap s(w)$, $(s \sqcap \mathbf{k})(w) = s(w) \sqcap \mathbf{k}$ for every $w \in A^*$. Since the disjunction and conjunction operations among MK-fuzzy languages are defined elementwise, we can easily show that properties of the structure $\left(K\left\langle\left\langle A^*\right\rangle\right\rangle, \sqcup, \sqcap, \widetilde{\mathbf{0}}, \widetilde{\mathbf{1}}\right)$ are inherited by the properties of the structure $(K, \sqcup, \sqcap, \mathbf{0}, \mathbf{1})$, hence $\left(K\left\langle\left\langle A^*\right\rangle\right\rangle, \sqcup, \sqcap, \widetilde{\mathbf{0}}, \widetilde{\mathbf{1}}\right)$ is a bimonoid. The *Cauchy product* rs *of* $r, s \in K\left\langle\left\langle A^*\right\rangle\right\rangle$ is defined as follows. For every $w = a_0 \dots a_{n-1} \in A^*$ with $a_0, \dots, a_{n-1} \in A$ we let

$$rs(w) = (r(\varepsilon) \sqcap s(a_0 \dots a_{n-1})) \sqcup (r(a_0) \sqcap s(a_1 \dots a_{n-1})) \sqcup \dots \sqcup (r(a_0 \dots a_{n-1}) \sqcap s(\varepsilon)).$$

Since disjunction and conjunction are not commutative, and they do not distribute over each other, the Cauchy product is not associative as we state in the next proposition.

Proposition 6. *The Cauchy product operation is not associative.*

Proof. Let $\mathbf{k} = (t, f, u, e) \in K$. By Proposition 3 we get $\mathbf{k} \sqcap \mathbf{0} = (0, t + f + u, 0, e)$ and we can easily see that the false value of the element in K resulting by applying n-times the disjunction operation on $\mathbf{k} \sqcap \mathbf{0}$ with itself, is $(t + f + u)^{n+1}$. Consider the constant MK-fuzzy languages $\widetilde{\mathbf{k}}, \widetilde{\mathbf{0}}$, and $\widetilde{\mathbf{1}}$, and the word $w = a_0a_1 \in A^*$. Then we have

$$\begin{aligned}
\widetilde{\mathbf{k}}(\widetilde{\mathbf{0}}\widetilde{\mathbf{1}})(w) &= \left(\widetilde{\mathbf{k}}(\varepsilon) \sqcap \widetilde{\mathbf{0}}\widetilde{\mathbf{1}}(a_0a_1)\right) \sqcup \left(\widetilde{\mathbf{k}}(a_0) \sqcap \widetilde{\mathbf{0}}\widetilde{\mathbf{1}}(a_1)\right) \sqcup \left(\widetilde{\mathbf{k}}(a_0a_1) \sqcap \widetilde{\mathbf{0}}\widetilde{\mathbf{1}}(\varepsilon)\right) \\
&= \left(\widetilde{\mathbf{k}}(\varepsilon) \sqcap \left(\left(\widetilde{\mathbf{0}}(\varepsilon) \sqcap \widetilde{\mathbf{1}}(a_0a_1)\right) \sqcup \left(\widetilde{\mathbf{0}}(a_0) \sqcap \widetilde{\mathbf{1}}(a_1)\right) \sqcup \left(\widetilde{\mathbf{0}}(a_0a_1) \sqcap \widetilde{\mathbf{1}}(\varepsilon)\right)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\sqcup \Big(\widetilde{\mathbf{k}}(a_0) \sqcap \Big(\Big(\widetilde{\mathbf{0}}(\varepsilon) \sqcap \widetilde{\mathbf{1}}(a_1)\Big) \sqcup \Big(\widetilde{\mathbf{0}}(a_1) \sqcap \widetilde{\mathbf{1}}(\varepsilon)\Big)\Big)\Big) \sqcup \Big(\widetilde{\mathbf{k}}(a_0a_1) \sqcap \Big(\widetilde{\mathbf{0}}(\varepsilon) \sqcap \widetilde{\mathbf{1}}(\varepsilon)\Big)\Big) \\
&= (\mathbf{k} \sqcap (\mathbf{0} \sqcup \mathbf{0} \sqcup \mathbf{0})) \sqcup (\mathbf{k} \sqcap (\mathbf{0} \sqcup \mathbf{0})) \sqcup (\mathbf{k} \sqcap \mathbf{0}) \\
&= (\mathbf{k} \sqcap \mathbf{0}) \sqcup (\mathbf{k} \sqcap \mathbf{0}) \sqcup (\mathbf{k} \sqcap \mathbf{0}) \\
&= \big(\ldots, (t+f+u)^3, \ldots, \ldots\big)\,.
\end{aligned}
$$

On the other hand, we get

$$
\begin{aligned}
(\widetilde{\mathbf{k}}\widetilde{\mathbf{0}})\widetilde{\mathbf{1}}(w) &= \Big(\widetilde{\mathbf{k}}\widetilde{\mathbf{0}}(\varepsilon) \sqcap \widetilde{\mathbf{1}}(a_0a_1)\Big) \sqcup \Big(\widetilde{\mathbf{k}}\widetilde{\mathbf{0}}(a_0) \sqcap \widetilde{\mathbf{1}}(a_1)\Big) \sqcup \Big(\widetilde{\mathbf{k}}\widetilde{\mathbf{0}}(a_0a_1) \sqcap \widetilde{\mathbf{1}}(\varepsilon)\Big) \\
&= \Big(\Big(\widetilde{\mathbf{k}}(\varepsilon) \sqcap \widetilde{\mathbf{0}}(\varepsilon)\Big) \sqcap \widetilde{\mathbf{1}}(a_0a_1)\Big) \sqcup \Big(\Big(\Big(\widetilde{\mathbf{k}}(\varepsilon) \sqcap \widetilde{\mathbf{0}}(a_0)\Big) \sqcup \Big(\widetilde{\mathbf{k}}(a_0) \sqcap \widetilde{\mathbf{0}}(\varepsilon)\Big)\Big) \sqcap \widetilde{\mathbf{1}}(a_1)\Big) \\
&\quad \sqcup \Big(\Big(\Big(\widetilde{\mathbf{k}}(\varepsilon) \sqcap \widetilde{\mathbf{0}}(a_0a_1)\Big) \sqcup \Big(\widetilde{\mathbf{k}}(a_0) \sqcap \widetilde{\mathbf{0}}(a_1)\Big) \sqcup \Big(\widetilde{\mathbf{k}}(a_0a_1) \sqcap \widetilde{\mathbf{0}}(\varepsilon)\Big)\Big) \sqcap \widetilde{\mathbf{1}}(\varepsilon)\Big) \\
&= (\mathbf{k} \sqcap \mathbf{0}) \sqcup ((\mathbf{k} \sqcap \mathbf{0}) \sqcup (\mathbf{k} \sqcap \mathbf{0})) \sqcup ((\mathbf{k} \sqcap \mathbf{0}) \sqcup (\mathbf{k} \sqcap \mathbf{0}) \sqcup (\mathbf{k} \sqcap \mathbf{0})) \\
&= \big(\ldots, (t+f+u)^6, \ldots, \ldots\big)\,.
\end{aligned}
$$

Hence, we get $\widetilde{\mathbf{k}}(\widetilde{\mathbf{0}}\widetilde{\mathbf{1}})(w) \neq (\widetilde{\mathbf{k}}\widetilde{\mathbf{0}})\widetilde{\mathbf{1}}(w)$ which implies that $\widetilde{\mathbf{k}}(\widetilde{\mathbf{0}}\widetilde{\mathbf{1}}) \neq (\widetilde{\mathbf{k}}\widetilde{\mathbf{0}})\widetilde{\mathbf{1}}$, and our proof is completed. ■

We assume now that the alphabet A is linearly ordered and let B be another alphabet. Then a homomorphism $h : A^* \to B^*$ is extended to a mapping $h : K\langle\langle A^*\rangle\rangle \to K\langle\langle B^*\rangle\rangle$ in the following way. For every $s \in K\langle\langle A^*\rangle\rangle$ and $v \in B^*$ we let $h(s)(v) = \bigsqcup_{w \in h^{-1}(v)} s(w)$ where in the definition of the disjunction we take into account the lexicographic order of the words $w \in h^{-1}(v)$. Finally, we assume that $h : A^* \to B^*$ is a strict alphabetic homomorphism, i.e., $h(a) \in B$ for every $a \in A$. Then, for every $r \in K\langle\langle B^*\rangle\rangle$ the MK-fuzzy language $h^{-1}(r) \in K\langle\langle A^*\rangle\rangle$ is determined by $h^{-1}(r)(w) = r(h(w))$ for every $w \in A^*$. We should note that for h^{-1} we do not require any order on the alphabet A.

3. MK-fuzzy automata

In this section we introduce the model of MK-fuzzy automata over A and K and investigate closure properties of the class of their behaviors. Moreover, we prove a Nivat-like theorem for recognizable MK-fuzzy languages.

Definition 7. *An* MK-fuzzy automaton over A and K *is a seven-tuple* $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ *where* Q *is the* finite state set *which is assumed to be linearly ordered,* I *is the* set of initial states,

$T \subseteq Q \times A \times Q$ *is the* set of transitions*,* F *is the* set of final states*,* $in : I \to K$ *is the* initial distribution*,* $wt : T \to K$ *is a mapping assigning* truth values *to the transitions of the automaton, and* $ter : F \to K$ *is the* final distribution.

Let $w = a_0 \dots a_{n-1}$ be a word over A with $a_0, \dots, a_{n-1} \in A$. A *path* $P_w^{(\mathcal{A})}$ (or simply P_w if the automaton is understood) *of* $\mathcal{A}$ *over* w is a sequence of transitions $P_w^{(\mathcal{A})} := ((q_i, a_i, q_{i+1}))_{0 \le i \le n-1}$, $(q_i, a_i, q_{i+1}) \in T$ for every $0 \le i \le n-1$, with $q_0 \in I$ and $q_n \in F$. The *weight of* $P_w^{(\mathcal{A})}$ is the truth value

$$weight\left(P_w^{(\mathcal{A})}\right) = in(q_0) \sqcap \bigsqcap_{0 \le i \le n-1} wt\,(q_i, a_i, q_{i+1}) \sqcap ter(q_n).$$

The set of paths of $\mathcal{A}$ over w can be linearly ordered in the following way. For two paths $P_w = ((q_i, a_i, q_{i+1}))_{0 \le i \le n-1}$ and $P'_w = \left((q'_i, a_i, q'_{i+1})\right)_{0 \le i \le n-1}$ we let

$$P_w \le P'_w \quad \text{iff} \quad q_0 \dots q_{n-1} \le_{lex} q'_0 \dots q'_{n-1}.$$

The *behavior of* $\mathcal{A}$ is the MK-fuzzy language $\|\mathcal{A}\| : A^* \to K$ and it is defined as follows. Let $w \in A^+$ and $\{P_{w,1}, \dots, P_{w,m}\}$ be the set of all paths of $\mathcal{A}$ over w. Furthermore, assume that $P_{w,1} \le \dots \le P_{w,m}$. Then, we set

$$\|\mathcal{A}\|(w) = weight(P_{w,1}) \sqcup \dots \sqcup weight(P_{w,m}).$$

If there are no paths of $\mathcal{A}$ over w, then we let $\|\mathcal{A}\|(w) = \mathbf{0}$. If $w = \varepsilon$, then

$$\|\mathcal{A}\|(\varepsilon) = (in(q_{i_1}) \sqcap ter(q_{i_1})) \sqcup \dots \sqcup (in(q_{i_m}) \sqcap ter(q_{i_m}))$$

where $I \cap F = \{q_{i_1}, \dots, q_{i_m}\}$ and $q_{i_1} \le \dots \le q_{i_m}$. If $I \cap F = \emptyset$, then we set $\|\mathcal{A}\|(\varepsilon) = \mathbf{0}$. An MK-fuzzy language $s : A^* \to K$ is called *recognizable* if there is an MK-fuzzy automaton $\mathcal{A}$ over A and K such that $s = \|\mathcal{A}\|$. We denote by $Rec(K, A)$ the class of all recognizable MK-fuzzy languages over A and K.

Remark 8. *By our definition above, we get that weight* $\left(P_w^{(\mathcal{A})}\right) = \mathbf{0}$ *whenever* $in(q_0) = \mathbf{0}$ *for every path* $P_w^{(\mathcal{A})} = ((q_i, a_i, q_{i+1}))_{0 \le i \le n-1}$ *of* $\mathcal{A}$ *over* $w = a_0 \dots a_{n-1}$*. Hence, in the sequel, we assume that* $in : I \to K \setminus \{\mathbf{0}\}$ *for every MK-fuzzy automaton* $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ *over* A *and* K*.*

Example 9. *Let* $\mathbf{k} \in K$*. Then the constant MK-fuzzy language* $\widetilde{\mathbf{k}}$ *is recognizable. Indeed, we consider the MK-fuzzy automaton* $\mathcal{A}_{\mathbf{k}} = (\{q\}, \{q\}, T, \{q\}, in, wt, ter)$ *with* $T = \{(q, a, q) \mid a \in A\}$ *and* $in(q) = \mathbf{k}$*,* $ter(q) = \mathbf{1}$*, and* $wt(q, a, q) = \mathbf{1}$ *for every* $a \in A$*. We trivially get* $\|\mathcal{A}\| = \widetilde{\mathbf{k}}$*.*

Proposition 10. *Let $L \subseteq A^*$ be a recognizable language. Then $\mathbf{1}_L \in Rec(K, A)$.*

Proof. We consider a deterministic finite automaton $\mathcal{A} = (Q, A, q_0, T, F)$ accepting L, and construct the MK-fuzzy automaton $\mathcal{A}' = (Q, \{q_0\}, T, F, in, wt, ter)$ where the weight mappings in, wt, ter assign the value $\mathbf{1}$ to every element of their domain. Then, for every $w \in L$ there is a unique successful path P_w of $\mathcal{A}$ over w. By construction of $\mathcal{A}'$, P_w is also the unique path of $\mathcal{A}'$ over w and $weight(P_w) = \mathbf{1}$, hence $\|\mathcal{A}'\|(w) = \mathbf{1}$. If $w \notin L$, then there is no successful path of $\mathcal{A}$ over w which in turn implies that there is no path of $\mathcal{A}'$ over w. Therefore $\|\mathcal{A}'\|(w) = \mathbf{0}$, and we are done. ■

Theorem 11. *The class $Rec(K, A)$ is closed under disjunction.*

Proof. Let $\mathcal{A}_1 = (Q_1, I_1, T_1, F_1, in_1, wt_1, ter_1)$, $\mathcal{A}_2 = (Q_2, I_2, T_2, F_2, in_2, wt_2, ter_2)$ be two MK-fuzzy automata over A and K. Without loss of generality, we assume that $Q_1 \cap Q_2 = \emptyset$. We define a linear order on $Q_1 \cup Q_2$ by preserving the orders of Q_1 and Q_2 and letting $\max Q_1 \leq \min Q_2$. We consider the MK-fuzzy automaton $\mathcal{A} = (Q, T, I, F, in, wt, ter)$ with $Q = Q_1 \cup Q_2$, $I = I_1 \cup I_2$, $T = T_1 \cup T_2$, $F = F_1 \cup F_2$, and in, wt, ter are defined respectively by

- $in(q) = \begin{cases} in_1(q) & \text{if } q \in I_1 \\ in_2(q) & \text{if } q \in I_2 \end{cases}$

  for every $q \in I$,

- $wt(q, a, q') = \begin{cases} wt_1(q, a, q') & \text{if } (q, a, q') \in T_1 \\ wt_2(q, a, q') & \text{if } (q, a, q') \in T_2 \end{cases}$

  for every $(q, a, q') \in T$, and

- $ter(q) = \begin{cases} ter_1(q) & \text{if } q \in F_1 \\ ter_2(q) & \text{if } q \in F_2 \end{cases}$

  for every $q \in F$.

Consider a word $w = a_0 \dots a_{n-1} \in A^*$ and a path $P_w^{(\mathcal{A})} = ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$ of $\mathcal{A}$ over w. By definition of T, the transitions of $P_w^{(\mathcal{A})}$ belong either to T_1 or to T_2, hence $P_w^{(\mathcal{A})}$ is either a path of $\mathcal{A}_1$ over w or a path of $\mathcal{A}_2$ over w. Conversely, every path of $\mathcal{A}_1$ (resp. of $\mathcal{A}_2$) over w is also a path of $\mathcal{A}$ over w. Taking into account the order of Q, we get

$$\|\mathcal{A}\|(w) = \|\mathcal{A}_1\|(w) \sqcup \|\mathcal{A}_2\|(w).$$

Hence, $\|\mathcal{A}\| = \|\mathcal{A}_1\| \sqcup \|\mathcal{A}_2\|$ which implies that $\|\mathcal{A}_1\| \sqcup \|\mathcal{A}_2\| \in Rec(K, A)$, as required. $\blacksquare$

Theorem 12. *Let $s \in Rec(K, A)$ and $L \subseteq A^*$ be a recognizable language. Then $\mathbf{1}_L \sqcap s \in Rec(K, A)$.*

Proof. Let $\mathcal{A}_1 = \left(Q_1, A, q_1^{(0)}, T_1, F_1\right)$ be a deterministic finite automaton accepting L and $\mathcal{A}_2 = (Q_2, I_2, T_2, F_2, in_2, wt_2, ter_2)$ an MK-fuzzy automaton over A and K accepting s. We define an arbitrary linear order $\leq$ on Q_1 and consider the MK-fuzzy automaton

$$\mathcal{A} = \left(Q_1 \times Q_2, \left\{q_1^{(0)}\right\} \times I_2, T, F_1 \times F_2, in, wt, ter\right)$$

with $T = \{((q_1, q_2), a, (q_1', q_2')) \mid (q_1, a, q_1') \in T_1 \text{ and } (q_2, a, q_2') \in T_2\}$ and

- $in\left(q_1^{(0)}, q_2\right) = in_2(q_2)$ for every $q_2 \in I_2$,
- $wt((q_1, q_2), a, (q_1', q_2')) = wt_2(q_2, a, q_2')$ for every $((q_1, q_2), a, (q_1', q_2')) \in T$,
- $ter(q_1, q_2) = ter_2(q_2)$ for every $(q_1, q_2) \in F_1 \times F_2$.

The state set $Q_1 \times Q_2$ is linearly ordered by

$$(q_1, q_2) \leq (q_1', q_2') \qquad \text{iff} \qquad ((q_2 < q_2') \text{ or } (q_2 = q_2' \ \text{ and } \ q_1 \leq q_1'))$$

for every $(q_1, q_2), (q_1', q_2') \in Q_1 \times Q_2$.
Let $w = a_0 \dots a_{n-1} \in A^*$ and $P_w^{(\mathcal{A})} = \left(\left((q_1^{(i)}, q_2^{(i)}), a_i, (q_1^{(i+1)}, q_2^{(i+1)})\right)\right)_{0 \leq i \leq n-1}$ be a path of $\mathcal{A}$ over w. By construction of $\mathcal{A}$ we get that $P_w^{(\mathcal{A}_1)} = \left(\left(q_1^{(i)}, a_i, q_1^{(i+1)}\right)\right)_{0 \leq i \leq n-1}$ is a successful path of $\mathcal{A}_1$ over w and $P_w^{(\mathcal{A}_2)} = \left(\left(q_2^{(i)}, a_i, q_2^{(i+1)}\right)\right)_{0 \leq i \leq n-1}$ is a path of $\mathcal{A}_2$ over w. In fact $P_w^{(\mathcal{A}_1)}$ is unique since $\mathcal{A}_1$ is deterministic and moreover $w \in L$. Trivially, we get $weight\left(P_w^{(\mathcal{A})}\right) = weight\left(P_w^{(\mathcal{A}_2)}\right)$. Let now $P_{w,1}^{(\mathcal{A})}, \dots, P_{w,m}^{(\mathcal{A})}$ be all the paths of $\mathcal{A}$ over w and assume that $P_{w,1}^{(\mathcal{A})} \leq \dots \leq P_{w,m}^{(\mathcal{A})}$. Then $P_{w,1}^{(\mathcal{A}_2)}, \dots, P_{w,m}^{(\mathcal{A}_2)}$ are all the paths of $\mathcal{A}_2$ over w. Moreover, by the order on $Q_1 \times Q_2$, we get $P_{w,1}^{(\mathcal{A}_2)} \leq \dots \leq P_{w,m}^{(\mathcal{A}_2)}$. Hence, we have

$$\begin{aligned}
\|\mathcal{A}\|(w) &= weight\left(P_{w,1}^{(\mathcal{A})}\right) \sqcup \dots \sqcup weight\left(P_{w,m}^{(\mathcal{A})}\right) \\
&= weight\left(P_{w,1}^{(\mathcal{A}_2)}\right) \sqcup \dots \sqcup weight\left(P_{w,m}^{(\mathcal{A}_2)}\right) \\
&= \|\mathcal{A}_2\|(w) \\
&= \mathbf{1}_L(w) \sqcap \|\mathcal{A}_2\|(w) \\
&= (\mathbf{1}_L \sqcap \|\mathcal{A}_2\|)(w).
\end{aligned}$$

If $w \notin L$, then $\mathbf{1}_L(w) = \mathbf{0}$ and there is not any successful path of $\mathcal{A}_1$ over w which in turn implies that there is not any path of $\mathcal{A}$ over w. Hence, $\|\mathcal{A}\|(w) = \mathbf{0}$, i.e., $\|\mathcal{A}\|(w) = (\mathbf{1}_L \sqcap \|\mathcal{A}_2\|)(w)$, and our proof is completed. ■

Theorem 13. *Let A be a linearly ordered alphabet and $h : A^* \to B^*$ a strict alphabetic homomorphism. Then $s \in Rec(K, A)$ implies $h(s) \in Rec(K, B)$.*

Proof. Let $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ be an MK-fuzzy automaton over A and K accepting s. We consider the MK-fuzzy automaton $\mathcal{B} = (A \times Q, \{\min A\} \times I, T', A \times F, in', wt', ter')$ over B and K with $T' = \{((a, q), b, (a'q')) \mid (q, a', q') \in T \text{ and } h(a') = b\}$. The weight mappings in', wt', ter' are defined respectively, by

- $in'(\overline{a}, q) = in(q)$, with $\overline{a} = \min A$ and every $q \in I$,
- $wt'((a, q), b, (a', q')) = wt(q, a', q')$, for every $((a, q), b, (a', q')) \in T'$, and
- $ter'(a, q) = ter(q)$, for every $(a, q) \in A \times F$.

Let $w = a_0 \ldots a_{n-1} \in A^+$ and $P_w^{(\mathcal{A})} = ((q_i, a_i, q_{i+1}))_{0 \le i \le n-1}$ be a path of $\mathcal{A}$ over w. By definition of the MK-fuzzy automaton $\mathcal{B}$ there is a unique path

$$P_{h(w)}^{(\mathcal{B})} = ((\overline{a}, q_0), h(a_0), (a_0, q_1))((a_0, q_1), h(a_1), (a_1, q_2)) \ldots ((a_{n-2}, q_{n-1}), h(a_{n-1}), (a_{n-1}, q_n))$$

of $\mathcal{B}$ over $h(w)$, and by a straightforward calculation we get $weight\left(P_{h(w)}^{(\mathcal{B})}\right) = weight\left(P_w^{(\mathcal{A})}\right)$. Conversely, let $v = b_0 \ldots b_{n-1} \in B^+$ and

$$P_v^{(\mathcal{B})} = ((\overline{a}, q_0), b_0, (a_0, q_1))((a_0, q_1), b_1, (a_1, q_2)) \ldots ((a_{n-2}, q_{n-1}), b_{n-1}, (a_{n-1}, q_n))$$

be a path of $\mathcal{B}$ over v. Then, $v = h(w)$ where $w = a_0 \ldots a_{n-1} \in A^+$. Moreover, $P_w^{(\mathcal{A})} = ((q_i, a_i, q_{i+1}))_{0 \le i \le n-1}$ is a path of $\mathcal{A}$ over w and $weight\left(P_v^{(\mathcal{B})}\right) = weight\left(P_w^{(\mathcal{A})}\right)$. Hence, for every $v \in B^+$, if $w_1, \ldots, w_m$ are all the words in A^+ such that $h(w_i) = v$ $(1 \le i \le m)$, then there is a one-to-one correspondence between the paths

$$P_{w_1,1}^{(\mathcal{A})}, \ldots, P_{w_1,j_1}^{(\mathcal{A})}, \ldots, P_{w_m,1}^{(\mathcal{A})}, \ldots, P_{w_m,j_m}^{(\mathcal{A})}$$

of $\mathcal{A}$, respectively over $w_1, \ldots, w_m$, and the paths

$$P_{v,1}^{(\mathcal{B})}, \ldots, P_{v,j_1}^{(\mathcal{B})}, P_{v,j_1+1}^{(\mathcal{B})}, \ldots, P_{v,j_1+j_2}^{(\mathcal{B})}, \ldots, P_{v,k}^{(\mathcal{B})}$$

of $\mathcal{B}$ over v, where $P^{(\mathcal{A})}_{w_l,r_l}$ corresponds to $P^{(\mathcal{B})}_{v,j_1+\ldots+j_{l-1}+r_l}$ for every $1 \le l \le m$ and $1 \le r_l \le j_l$. Then we get $weight\left(P^{(\mathcal{A})}_{w_l,r_l}\right) = weight\left(P^{(\mathcal{B})}_{v,j_1+\ldots+j_{l-1}+r_l}\right)$. Moreover, if $w_1 \le \ldots \le w_m$ and

$$P^{(\mathcal{A})}_{w_1,1} \le \ldots \le P^{(\mathcal{A})}_{w_1,j_1}, \ldots, P^{(\mathcal{A})}_{w_m,1} \le \ldots \le P^{(\mathcal{A})}_{w_m,j_m},$$

then

$$P^{(\mathcal{B})}_{v,1} \le \ldots \le P^{(\mathcal{B})}_{v,j_1} \le P^{(\mathcal{B})}_{v,j_1+1} \le \ldots \le P^{(\mathcal{B})}_{v,j_1+j_2} \le \ldots \le P^{(\mathcal{B})}_{v,k}.$$

Hence we have

$$\begin{aligned} h(s)(v) &= \bigsqcup_{w\in h^{-1}(v)} s(w) = s(w_1) \sqcup \ldots \sqcup s(w_m) \\ &= \bigsqcup_{1\le r_1\le j_1} weight\left(P^{(\mathcal{A})}_{w_1,r_1}\right) \sqcup \ldots \sqcup \bigsqcup_{1\le r_m\le j_m} weight\left(P^{(\mathcal{A})}_{w_m,r_m}\right) \\ &= \bigsqcup_{1\le i\le k} weight\left(P^{(\mathcal{B})}_{v,i}\right) = \|\mathcal{B}\|(v). \end{aligned}$$

Next let $s(\varepsilon) \ne \mathbf{0}$ and assume that $I \cap F = \{q_{i_1}, \ldots, q_{i_m}\}$. Then $(\{\min A\} \times I) \cap (A \times F) = \{(\min A, q_{i_1}), \ldots, (\min A, q_{i_m})\}$ and by definition of in' and ter' we get $\|\mathcal{A}\|(\varepsilon) = \|\mathcal{B}\|(\varepsilon)$. Since $h(s)(\varepsilon) = s(\varepsilon)$, we finally conclude that $h(s) = \|\mathcal{B}\|$, i.e, $h(s) \in Rec(K, B)$, and we are done. ■

Theorem 14. *Let $h : A^* \to B^*$ be a strict alphabetic homomorphism. Then $s \in Rec(K, B)$ implies $h^{-1}(s) \in Rec(K, A)$.*

Proof. Let $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ be an MK-fuzzy automaton over B and K accepting s. We consider the MK-fuzzy automaton $\mathcal{A}' = (Q, I, T', F, in, wt', ter)$ over A and K, where $T' = \{(q, a, q') \in Q \times A \times Q \mid (q, h(a), q') \in T\}$ and the mapping $wt' : T' \to K$ is defined by $wt'(q, a, q') = wt(q, h(a), q')$ for every $(q, a, q') \in T'$.

Let $w = a_0 \ldots a_{n-1} \in A^+$ and $P^{(\mathcal{A})}_{h(w)} = ((q_i, h(a_i), q_{i+1}))_{0\le i\le n-1}$ be a path of $\mathcal{A}$ over $h(w)$. By construction of $\mathcal{A}'$, there is a path $P^{(\mathcal{A}')}_{w} = ((q_i, a_i, q_{i+1}))_{0\le i\le n-1}$ of $\mathcal{A}$ over w, and vice versa. Trivially $weight\left(P^{(\mathcal{A}')}_{w}\right) = weight\left(P^{(\mathcal{A})}_{h(w)}\right)$. Furthermore, if $P^{(\mathcal{A})}_{h(w),1}, \ldots, P^{(\mathcal{A})}_{h(w),m}$ are all the paths of $\mathcal{A}$ over $h(w)$ and

$$P^{(\mathcal{A})}_{h(w),1} \le \ldots \le P^{(\mathcal{A})}_{h(w),m},$$

then for the corresponding paths $P^{(\mathcal{A}')}_{w,1}, \ldots, P^{(\mathcal{A}')}_{w,m}$ of $\mathcal{A}'$ over w, we get

$$P^{(\mathcal{A}')}_{w,1} \le \ldots \le P^{(\mathcal{A}')}_{w,m}.$$

We conclude that $\|\mathcal{A}'\|(w) = \|\mathcal{A}\|(h(w))$, i.e., $\|\mathcal{A}'\|(w) = h^{-1}(s)(w)$ for every word $w \in A^+$. By construction of $\mathcal{A}'$, we get $\|\mathcal{A}'\|(\varepsilon) = \|\mathcal{A}\|(\varepsilon) = s(\varepsilon) = h^{-1}(s)(\varepsilon)$. Hence, $\|\mathcal{A}'\| = h^{-1}(s)$, and our proof is completed. ■

The subsequent definition refers to normalized MK-fuzzy automata and differs from the corresponding one of weighted automata over semirings.

Definition 15. *An MK-fuzzy automaton $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ is called* normalized *if $I \cap F = \emptyset$, $in(q) = \mathbf{1}$ for every $q \in I$, $ter(q) = \mathbf{1}$ for every $q \in F$, $(q, a, q') \notin T$ for every $q \in Q, a \in A$, $q' \in I$, and $(q, a, q') \notin T$ for every $q \in F, a \in A$, $q' \in Q$.*

For normalized automata, we simply write $\mathcal{A} = (Q, I, T, F, wt)$, and if $I = \{q_{in}\}$, then $\mathcal{A} = (Q, q_{in}, T, F, wt)$. It should be clear that $\|\mathcal{A}\|(\varepsilon) = \mathbf{0}$ whenever $\mathcal{A}$ is a normalized MK-fuzzy automaton over A and K.

Proposition 16. *For every MK-fuzzy automaton $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ we can effectively construct a normalized MK-fuzzy automaton $\mathcal{A}'$ such that $\|\mathcal{A}'\|(w) = \|\mathcal{A}\|(w)$ for every $w \in A^+$.*

Proof. Firstly, we show that we can construct an MK-fuzzy automaton $\overline{\mathcal{A}} = (\overline{Q}, \overline{I}, \overline{T}, \overline{F}, \overline{in}, \overline{wt}, \overline{ter})$ such that $\overline{I} \cap \overline{F} = \emptyset$ and $\|\overline{\mathcal{A}}\|(w) = \|\mathcal{A}\|(w)$ for every $w \in A^+$. If $I \cap F = \emptyset$, then we have nothing to prove. Let $q \in I \cap F$. We consider a new state $\overline{q}$ and the MK-fuzzy automaton $\overline{\mathcal{A}} = (\overline{Q}, \overline{I}, \overline{T}, \overline{F}, \overline{in}, \overline{wt}, \overline{ter})$ with $\overline{Q} = Q \cup \{\overline{q}\}$, $\overline{I} = I$, $\overline{F} = (F \setminus q) \cup \{\overline{q}\}$, $\overline{T} = T \cup \{(p, a, \overline{q}) \mid (p, a, q) \in T\}$, $\overline{in} = in$, $\overline{ter}(p) = ter(p)$ for every $p \in F \setminus \{q\}$, and $\overline{ter}(\overline{q}) = ter(q)$, $\overline{wt}(p, a, p') = wt(p, a, p')$ for every $(p, a, p') \in T$, and $\overline{wt}(p, a, \overline{q}) = wt(p, a, q)$ for every $(p, a, \overline{q}) \in T' \setminus T$. Furthermore, the order on Q is extended to an order on $\overline{Q}$ by letting $p \leq q \leq \overline{q} \leq p'$ whenever $p \leq q \leq p'$. By construction of $\overline{\mathcal{A}}$, we get that for every $w \in A^+$ and path P_w of $\mathcal{A}$ over w, terminating at q, there exists a path $\overline{P}_w$ of $\overline{\mathcal{A}}$ over w terminating at $\overline{q}$, and vice versa. Moreover, $weight(P_w) = weight(\overline{P}_w)$ and if $P_{w,1} \leq P_w \leq P_{w,2}$, then $P_{w,1} \leq \overline{P}_w \leq P_{w,2}$. Hence, $\|\overline{\mathcal{A}}\| = \|\mathcal{A}\|$. If $\overline{I} \cap \overline{F} \neq \emptyset$, then we repeat the same process.

Therefore, we assume in the sequel that $I \cap F = \emptyset$. We consider the MK-fuzzy automaton $\mathcal{A}' = (Q', I', T', F', in', wt', ter')$ where $I' = \{q' \mid q \in I\}$ and $F' = \{p' \mid p \in F\}$ are copy states of I and F respectively, $Q' = Q \cup I' \cup F'$, and $T' = T \cup \{(q', a, p) \mid (q, a, p) \in T \text{ and } q \in I\} \cup \{(q, a, p') \mid (q, a, p) \in T \text{ and } p \in F\} \cup \{(q', a, p') \mid (q, a, p) \in T \text{ and } q \in I, p \in F\}$. We extend on Q' the linear order on Q as follows. For every $q' \in I'$ (which corresponds to $q \in I$) we let $q_1 \leq q' \leq q \leq q_2$ whenever

$q_1 \leq q \leq q_2$. Similarly, for every $p' \in F'$ (which corresponds to $p \in F$) we let $p_1 \leq p' \leq p \leq p_2$ whenever $p_1 \leq p \leq p_2$. The weight mappings in', wt', ter' are defined by

- $in'(q') = \mathbf{1}$ for every $q' \in I'$,

- $wt'(\overline{q}, a, \overline{p}) = \begin{cases} wt(q,a,p) & \text{if } (\overline{q}, a, \overline{p}) = (q,a,p) \in T \\ in(q) \sqcap wt(q,a,p) & \text{if } (\overline{q}, a, \overline{p}) = (q',a,p) \text{ with } q' \in I', p \in Q \\ wt(q,a,p) \sqcap ter(p) & \text{if } (\overline{q}, a, \overline{p}) = (q,a,p') \text{ with } q \in Q, p' \in F' \\ in(q) \sqcap wt(q,a,p) \sqcap ter(p) & \text{if } (\overline{q}, a, \overline{p}) = (q',a,p') \text{ with } q' \in I', p' \in F' \end{cases}$

  for every $(\overline{q}, a, \overline{p}) \in T'$, and

- $ter'(p') = \mathbf{1}$ for every $p' \in F'$.

Let now $w = a_0 \ldots a_{n-1} \in A^+$ with $n > 1$, and $P_w = ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$ be a path of $\mathcal{A}$ over w. Then, there is a path $P'_w = (q'_0, a_0, q_1)\,((q_i, a_i, q_{i+1}))_{1 \leq i \leq n-2}\,(q_{n-1}, a_{n-1}, q'_n)$ of $\mathcal{A}'$ over w and vice versa. Moreover, $weight(P_w) = weight(P'_w)$. Furthermore, the order on Q' implies

$$P_{w,1} \leq \ldots \leq P_{w,m}$$

iff

$$P'_{w,1} \leq \ldots \leq P'_{w,m}$$

where $P_{w,1}, \ldots, P_{w,m}$ are all the paths of $\mathcal{A}$ over w. Hence $\|\mathcal{A}\|(w) = \|\mathcal{A}'\|(w)$. If $w = a$, with $a \in A$ and $P_a = (q, a, p)$ is a path of $\mathcal{A}$ over a, then $P'_a = (q', a, p')$ is a path of $\mathcal{A}'$ over a, and by definition again we get $weight(P_a) = weight(P'_a)$, hence $\|\mathcal{A}\|(a) = \|\mathcal{A}'\|(a)$. We conclude $\|\mathcal{A}\|(w) = \|\mathcal{A}'\|(w)$ for every $w \in A^+$, and our proof is completed. ■

Next, we show a Nivat-like decomposition theorem for recognizable MK-fuzzy languages. The fundamental Nivat's theorem [21] states a relation among rational transductions and rational languages. A Nivat-like result was proved for weighted automata over semirings in [8]. We need some preliminary matter. Let B be an alphabet and $g : B \to K$ a mapping. Then g can be extended to an MK-fuzzy language $g : B^* \to K$ by $g(b_0 \ldots b_{n-1}) = \prod_{0 \leq i \leq n-1} g(b_i)$ for every $b_0 \ldots b_{n-1} \in B^+$, $b_0, \ldots, b_{n-1} \in B$, and $g(\varepsilon) = \mathbf{1}$. Then, for a language $L \subseteq B^+$ we define the MK-fuzzy language $L \cap g$ by $(L \cap g)(w) = g(w)$ if $w \in L$ and $(L \cap g)(w) = \mathbf{0}$ otherwise, for every $w \in B^*$. It should be clear that $L \cap g = \mathbf{1}_L \sqcap g$. Now we are ready to state our Nivat-like theorem.

Theorem 17. *Let A be a linearly ordered alphabet and s an MK-fuzzy language over A and K with $s(\varepsilon) = \mathbf{0}$. Then s is recognizable iff there is a linearly ordered alphabet B, a recognizable language $L \subseteq B^+$, a mapping $g : B \to K$, and a strict alphabetic homomorphism $h : B^* \to A^*$ such that $s = h(L \cap g)$.*

Proof. We prove firstly the implication " $\Longleftarrow$ ". The MK-fuzzy language g is recognizable. Indeed, consider the MK-fuzzy automaton $\mathcal{G} = (\{q\}, \{q\}, T, \{q\}, in, wt, ter)$ over B and K, with $in(q) = ter(q) = \mathbf{1}$ and $wt(q, b, q) = g(b)$ for every $b \in B$. Trivially $\|\mathcal{G}\| = g$. Then, by Proposition 10 and Theorem 12 the MK-fuzzy language $\mathbf{1}_L \sqcap g$ is recognizable and hence, $h(L \cap g)$ is recognizable by Theorem 13.

Conversely, let $s \in Rec(K, A)$ with $s(\varepsilon) = \mathbf{0}$ and $\mathcal{A} = (Q, I, T, in, wt, ter)$ be an MK-fuzzy automaton accepting s. We set $B = T$ and consider the finite automaton $\mathcal{B} = (Q, B, I, T', F)$ with $T' = \{(q, (q, a, q'), q') \mid (q, a, q') \in T\}$. It can be easily seen that $L(\mathcal{B}) = \{P_w \mid w \in A^+ \text{ and } P_w \text{ path of } \mathcal{A} \text{ over } w\} \cup C$, where $C = \{\varepsilon\}$ if $I \cap F \neq \emptyset$ and $C = \emptyset$ otherwise. We let $L = L(\mathcal{B}) \setminus \{\varepsilon\}$ and define the mapping $g : B \to K$ by $g(q, a, q') = wt(q, a, q')$ for every $(q, a, q') \in B$. Since A and Q are linearly ordered the set $Q \times A \times Q$ is also linearly ordered and thus B is linearly ordered as well. We consider the strict alphabetic homomorphism $h : B^* \to A^*$ by $h(q, a, q') = a$ for every $(q, a, q') \in B$. Then, for every $w \in A^+$ we get

$$h(L \cap g)(w) = \bigsqcup_{v \in h^{-1}(w)} (L \cap g)(v) = \bigsqcup_{\substack{v \in h^{-1}(w) \\ v \in L}} g(v) = \bigsqcup_{P_w} weight(P_w) = \|\mathcal{A}\|(w),$$

i.e., $h(L \cap g)(w) = \|\mathcal{A}\|$ as required, and our proof is completed. ■

In the sequel, we deal with the deterministic counterpart of our model. An MK-fuzzy automaton $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ over A and K is called *deterministic* if $I = \{q_0\}$ and for every $q \in Q, a \in A$ there is at most one $q' \in Q$ such that $(q, a, q') \in T$. Then for every word $w \in A^*$ there is at most one path P_w of $\mathcal{A}$ over w, which in turn implies that we can relax the order relation of Q. Nevertheless, in the sequel, sometimes we will need the state set of a deterministic MK-fuzzy automaton to be ordered. A deterministic MK-fuzzy automaton $\mathcal{A}$ is simply written as $\mathcal{A} = (Q, q_0, T, F, in, wt, ter)$. An MK-fuzzy language $s \in K\langle\langle A^* \rangle\rangle$ is called *deterministically recognizable* if there is a deterministic MK-fuzzy automaton $\mathcal{A}$ over A and K such that $s = \|\mathcal{A}\|$. We denote by $DRec(K, A)$ the class of all deterministically recognizable MK-fuzzy languages over A and K. An MK-fuzzy automaton $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ is called *unambiguous* if for every word $w \in A^*$ there is at most one path P_w of $\mathcal{A}$ over A. Clearly, every deterministic MK-fuzzy automaton is unambiguous as well, but the converse is not always true.

Theorem 18. *Let $s \in DRec(K, A)$ and $\mathbf{k} \in K$. Then $s \sqcap \mathbf{k} \in DRec(K, A)$ and $\mathbf{k} \sqcap s \in Rec(K, A)$.*

Proof. Let $\mathcal{A} = (Q, q_0, T, F, in, wt, ter)$ be a deterministic MK-fuzzy automaton accepting s. We consider the deterministic MK-fuzzy automaton $\overline{\mathcal{A}} = (Q, q_0, T, F, in, wt, \overline{ter})$ with $\overline{ter}(q) = ter(q) \sqcap \mathbf{k}$ for every $q \in F$. Trivially, $\|\overline{\mathcal{A}}\| = s \sqcap \mathbf{k}$.

Next we consider the MK-fuzzy automaton $\mathcal{A}' = (Q', I', T', F', in', wt', ter')$ with

- $Q' = Q \cup \overline{Q} \cup \widetilde{Q} \cup \{r\} \cup C$ where $\overline{Q} = \{\overline{q} \mid q \in Q\}$ and $\widetilde{Q} = \{\widetilde{q} \mid q \in Q\}$ are copies of Q, r is a new state and $C = \{\overline{\overline{q_0}}\}$ if $q_0 \notin F$ where $\overline{\overline{q_0}}$ is a new state, and $C = \emptyset$ otherwise; the sets $Q, \overline{Q}, \widetilde{Q}, \{r\}$, and C are considered pairwise disjoint,
- $I' = \{q_0, \overline{q_0}, \widetilde{q_0}\} \cup C$,
- $T' = T \cup \{(\overline{q}, a, \overline{p}) \mid (q, a, p) \in T\} \cup \{(\overline{q}, a, r) \mid \overline{q} \in \overline{Q}$ and there is no p such that $(q, a, p) \in T\} \cup \{(r, a, r) \mid a \in A\} \cup \{(\widetilde{q}, a, \widetilde{p}) \mid (q, a, p) \in T\}$,
- $F' = F \cup \{r\} \cup C \cup \{\widetilde{q} \mid q \in Q \setminus F\}$,
- $in'(q') = \begin{cases} \mathbf{k} \sqcap in(q_0) & \text{if } q' = q_0 \\ \mathbf{k} & \text{if } q' = \overline{q_0} \text{ or } q' = \widetilde{q_0} \text{ or } (C \neq \emptyset \text{ and } q' = \overline{\overline{q_0}}) \end{cases}$

  for every $q' \in I'$,
- $wt'(q', a, p') = \begin{cases} wt(q', a, p') & \text{if } (q', a, p') \in T \\ \mathbf{1} & \text{if } q', p' \in \overline{Q} \text{ or } (q' \in \overline{Q} \text{ and } p' = r) \text{ or } q' = p' = r \text{ or } q', p' \in \widetilde{Q} \end{cases}$

  for every $(q', a, p') \in T'$, and
- $ter'(q') = \begin{cases} ter(q') & \text{if } q' \in F \\ \mathbf{0} & \text{if } q' = r \text{ or } (C \neq \emptyset \text{ and } q' = \overline{\overline{q_0}}) \text{ or } q' \in \{\widetilde{q} \mid q \in Q \setminus F\} \end{cases}$

  for every $q' \in F'$.

We claim that $\|\mathcal{A}'\|(w) = \mathbf{k} \sqcap s(w)$ for every $w \in A^*$. Indeed, let $w = a_0 \ldots a_{n-1} \in A^+$ and assume first that there is a unique path $P_w^{(\mathcal{A})} = ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$ of $\mathcal{A}$ over w. Then $P_w^{(\mathcal{A})}$ is also the unique path of $\mathcal{A}'$ over w and trivially its weight in $\mathcal{A}'$ equals $\mathbf{k} \sqcap weight\left(P_w^{(\mathcal{A})}\right)$, i.e., $\|\mathcal{A}'\|(w) = \mathbf{k} \sqcap \mathcal{A}(w)$. If $q_0 \in F$, then $\|\mathcal{A}'\|(\varepsilon) = \mathbf{k} \sqcap in(q_0) \sqcap ter(q_0) = \mathbf{k} \sqcap \|\mathcal{A}\|(\varepsilon)$.

Next let $v = b_0 \dots b_{m-1} \in A^+$ and assume that there is not any path of $\mathcal{A}$ over v, i.e., $\|\mathcal{A}\|(v) = \mathbf{0}$. This implies that:

- There is an index $0 \leq j \leq m-1$ such that $(q_0, b_0, q_1) \dots (q_{j-1}, b_{j-1}, q_j)$ is a sequence of transitions of $\mathcal{A}$ over $b_0 \dots b_{j-1}$ and there is not any transition of the form (q_j, b_j, q_{j+1}) in T. Then, there is a unique path

$$P_v^{(\mathcal{A}')} = (\overline{q_0}, b_0, \overline{q_1}) \dots (\overline{q_{j-1}}, b_{j-1}, \overline{q_j})(\overline{q_j}, b_j, r) \dots (r, b_{m-1}, r)$$

of $\mathcal{A}'$ over v and

$$weight\left(P_v^{(\mathcal{A}')}\right) = \mathbf{k} \sqcap \mathbf{1} \sqcap \dots \sqcap \mathbf{1} \sqcap \mathbf{0} = \mathbf{k} \sqcap \mathbf{0},$$

i.e., $\|\mathcal{A}'\|(v) = \mathbf{k} \sqcap \|\mathcal{A}\|(v)$.

or

- There is a sequence of transitions $(q_0, b_0, q_1) \dots (q_{m-1}, b_{m-1}, q_m)$ of $\mathcal{A}$ over v and $q_m \notin F$. Then, there is a unique path

$$P_v^{(\mathcal{A}')} = (\widetilde{q_0}, b_0, \widetilde{q_1}) \dots (\widetilde{q_{m-1}}, b_{m-1}, \widetilde{q_m})$$

of $\mathcal{A}'$ over v and

$$weight\left(P_v^{(\mathcal{A}')}\right) = \mathbf{k} \sqcap \mathbf{1} \sqcap \dots \sqcap \mathbf{1} \sqcap \mathbf{0} = \mathbf{k} \sqcap \mathbf{0},$$

i.e., $\|\mathcal{A}'\|(v) = \mathbf{k} \sqcap \|\mathcal{A}\|(v)$.

Finally, let $q_0 \notin F$, hence $\|\mathcal{A}\|(\varepsilon) = \mathbf{0}$. Then $C = \{\overline{\overline{q_0}}\}$ and $\|\mathcal{A}'\|(\varepsilon) = in'(\overline{\overline{q_0}}) \sqcap ter'(\overline{\overline{q_0}}) = \mathbf{k} \sqcap \mathbf{0} = \mathbf{k} \sqcap \|\mathcal{A}\|(\varepsilon)$, and our proof is completed. ■

Corollary 19. *Let $L \subseteq A^*$ be a recognizable language and $\mathbf{k} \in K$. Then $\mathbf{1}_L \sqcap \mathbf{k} \in DRec(K, A)$ and $\mathbf{k} \sqcap \mathbf{1}_L \in Rec(K, A)$.*

Proof. The MK-fuzzy automaton accepting $\mathbf{1}_L$ (cf. Proposition 10) is deterministic, hence we obtain our result by Theorem 18. ■

Next, we investigate the closure of the class of deterministically recognizable MK-fuzzy languages under Cauchy product. More precisely, we show that the Cauchy product of two deterministically recognizable MK-fuzzy languages is a recognizable MK-fuzzy language. For this we shall need the subsequent intermediate results.

Proposition 20. *For every deterministic MK-fuzzy automaton $\mathcal{A} = (Q, q_0, T, F, in, wt, ter)$ we can effectively construct a normalized unambiguous MK-fuzzy automaton $\mathcal{A}'$, with one initial state, such that $\|\mathcal{A}'\|(w) = \|\mathcal{A}\|(w)$ for every $w \in A^+$.*

Proof. We follow the proof of Proposition 16. Since the MK-fuzzy automaton $\mathcal{A}$ is deterministic, we can easily see that the derived normalized MK-fuzzy automaton $\mathcal{A}'$ is unambiguous with one initial state. ■

Proposition 21. *Let $s \in K\langle\langle A^* \rangle\rangle$ and $\mathbf{k} \in K$. If s is accepted by a normalized unambiguous MK-fuzzy automaton, then $s \sqcap \mathbf{k}$ is accepted also by a normalized unambiguous MK-fuzzy automaton.*

Proof. Let $\mathcal{A} = (Q, I, T, F, wt)$ be a normalized unambiguous MK-fuzzy automaton accepting s. We consider the MK-fuzzy automaton $\mathcal{A}' = (Q, I, T, F, wt')$ with

$$wt'((q, a, q')) = \begin{cases} wt(q, a, q') \sqcap \mathbf{k} & \text{if } q' \in F \\ wt(q, a, q') & \text{otherwise} \end{cases},$$

for every $(q, a, q') \in T$.

By construction, the automaton $\mathcal{A}'$ is normalized. Let now $w \in A^+$ and assume that there is a path $P_w^{(\mathcal{A}')}$ of $\mathcal{A}'$ over w. Then $P_w^{(\mathcal{A}')}$ is also a path of $\mathcal{A}$ over w. Since $\mathcal{A}$ is unambiguous, we conclude that $P_w^{(\mathcal{A}')}$ is unique which implies that the normalized MK-fuzzy automaton $\mathcal{A}'$ is also unambiguous. Furthermore, by a standard computation we get $weight\left(P_w^{(\mathcal{A}')}\right) = weight\left(P_w^{(\mathcal{A})}\right) \sqcap \mathbf{k}$, i.e., $\|\mathcal{A}'\|(w) = \|\mathcal{A}\|(w) \sqcap \mathbf{k}$. On the other hand, $\|\mathcal{A}\|(\varepsilon) = \mathbf{0}$ and $\|\mathcal{A}'\|(\varepsilon) = \mathbf{0}$, since both $\mathcal{A}$ and $\mathcal{A}'$ are normalized, hence $\|\mathcal{A}'\|(\varepsilon) = \|\mathcal{A}'\|(\varepsilon) \sqcap \mathbf{k}$. Therefore, we conclude $\|\mathcal{A}'\| = s \sqcap \mathbf{k}$, as required. ■

Theorem 22. *Let $r, s \in DRec(K, A)$. Then $rs \in Rec(K, A)$.*

Proof. Since $r, s \in DRec(K, A)$, there are deterministic MK-fuzzy automata accepting them. Then, by Proposition 20, we can effectively construct normalized unambiguous MK-fuzzy automata $\mathcal{A}_1 = \left(Q_1, q_{in}^{(1)}, T_1, F_1, wt_1\right)$ and $\mathcal{A}_2 = \left(Q_2, q_{in}^{(2)}, T_2, F_2, wt_2\right)$ such that $\|\mathcal{A}_1\|(w) = r(w)$ and $\|\mathcal{A}_2\|(w) = s(w)$ for every $w \in A^+$. We consider a copy $\overline{Q_1} = \{\overline{q} \mid q \in Q_1\}$ of Q_1, a copy $\overline{Q_2} = \{\overline{q} \mid q \in Q_2\}$ of Q_2 and a new state r. Without loss of generality we assume the sets Q_1, Q_2, $\overline{Q_1}$, $\overline{Q_2}$, and $\{r\}$ pairwise disjoint and consider the MK-fuzzy automaton $\mathcal{A} = \left(Q, \left\{q_{in}^{(1)}, \overline{q_{in}^{(1)}}\right\}, T, F, in, wt, ter\right)$ with

- $Q = (Q_1 \setminus F_1) \cup Q_2 \cup (\overline{Q_1} \setminus \overline{F_1}) \cup \overline{Q_2} \cup \{r\}$
  where $\overline{F_1} = \{\overline{q} \mid q \in F_1\}$,

- $T = \left\{\left(q^{(1)}, a, p^{(1)}\right) \in T_1 \mid p^{(1)} \notin F_1\right\} \cup T_2 \cup$
  $\left\{\left(q^{(1)}, a, q_{in}^{(2)}\right) \mid \text{ there exists } p^{(1)} \in F_1 \text{ such that } \left(q^{(1)}, a, p^{(1)}\right) \in T_1\right\} \cup$
  $\left\{\left(\overline{q^{(1)}}, a, \overline{p^{(1)}}\right) \mid \left(q^{(1)}, a, p^{(1)}\right) \in T_1 \text{ and } \overline{p^{(1)}} \notin \overline{F_1}\right\} \cup$
  $\left\{\left(\overline{q^{(1)}}, a, \overline{q_{in}^{(2)}}\right) \mid \text{ there exists } p^{(1)} \in F_1 \text{ such that } \left(q^{(1)}, a, p^{(1)}\right) \in T_1\right\} \cup$
  $\left\{\left(\overline{q^{(2)}}, a, \overline{p^{(2)}}\right) \mid \left(q^{(2)}, a, p^{(2)}\right) \in T_2\right\} \cup$
  $\left\{\left(\overline{q^{(2)}}, a, r\right) \mid \overline{q^{(2)}} \in \overline{Q_2} \text{ and } \left(q^{(2)}, a, p^{(2)}\right) \notin T_2 \text{ for every } p^{(2)} \in Q_2\right\} \cup$
  $\{(r, a, r) \mid a \in A\}$,

- $F = F_2 \cup \{r\}$,

- $in\left(q_{in}^{(1)}\right) = in\left(\overline{q_{in}^{(1)}}\right) = \mathbf{1}$,

- $$wt(q,a,p) = \begin{cases} wt_1(q,a,p) & \text{if } (q,a,p) \in T_1 \\ wt_2(q,a,p) & \text{if } (q,a,p) \in T_2 \\ wt_1(q,a,p^{(1)}) & \text{if } q \in Q_1 \setminus F_1,\ p = q_{in}^{(2)},\ p^{(1)} \in F_1, \text{ and } \left(q,a,p^{(1)}\right) \in T_1 \\ wt_1\left(q^{(1)},a,p^{(1)}\right) & \text{if } (q,a,p) = \left(\overline{q^{(1)}}, a, \overline{p^{(1)}}\right) \text{ and } \left(q^{(1)},a,p^{(1)}\right) \in T_1 \\ wt_1\left(q^{(1)},a,p^{(1)}\right) & \text{if } q = \overline{q^{(1)}} \in \overline{Q_1} \setminus \overline{F_1},\ p = \overline{q_{in}^{(2)}},\ \overline{p^{(1)}} \in \overline{F_1}, \text{ and} \\ & \quad \left(q^{(1)},a,p^{(1)}\right) \in T_1 \\ \mathbf{0} & \text{if } q,p \in \overline{Q_2} \cup \{r\} \end{cases}$$

  for every $(q,q,p) \in T$, and

- $$ter(q) = \begin{cases} \mathbf{1} & \text{if } q \in F_2 \\ \mathbf{0} & \text{otherwise} \end{cases}$$
  for every $q \in F$.

We should note that in case $p = q_{in}^{(2)}$ above, the value $wt(q,a,p)$ is well-defined. Indeed, since the original MK-fuzzy automaton accepting r is deterministic, by construction of $\mathcal{A}_1$, we get that there is at most one $p^{(1)} \in F_1$ such that $(q,a,p^{(1)}) \in T_1$. A similar argument holds for the case $p = \overline{q_{in}^{(2)}}$.

We define a linear order on Q as follows. We preserve the orders of Q_1 and Q_2 and define a linear order on $\overline{Q_1}$ (resp. $\overline{Q_2}$) by letting $\overline{q^{(1)}} \leq \overline{p^{(1)}}$ (resp. $\overline{q^{(2)}} \leq \overline{p^{(2)}}$) iff $q^{(1)} \leq p^{(1)}$ (resp. $q^{(2)} \leq p^{(2)}$) for every $\overline{q^{(1)}}, \overline{p^{(1)}} \in \overline{Q_1}$ (resp. $\overline{q^{(2)}}, \overline{p^{(2)}} \in \overline{Q_2}$). Then we set $\max Q_2 \leq \min Q_1, \max Q_1 \leq \min \overline{Q_2}$, and $\max \overline{Q_2} \leq r \leq \min \overline{Q_1}$.

Let $w = a_0 \ldots a_{n-1} \in A^+$ and $P_w^{(\mathcal{A})} = ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$ be a path of $\mathcal{A}$ over w. By construction of T we get that $n > 1$ and distinguish the following cases:

i) $q_0 = q_{in}^{(1)}$. Then, by definition of T, we get $q_n \in F_2$ and there is an index $0 < j < n$ such that $q_j = q_{in}^{(2)}$, $q_1, \ldots, q_{j-1} \in Q_1 \setminus F_1$, and $q_{j+1}, \ldots, q_{n-1} \in Q_2$. This in turn implies that there is a path $P_{a_0 \ldots a_{j-1}}^{(\mathcal{A}_1)} = \left(q_{in}^{(1)}, a_0, q_1\right) ((q_i, a_i, q_{i+1}))_{1 \leq i \leq j-2} \left(q_{j-1}, a_{j-1}, p^{(1)}\right)$ of $\mathcal{A}_1$ over $a_0 \ldots a_{j-1}$, with $p^{(1)} \in F_1$, and a path $P_{a_j \ldots a_{n-1}}^{(\mathcal{A}_2)} = \left(q_{in}^{(2)}, a_j, q_{j+1}\right) ((q_i, a_i, q_{i+1}))_{j+1 \leq i \leq n-1}$ of $\mathcal{A}_2$ over $a_j \ldots a_{n-1}$. Since the MK-fuzzy automata $\mathcal{A}_1$ and $\mathcal{A}_2$ are unambiguous, these paths are unique. Furthermore, we get

$$\begin{aligned}
weight\left(P_w^{(\mathcal{A})}\right) &= \bigsqcap_{0 \leq i \leq n-1} wt(q_i, a_i, q_{i+1}) \\
&= wt_1\left(q_{in}^{(1)}, a_0, q_1\right) \sqcap \ldots \sqcap wt_1\left(q_{j-1}, a_{j-1}, p^{(1)}\right) \\
&\qquad \sqcap wt_2\left(q_{in}^{(2)}, a_j, q_{j+1}\right) \sqcap \ldots \sqcap wt_2\left(q_{n-1}, a_{n-1}, q_n\right) \\
&= weight\left(P_{a_0 \ldots a_{j-1}}^{(\mathcal{A}_1)}\right) \sqcap weight\left(P_{a_j \ldots a_{n-1}}^{(\mathcal{A}_2)}\right) \\
&= \|\mathcal{A}_1\|(a_0 \ldots a_{j-1}) \sqcap \|\mathcal{A}_2\|(a_j \ldots a_{n-1})
\end{aligned}$$

where the last equality holds by the uniqueness of the paths $P_{a_0 \ldots a_{j-1}}^{(\mathcal{A}_1)}$ and $P_{a_j \ldots a_{n-1}}^{(\mathcal{A}_2)}$.

ii) $q_0 = \overline{q_{in}^{(1)}}$. Again, by definition of T, we get $q_n = r$ and there is an index $0 < j < n$ such that $q_j = \overline{q_{in}^{(2)}}$, $q_1 = \overline{q_1^{(1)}}, \ldots, q_{j-1} = \overline{q_{j-1}^{(1)}}$ where $\overline{q_1^{(1)}}, \ldots, \overline{q_{j-1}^{(1)}} \in \overline{Q_1} \setminus \overline{F_1}$, and $q_{j+1}, \ldots, q_{n-1} \in \overline{Q_2} \cup \{r\}$. This means that there is a (unique) path $P_{a_0 \ldots a_{j-1}}^{(\mathcal{A}_1)} = \left(q_{in}^{(1)}, a_0, q_1\right) ((q_i, a_i, q_{i+1}))_{1 \leq i \leq j-2}$ $\left(q_{j-1}, a_{j-1}, p^{(1)}\right)$ of $\mathcal{A}_1$ over $a_0 \ldots a_{j-1}$, with $p^{(1)} \in F_1$, and there is not any path of $\mathcal{A}_2$ over $a_j \ldots a_{n-1}$, hence $\|\mathcal{A}_2\|(a_j \ldots a_{n-1}) = \mathbf{0}$. Moreover, we get

$$\begin{aligned}
weight\left(P_w^{(\mathcal{A})}\right) &= \bigsqcap_{0 \leq i \leq n-1} wt(q_i, a_i, q_{i+1}) \\
&= wt_1\left(\overline{q_{in}^{(1)}}, a_0, q_1\right) \sqcap \ldots \sqcap wt_1\left(q_{j-1}, a_{j-1}, \overline{p^{(1)}}\right) \sqcap \mathbf{0} \ldots \sqcap \mathbf{0} \\
&= weight\left(P_{a_0 \ldots a_{j-1}}^{(\mathcal{A}_1)}\right) \sqcap \mathbf{0} \\
&= \|\mathcal{A}_1\|(a_0 \ldots a_{j-1}) \sqcap \|\mathcal{A}_2\|(a_j \ldots a_{n-1})
\end{aligned}$$

where the last equality holds by the uniqueness of the path $P_{a_0 \ldots a_{j-1}}^{(\mathcal{A}_1)}$.

Conversely, keeping the above notations, we distinguish the following cases:

i)′ There is an index $0 < j < n$ such that there are (unique) paths $P^{(\mathcal{A}_1)}_{a_0 \ldots a_{j-1}}$ of $\mathcal{A}_1$ over $a_0 \ldots a_{j-1}$ and $P^{(\mathcal{A}_2)}_{a_j \ldots a_{n-1}}$ of $\mathcal{A}_2$ over $a_j \ldots a_{n-1}$, hence $\|\mathcal{A}_1\|(a_0 \ldots a_{j-1}) = weight\left(P^{(\mathcal{A}_1)}_{a_0 \ldots a_{j-1}}\right)$ and $\|\mathcal{A}_2\|(a_j \ldots a_{n-1}) = weight\left(P^{(\mathcal{A}_2)}_{a_j \ldots a_{n-1}}\right)$. Then, by definition of T, there is a path $P^{(\mathcal{A})}_w$ of $\mathcal{A}$ over w with

$$\begin{aligned} weight\left(P^{(\mathcal{A})}_w\right) &= weight\left(P^{(\mathcal{A}_1)}_{a_0 \ldots a_{j-1}}\right) \sqcap weight\left(P^{(\mathcal{A}_2)}_{a_j \ldots a_{n-1}}\right) \\ &= \|\mathcal{A}_1\|(a_0 \ldots a_{j-1}) \sqcap \|\mathcal{A}_2\|(a_j \ldots a_{n-1}). \end{aligned}$$

ii)′ There is an index $0 < j < n$ such that there is a (unique) path $P^{(\mathcal{A}_1)}_{a_0 \ldots a_{j-1}} = \left(q^{(1)}_{in}, a_0, q_1\right)$ $((q_i, a_i, q_{i+1}))_{1 \leq i \leq j-2} \left(q_{j-1}, a_{j-1}, p^{(1)}\right)$ of $\mathcal{A}_1$ over $a_0 \ldots a_{j-1}$, with $p^{(1)} \in F_1$ and there is not any path of $\mathcal{A}_2$ over $a_j \ldots a_{n-1}$. Thus $\|\mathcal{A}_1\|(a_0 \ldots a_{j-1}) = weight\left(P^{(\mathcal{A}_1)}_{a_0 \ldots a_{j-1}}\right)$ and $\|\mathcal{A}_2\|(a_j \ldots, a_{n-1}) = \mathbf{0}$. By definition of T, there is a path

$$\begin{aligned} P^{(\mathcal{A})}_w = &\left(\overline{q^{(1)}_{in}}, a_0, \overline{q_1}\right)((\overline{q_i}, a_i, \overline{q_{i+1}}))_{1 \leq i \leq j-2}\left(\overline{q_{j-1}}, a_{j-1}, \overline{q^{(2)}_{in}}\right) \\ &\left(\overline{q^{(2)}_{in}}, a_j, \overline{q_{j+1}}\right)((\overline{q_k}, a_k, \overline{q_{k+1}}))_{j+1 \leq k \leq n-1} \end{aligned}$$

of $\mathcal{A}$ over w with $\overline{q^{(1)}_{in}}, \overline{q_1}, \ldots, \overline{q_{j-1}} \in \overline{Q_1} \setminus \overline{F_1}$, $\overline{q^{(2)}_{in}} \in \overline{Q_2}$, $\overline{q_{j+1}}, \ldots, \overline{q_{n-1}} \in \overline{Q_2} \cup \{r\}$, and $\overline{q_n} = r$. Moreover, we get

$$\begin{aligned} weight\left(P^{(\mathcal{A})}_w\right) &= weight\left(P^{(\mathcal{A})}_{a_0 \ldots a_{j-1}}\right) \sqcap \mathbf{0} \\ &= \|\mathcal{A}_1\|(a_0 \ldots a_{j-1}) \sqcap \|\mathcal{A}_2\|(a_j \ldots a_{n-1}). \end{aligned}$$

We conclude that for every word $w = a_0 \ldots a_{n-1} \in A^+$ with $n > 1$, the existence of a path $P^{(\mathcal{A})}_w$ of $\mathcal{A}$ over w implies the existence of an index $0 < j < n$ such that $weight\left(P^{(\mathcal{A})}_w\right) = \|\mathcal{A}_1\|(a_0 \ldots a_{j-1}) \sqcap \|\mathcal{A}_2\|(a_j \ldots a_{n-1})$, and vice versa.

Next let us assume that there is not any path of $\mathcal{A}$ over $w = a_0 \ldots a_{n-1}$ with $n > 1$, hence $\|\mathcal{A}\|(w) = \mathbf{0}$. This implies, that there is not any path of $\mathcal{A}_1$ over any prefix of w. Therefore, we get

$$r(a_0) = r(a_0 a_1) = \ldots = r(a_0 \ldots a_{n-2}) = \mathbf{0}.$$

The inverse implication trivially holds, therefore we conclude that

$$\|\mathcal{A}\|(w) = (r(a_0) \sqcap s(a_1 \ldots a_{n-1})) \sqcup \ldots \sqcup (r(a_0 \ldots a_{n-2}) \sqcap s(a_{n-1}))$$

for every $w = a_0 \dots a_{n-1} \in A^+$ with $n > 1$.

Next, by Theorem 18, the series $r(\varepsilon) \sqcap s$ is recognizable, hence by Proposition 16 there is a normalized MK-fuzzy automaton $\mathcal{A}_3$ such that $\|\mathcal{A}_3\|(w) = (r(\varepsilon) \sqcap s)(w)$ for every $w \in A^+$, and $\|\mathcal{A}_3\|(\varepsilon) = \mathbf{0}$. Furthermore, by Corollary 19 and Proposition 21 respectively, the MK-fuzzy languages $\overline{\varepsilon} \sqcap r(\varepsilon) \sqcap s(\varepsilon)$ and $\|\mathcal{A}_1\| \sqcap s(\varepsilon)$ are recognizable. Since

$$rs = (\overline{\varepsilon} \sqcap r(\varepsilon) \sqcap s(\varepsilon)) \sqcup \|\mathcal{A}_3\| \sqcup \|\mathcal{A}\| \sqcup (\|\mathcal{A}_1\| \sqcap s(\varepsilon)),$$

we conclude our proof by Theorem 11. ∎

Proposition 23. *Let $s \in DRec(K, A)$. Then the strong support of s is a recognizable language.*

Proof. Let $\mathcal{A} = (Q, q_0, T, F, in, wt, ter)$ be a deterministic MK-fuzzy automaton over A and K accepting s. Assume firstly that $t(in(q_0)) = 0$. Then for every word $w \in A^*$, if there is a path $P_w^{(\mathcal{A})}$ (which is unique) of $\mathcal{A}$ over w, we get $t\left(weight\left(P_w^{(\mathcal{A})}\right)\right) = 0$. This implies that $\mathrm{stgsupp}(s) = \emptyset$ which is recognizable.

Next let $t(in(q_0)) \neq 0$. We consider the finite automaton $\mathcal{A}' = (Q, A, q_0, T', F')$ with $T' = \{(q, a, q') \in T \mid t(wt((q, a, q'))) \neq 0\}$ and $F' = \{q \in F \mid t(ter(q)) \neq 0\}$. By definition the automaton $\mathcal{A}'$ is deterministic. Let $w = a_0 \dots a_{n-1} \in A^+$ being accepted by $\mathcal{A}'$. Hence, there is a unique successful path $P_w^{(\mathcal{A}')} = ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$ of $\mathcal{A}'$ over w. By construction of $\mathcal{A}'$, we get that $P_w^{(\mathcal{A}')}$ is also a path of $\mathcal{A}$ over w. Moreover it holds

$$t\left(weight\left(P_w^{(\mathcal{A}')}\right)\right) = t(in(q_0)) \cdot \prod_{0 \leq i \leq n-1} t(wt(q_i, a_i, q_{i+1})) \cdot t(ter(q_n))$$

and by our assumption we get $t\left(weight\left(P_w^{(\mathcal{A}')}\right)\right) \neq 0$ which in turn implies that $w \in \mathrm{stgsupp}(s)$. If $\varepsilon \in L(\mathcal{A}')$, then $q_0 \in F'$, and hence $t(in(q_0)) \cdot t(ter(q_0)) \neq 0$, i.e., $\varepsilon \in \mathrm{stgsupp}(s)$.

Conversely assume that $w = a_0 \dots a_{n-1} \in A^+$ is in $\mathrm{stgsupp}(s)$. Then, there is a unique path $P_w^{(\mathcal{A})} = ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$ of $\mathcal{A}$ over w with $t\left(weight\left(P_w^{(\mathcal{A})}\right)\right) \neq 0$. By construction of the finite automaton $\mathcal{A}'$, we get that $P_w^{(\mathcal{A})}$ is a successful path of $\mathcal{A}'$ over w, and thus $w \in L(\mathcal{A}')$. If $\varepsilon \in \mathrm{stgsupp}(s)$, then $q_0 \in F$ and $t(ter(q_0)) \neq 0$, hence $q_0 \in F'$, i.e., $\varepsilon \in L(\mathcal{A}')$, and our proof is completed. ∎

4. MK-fuzzy monadic second order logic

In this section we introduce our MK-fuzzy monadic second order (MSO for short) logic and we prove the fundamental theorem of Büchi [3], Elgot [12], and Trakhtenbrot [24] in the setup of

MK-fuzzy languages. We need to recall the definition of syntax and semantics of MSO logic (cf. for instance [23]).

The syntax of MSO logic formulas over A is given by the grammar

$$\phi ::= \mathit{true} \mid P_a(x) \mid x \le x' \mid x \in X \mid \neg\phi \mid \phi \vee \phi \mid \exists x \centerdot \phi \mid \exists X \centerdot \phi$$

where $a \in A$ and we let $\mathit{false} = \neg \mathit{true}$. The set $\mathit{free}(\phi)$ of free variables of an MSO logic formula ϕ is defined as usual. In order to define the semantics of MSO logic formulas we need the notions of the extended alphabet and valid assignment. Let $\mathcal{V}$ be a finite set of first and second order variables. For every word $w = w(0)\dots w(n-1) \in A^*$ we let $\mathit{dom}(w) = \{0,\dots,n-1\}$. A $(\mathcal{V},w)$-*assignment* σ is a mapping associating first order variables from $\mathcal{V}$ to elements of $\mathit{dom}(w)$, and second order variables from $\mathcal{V}$ to subsets of $\mathit{dom}(w)$. If x is a first order variable and $i \in \mathit{dom}(w)$, then $\sigma[x \to i]$ denotes the $(\mathcal{V} \cup \{x\}, w)$-assignment which associates i to x and coincides with σ on $\mathcal{V} \setminus \{x\}$. For a second order variable X and $I \subseteq \mathit{dom}(w)$, the notation $\sigma[X \to I]$ has a similar meaning. We shall encode pairs of the form (w,σ), where $w \in A^*$ and σ is a $(\mathcal{V},w)$-assignment, using the extended alphabet $A_{\mathcal{V}} = A \times \{0,1\}^{\mathcal{V}}$. Indeed, every word in $A_{\mathcal{V}}^*$ can be considered as a pair (w,σ) where w is the projection over A and σ is the projection over $\{0,1\}^{\mathcal{V}}$. Then σ is a valid assignment if for every first order variable $x \in \mathcal{V}$ the x-row contains exactly one 1. In this case, σ is the $(\mathcal{V},w)$-assignment such that for every first order variable $x \in \mathcal{V}$, $\sigma(x)$ is the position of the 1 on the x-row, and for every second order variable $X \in \mathcal{V}$, $\sigma(X)$ is the set of positions labelled with 1 along the X-row. It is well-known that

$$N_{\mathcal{V}} = \{(w,\sigma) \in A_{\mathcal{V}}^* \mid \sigma \text{ is a valid } (\mathcal{V},w)\text{-assignment}\}$$

is a recognizable language. For every $(w,\sigma) \in \mathcal{N}_{\mathcal{V}}$ we define the satisfaction relation $(w,\sigma) \models \phi$ by induction on the structure of ϕ, as follows:

$(w,\sigma) \models \mathit{true}$, $\qquad$ $(w,\sigma) \models x \in X$ iff $\sigma(x) \in \sigma(X)$,

$(w,\sigma) \models P_a(x)$ iff $w(\sigma(x)) = a$, $\qquad$ $(w,\sigma) \models \neg\phi$ iff $(w,\sigma) \not\models \phi$,

$(w,\sigma) \models x \le x'$ iff $\sigma(x) \le \sigma(x')$, $\qquad$ $(w,\sigma) \models \phi \vee \phi'$ iff $(w,\sigma) \models \phi$ or $(w,\sigma) \models \phi'$,

$(w,\sigma) \models \exists x \centerdot \phi$ iff there exists an $i \in \mathit{dom}(w)$ such that $(w,\sigma[x \to i]) \models \phi$,

$(w,\sigma) \models \exists X \centerdot \phi$ iff there exists an $I \subseteq \mathit{dom}(w)$ such that $(w,\sigma[X \to I]) \models \phi$.

If $(w,\sigma) \in A_{\mathcal{V}}^* \setminus \mathcal{N}_{\mathcal{V}}$, then we let $(w,\sigma) \not\models \phi$.

We denote by $L(\phi)$ the language of an MSO logic sentence ϕ, i.e., $L(\phi) = \{w \in A^* \mid w \models \phi\}$.

Remark 24. *For the definition of the semantics of our MK-fuzzy* MSO *logic, we shall need the power set $\mathcal{P}(dom(w))$ to be linearly ordered for every word $w \in A^*$. Let $w = a_0 \ldots a_{n-1} \in A^*$, hence $dom(w) = \{0, \ldots, n-1\}$. We define the linear order $\leq$ on $\mathcal{P}(dom(w))$ in the following way. Let $I = \{i_1, \ldots, i_m\}, J = \{j_1, \ldots, j_k\} \in \mathcal{P}(dom(w))$ and assume that $0 \leq i_1 < \ldots < i_m \leq n-1$ and $0 \leq j_1 < \ldots < j_k \leq n-1$. Then we consider the words $v_I = i_1 \ldots i_m, v_J = j_1 \ldots j_k \in dom(w)^*$. Clearly, there is a one-to-one correspondence among the subsets of $dom(w)$, and the words of $dom(w)^*$ with length at most n and their letters being pairwise disjoint. The empty set corresponds to the empty word. Now, for every $I, J \in \mathcal{P}(dom(w))$ we set $I \leq J$ iff $v_I \leq_{lex} v_J$.*

Definition 25. *The syntax of formulas of the* MK-fuzzy MSO logic over A and K *is given by the grammar*

$$\varphi ::= \mathbf{k} \mid \phi \mid \varphi \oplus \varphi \mid \varphi \otimes \varphi \mid \bigoplus\nolimits_x \centerdot \varphi \mid \bigoplus\nolimits_X \centerdot \varphi \mid \bigotimes\nolimits_x \centerdot \varphi$$

where $\mathbf{k} \in K$, $a \in A$, and ϕ denotes an MSO *logic formula.*

We denote by $MSO(K, A)$ the set of all MK-fuzzy MSO logic formulas φ over A and K. We represent the semantics of a formula $\varphi \in MSO(K, A)$ as an MK-fuzzy language $\|\varphi\| \in K \langle\langle A^* \rangle\rangle$. For the semantics of an MSO logic formula ϕ we use the satisfaction relation as defined above. Therefore, the semantics of an MSO logic formula ϕ gets only the values $\mathbf{0}$ and $\mathbf{1}$.

Definition 26. *Let $\varphi \in MSO(K, A)$ and $\mathcal{V}$ be a finite set of variables with free$(\varphi) \subseteq \mathcal{V}$. The semantics of φ is an MK-fuzzy language $\|\varphi\|_{\mathcal{V}} \in K \langle\langle A^*_{\mathcal{V}} \rangle\rangle$. Consider an element $(w, \sigma) \in A^*_{\mathcal{V}}$. If $(w, \sigma) \notin N_{\mathcal{V}}$, then we let $\|\varphi\|_{\mathcal{V}}(w, \sigma) = \mathbf{0}$. Otherwise, we define $\|\varphi\|_{\mathcal{V}}(w, \sigma) \in K$, inductively on the structure of φ, as follows:*

- $\|\mathbf{k}\|_{\mathcal{V}}(w, \sigma) = \mathbf{k}$,
- $\|\phi\|_{\mathcal{V}}(w, \sigma) = \begin{cases} \mathbf{1} & \text{if } (w, \sigma) \models \phi \\ \mathbf{0} & \text{otherwise} \end{cases}$,
- $\|\varphi \oplus \psi\|_{\mathcal{V}}(w, \sigma) = \|\varphi\|_{\mathcal{V}}(w, \sigma) \sqcup \|\psi\|_{\mathcal{V}}(w, \sigma)$,
- $\|\varphi \otimes \psi\|_{\mathcal{V}}(w, \sigma) = \|\varphi\|_{\mathcal{V}}(w, \sigma) \sqcap \|\psi\|_{\mathcal{V}}(w, \sigma)$,
- $\|\bigoplus_x \centerdot \varphi\|_{\mathcal{V}}(w, \sigma) = \bigsqcup_{0 \leq i \leq |w|-1} \|\varphi\|_{\mathcal{V} \cup \{x\}}(w, \sigma[x \to i])$,

- $\left\|\bigotimes_x \centerdot \varphi\right\|_{\mathcal{V}}(w,\sigma) = \bigsqcap_{0\leq i\leq |w|-1} \|\varphi\|_{\mathcal{V}\cup\{x\}}(w,\sigma[x\to i])$,
- $\left\|\bigoplus_X \centerdot \varphi\right\|_{\mathcal{V}}(w,\sigma) = \bigsqcup_{I\subseteq dom(w)} \|\varphi\|_{\mathcal{V}\cup\{X\}}(w,\sigma[X\to I])$

where the operator $\bigsqcup_{I\subseteq dom(w)}$ *is applied on the ascending order according to the relation* $\leq$ *as defined in Remark 24.*

We simply denote $\|\varphi\|_{free(\varphi)}$ by $\|\varphi\|$, hence if φ is a sentence, then $\|\varphi\| \in K\langle\langle A^*\rangle\rangle$.

Lemma 27. *[6] Let A be a linearly ordered alphabet, $\varphi \in MSO(K,A)$, and $\mathcal{V}$ be a finite set of variables containing free(φ). Then*

$$\|\varphi\|_{\mathcal{V}}(w,\sigma) = \|\varphi\|\left(w,\sigma|_{free(\varphi)}\right)$$

for every $(w,\sigma)\in N_{\mathcal{V}}$. Furthermore $\|\varphi\|_{\mathcal{V}}$ is recognizable iff $\|\varphi\|$ is recognizable.

Proof. We extend the order on A to a linear order on $A_{\mathcal{V}}$ and apply the proof of Prop. 3.3. in [6] using our Theorems 12–14. ∎

For first order variables x,y,z, second order variables $X_1,\ldots,X_m$, and $\mathbf{k}\in K$ let

$$\text{first}(y) := \forall x \centerdot y \leq x, \qquad (y = x+1) := ((x\leq y)\wedge\neg(y\leq x)\wedge\forall z \centerdot (z\leq x\vee y\leq z)),$$

$$\text{last}(y) := \forall x \centerdot x\leq y, \qquad \text{partition}(X_1,\ldots,X_m) := \forall x \centerdot \bigvee_{i=1,\ldots,m}\left((x\in X_i)\wedge\bigwedge_{j\neq i}\neg(x\in X_j)\right),$$

$$x\in X\to\mathbf{k} := \neg(x\in X)\oplus((x\in X)\otimes\mathbf{k}).$$

Next we define a fragment of our MK-fuzzy MSO logic.

Definition 28. *A formula $\varphi \in MKO(K,A)$ will be called* restricted *if whenever it contains a subformula $\psi\otimes\psi'$, then ψ is a (boolean)* MSO *logic formula, and whenever it contains a subformula of the form $\bigotimes_x \centerdot \psi$, then ψ is of the form $\bigoplus_{1\leq i\leq m}((x\in X_i)\to\mathbf{k_i})$, where $\mathbf{k_i}\in K$ for every $1\leq i\leq m$.*

We shall denote by $RMSO(K,A)$ the class of all restricted MK-fuzzy MSO logic formulas over A and K. An MK-fuzzy language $s\in K\langle\langle A^*\rangle\rangle$ is called RMSO-*definable* if there is a sentence $\varphi\in RMSO(K,A)$ such that $s=\|\varphi\|$. The main result of this section is the subsequent theorem which follows from Theorems 30 and 31 below.

Theorem 29. *Let A be a linearly ordered alphabet and $s\in K\langle\langle A^*\rangle\rangle$. Then s is recognizable iff it is* RMSO-*definable.*

Theorem 30. *Let A be a linearly ordered alphabet. If an MK-fuzzy language $s \in K\langle\langle A^*\rangle\rangle$ is* RMSO*-definable, then it is recognizable.*

Proof. Let $\varphi \in RMSO(K, A)$ such that $s = \|\varphi\|$. We show by induction on the structure of φ that $\|\varphi\| \in Rec(K, A)$.

Assume first that $\varphi = \phi$ is a (boolean) MSO logic formula. Then $L(\phi)$ is a recognizable language hence, by Proposition 10 the MK-fuzzy language $\|\varphi\| = \mathbf{1}_{L(\phi)}$ is recognizable. Let $\varphi = \mathbf{k}$. By definition $\|\mathbf{k}\| = \widetilde{\mathbf{k}}$, hence $\|\mathbf{k}\|$ is recognizable by Example 9.

Next let $\varphi = \psi \oplus \psi'$ such that $\|\psi\|, \|\psi'\|$ are recognizable. We let $\mathcal{V} = free(\psi) \cup free(\psi')$. Then, by Lemma 27, the MK-fuzzy languages $\|\psi\|_{\mathcal{V}}, \|\psi'\|_{\mathcal{V}}$ are recognizable. Since $\|\psi \oplus \psi'\| = \|\psi\|_{\mathcal{V}} \sqcup \|\psi'\|_{\mathcal{V}}$, we conclude our claim by Theorem 11.

Assume now that $\varphi = \psi \otimes \psi'$ such that $\psi = \phi$ is a (boolean) MSO logic formula and $\|\psi'\|$ is recognizable. We let $\mathcal{V} = free(\psi) \cup free(\psi')$. Then, the language $L(\phi)$ is recognizable, $\|\psi\|_{\mathcal{V}} = \mathbf{1}_{L(\phi)}$, and by Lemma 27 the MK-fuzzy language $\|\psi'\|_{\mathcal{V}}$ is recognizable. Since $\|\psi \otimes \psi'\| = \|\psi\|_{\mathcal{V}} \sqcap \|\psi'\|_{\mathcal{V}}$ we get our result by Theorem 12.

Let $\varphi = \bigoplus_x \centerdot \psi$ such that $\|\psi\|$ is a recognizable MK-fuzzy language and let $\mathcal{V} = free(\varphi)$. We extend the order on $A_{\mathcal{V}}$ to a linear order on $A_{\mathcal{V}\cup\{x\}}$ by letting $(a, r[x = 1]) \leq (a, r[x = 0])$ for every $(a, r) \in A_{\mathcal{V}}$. Then, we follow the proof of Lm. 4.3. in [6] taking into account our Theorem 13 and show that $\|\varphi\|$ is recognizable.

Next let $\varphi = \bigoplus_X \centerdot \psi$ such that $\|\psi\|$ is a recognizable MK-fuzzy language and let $\mathcal{V} = free(\varphi)$. We extend the order on $A_{\mathcal{V}}$ to a linear order on $A_{\mathcal{V}\cup\{X\}}$ by letting $(a, r[X = 1]) \leq (a, r[X = 0])$ for every $(a, r) \in A_{\mathcal{V}}$. Then, we follow the proof of Lm. 4.3. in [6] taking into account our Theorem 13 and show that $\|\varphi\|$ is recognizable.

Finally, let $\varphi = \bigotimes_x \centerdot \left(\bigoplus_{1\leq i\leq m} ((x \in X_i) \to \mathbf{k_i}) \right)$ where $\mathbf{k_i} \in K$ for every $1 \leq i \leq m$. We consider the deterministic MK-fuzzy automaton $\mathcal{A} = (\{q\}, q, T, \{q\}, in, wt, ter)$ over $A_{\{X_1,\ldots,X_m\}}$ and K, with $T = \left\{(q, (a, r), q) \mid a \in A, r \in \{0, 1\}^{\{X_1,\ldots,X_m\}}\right\}$. The weight mappings are defined by $in(q) = ter(q) = \mathbf{1}$ and $wt(q, (a, r), q) = \bigsqcup_{1\leq i\leq m} (\mathbf{r}(X_i) \sqcap \mathbf{k_i})$ for every $a \in A$ and $r \in \{0, 1\}^{\{X_1,\ldots,X_m\}}$, where $\mathbf{r}(X_i) = \mathbf{1}$ if $r(X_i) = 1$ and $\mathbf{r}(X_i) = \mathbf{0}$ otherwise. Let $(w, \sigma) \in N_{\{X_1,\ldots,X_m\}} \left(= A^*_{\{X_1,\ldots,X_m\}}\right)$, and assume $(w, \sigma) = (a_0, r_0) \ldots (a_{n-1}, r_{n-1})$ where $w = a_0 \ldots a_{n-1} \in A^*$ and $r_j \in \{0, 1\}^{\{X_1,\ldots,X_m\}}$ for every $0 \leq j \leq n - 1$. Then, there is a unique path $P_{(w,\sigma)}$ of $\mathcal{A}$ over (w, σ). Moreover, we have

$$\|\mathcal{A}\|(w, \sigma) = weight(P_{(w,\sigma)})$$

$$
\begin{aligned}
&= \bigsqcap_{0\le j\le n-1}\left(\bigsqcup_{1\le i\le m}(\mathbf{r_j}(X_i)\sqcap \mathbf{k_i})\right)\\
&= \bigsqcap_{0\le j\le n-1}\left(\left\|\left(\bigoplus_{1\le i\le m}((x\in X_i)\to \mathbf{k_i})\right)\right\|_{\{x\}}(w,\sigma[x\to j])\right)\\
&= \left\|\bigotimes_x \centerdot \left(\bigoplus_{1\le i\le m}((x\in X_i)\to \mathbf{k_i})\right)\right\|(w,\sigma)\\
&= \|\varphi\|(w,\sigma).
\end{aligned}
$$

Therefore, $\|\mathcal{A}\| = \|\varphi\|$, which implies that $\|\varphi\| \in \mathit{Rec}\left(K, A_{\{X_1,\dots,X_m\}}\right)$, and this concludes our proof. ■

Theorem 31. *Let A be a linearly ordered alphabet. If an MK-fuzzy language $s \in K\langle\langle A^*\rangle\rangle$ is recognizable, then it is* RMSO*-definable.*

Proof. Let $\mathcal{A} = (Q, I, T, F, in, wt, ter)$ be an MK-fuzzy automaton over A and K, and let $\|\mathcal{A}\|(\varepsilon) = \mathbf{0}$. By Proposition 16, we can assume that $\mathcal{A}$ is normalized. We intend to show that $\|\mathcal{A}\|$ is an RMSO-definable MK-fuzzy language. For this, we can follow the proof of Thm. 5.5. in [6]. Nevertheless, in our case we have, in addition, to take care for the order of the paths of $\mathcal{A}$ over any word $w \in A^+$, as well as the order of the corresponding assignments. For every transition $(p, a, q) \in T$, we consider a second order variable $X_{p,a,q}$ and we let $\mathcal{V} = \{X_{p,a,q} \mid (p, a, q) \in T\}$. Let $m = |T|$. We define an enumeration $X_1, \dots, X_m$ of $\mathcal{V}$, preserving the order of the corresponding transitions in T. We let

$$
\begin{aligned}
\psi(X_1,\dots,X_m) := \text{partition}(X_1,\dots,X_m) \wedge \bigwedge_{(p,a,q)\in T} \forall x \centerdot ((x\in X_{p,a,q}) \to P_a(x)) \wedge \\
\forall x \centerdot \forall y \centerdot \left((y = x+1) \to \bigvee_{(p,a,q),(q,b,r)\in T} (x\in X_{p,a,q}) \wedge (y \in X_{q,b,r})\right) \wedge \\
\exists z \centerdot \left(\mathit{first}(z) \wedge \bigvee_{\substack{(p,a,q)\in T\\ p\in I}} z\in X_{p,a,q}\right) \wedge \exists z' \centerdot \left(\mathit{last}(z') \wedge \bigvee_{\substack{(p,a,q)\in T\\ q\in F}} z'\in X_{p,a,q}\right).
\end{aligned}
$$

Let $w = a_0 \dots a_{n-1} \in A^+$. We define a linear order on the set of all $(\mathcal{V}, w)$-assignments satisfying ψ in the following way. For two such assignments σ and σ', we let $\sigma \le \sigma'$ iff there exists $k \in dom(w)$, with $0 \leq k \leq n-1$, such that $k \in \sigma(X_{i_k}) \cap \sigma'(X_{i'_k})$ with $i_k \leq i'_k$ and $j \in \sigma(X_{i_j}) \cap \sigma'(X_{i_j})$ for every $0 \leq j < k$. Trivially $\leq$ is a linear order. On the other hand, for every path P_w of $\mathcal{A}$ over w there exists a unique $(\mathcal{V}, w)$-assignment σ_{P_w} satisfying ψ, i.e., $\|\psi\|(w, \sigma_{P_w}) = \mathbf{1}$ and vice versa (cf. Thm. 5.5. in [6]). Then, we can easily get that $P_w \leq P'_w$ iff $\sigma_{P_w} \leq \sigma_{P'_w}$. Next, we consider the formula

$$\varphi(X_1, \ldots, X_m) := \psi(X_1, \ldots, X_m) \otimes \bigotimes\nolimits_x \centerdot \left(\bigoplus_{(p,a,q) \in T} (x \in X_{p,a,q}) \to wt(p, a, q) \right).$$

Let now $w = a_0 \ldots a_{n-1} \in A^+$, $P_w = ((q_i, a_i, q_{i+1}))_{0 \leq i \leq n-1}$ a path of $\mathcal{A}$ over w, and σ_{P_w} the corresponding $(\mathcal{V}, w)$-assignment. Then, we get

$$\|\varphi\|_{\mathcal{V}} (w, \sigma_{P_w}) = \prod_{0 \leq i \leq n-1} wt\,(q_i, a_i, q_{i+1}) = weight\,(P_w)\,.$$

Finally, we consider the restricted MK-fuzzy MSO logic sentence

$$\xi = \bigoplus\nolimits_{X_1} \ldots \bigoplus\nolimits_{X_m} \centerdot \varphi(X_1, \ldots, X_m).$$

Then for every $w \in A^+$, due to the above bijection of paths of $\mathcal{A}$ over w and $(\mathcal{V}, w)$-assignments which preserves the order, we get

$$\begin{aligned}
\|\xi\|\,(w) &= \bigsqcup_{\sigma\ (\mathcal{V},w)\text{-assignment}} \|\varphi\|_{\mathcal{V}}\,(w, \sigma) \\
&= \bigsqcup_{P_w} \|\varphi\|_{\mathcal{V}}\,(w, \sigma_{P_w}) \\
&= \bigsqcup_{P_w} weight(P_w) = \\
&= \|\mathcal{A}\|(w).
\end{aligned}$$

Hence, $\|\mathcal{A}\| = \|\xi\|$, i.e., $\|\mathcal{A}\|$ is RMSO-definable.

Next let $\|\mathcal{A}\|(\varepsilon) = \mathbf{k} \neq \mathbf{0}$. Then, by Proposition 16, we consider the MK-fuzzy automaton $\mathcal{A}'$ such that $\|\mathcal{A}'\|(w) = \|\mathcal{A}\|(w)$ for every $w \in A^+$. By what we have shown previously, there exists a restricted MK-fuzzy MSO logic sentence ξ' such that $\|\mathcal{A}'\| = \|\xi'\|$. We let

$$\xi = \xi' \oplus (\forall x \centerdot \neg(x \leq x) \otimes \mathbf{k})\,.$$

Then ξ is a restricted MK-fuzzy MSO logic sentence, and we get $\|\forall x \centerdot \neg(x \leq x) \otimes \mathbf{k}\|(w) = \mathbf{0}$ for every $w \in A^+$, and $\|\forall x \centerdot \neg(x \leq x) \otimes \mathbf{k}\|(\varepsilon) = \mathbf{k}$ (cf. [6]). Hence $\|\mathcal{A}\| = \|\xi\|$, and this concludes our proof. ■

5. Conclusion

We introduced the bimonoid K related to the fuzzification of MK-logic, and investigated MK-fuzzy automata over K. Our models are inspired by real practical applications being in development within the project LogicGuard [17, 18, 15, 4]. We proved properties of the class of MK-fuzzy languages accepted by MK-fuzzy automata as well as by their deterministic counterpart. We introduced an MK-fuzzy MSO logic and established a Büchi type theorem for the class of MK-fuzzy recognizable languages.

It is worth noting that our results can be generalized to weighted automata over any bimonoid $(K, +, \cdot, 0, 1)$ with the additional property that $0 \cdot k = 0$ for every $k \in K$. Indeed, one can replace $\sqcup$ by $+$ and $\sqcap$ by $\cdot$.

Several problems remain open and they are under investigation, for instance, whether the class of recognizable MK-fuzzy languages is closed under MK-conjunction, Cauchy product and star operation, as well as whether the class of deterministically recognizable MK-fuzzy languages is closed under MK-disjunction and conjunction, Cauchy product, and star operation. Furthermore, due to the four-valued elements of K, there are several notions of supports and it is greatly desirable for applications to check which of them constitute recognizable languages. It should be clear from the proofs of our results, that the usual constructions on semiring-weighted automata cannot be always applied, even with modifications, when the weight structure is just a bimonoid. For instance, our bimonoid K is zero-sum free and zero-divisor free. Nevertheless, one can not show that the support $\mathrm{supp}(s) = \{w \in A^* \mid s(w) \neq \mathbf{0}\}$ of a recognizable (even deterministically recognizable) MK-fuzzy language s over A and K is a recognizable language following the usual construction on weighted automata (cf. for instance [8]). In our future research we intend also to study MK-fuzzy automata models over infinite words.

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