Uniform Continuity of Sum of Functions

Definitions

Uniform continuity

$$(D_1)\quad \underset{f:\mathbb{R}\longrightarrow\mathbb{R}}{\forall}\left(U[f] \Leftrightarrow \underset{\substack{\epsilon\in\mathbb{R}\\ \epsilon>0}}{\forall}\ \underset{\substack{\delta\in\mathbb{R}\\ \delta>0}}{\exists}\ \underset{\substack{x,y\in\mathbb{R}\\ \mathrm{Abs}[x-y]<\delta}}{\forall} \mathrm{Abs}[f[x]-f[y]]<\epsilon\right)$$

Sum of functions

$$(D_2)\quad \underset{f_1,f_2:\mathbb{R}\longrightarrow\mathbb{R}}{\forall}\left((f_1+f_2):\mathbb{R}\longrightarrow\mathbb{R} \wedge \underset{x\in\mathbb{R}}{\forall}(f_1+f_2)[x] = (f_1[x]+f_2[x])\right)$$

Sum of uniformly continuous functions is uniformly continuous

Formula to prove

$$(G)\quad \underset{\substack{f_1,f_2:\mathbb{R}\longrightarrow\mathbb{R}\\ U[f_1],\,U[f_2]}}{\forall} U[f_1+f_2]$$

Proof presentation

For proving (G) we take $f_1,\ f_2:\mathbb{R}\longrightarrow\mathbb{R}$ arbitrary but fixed, we assume :

(A_1) $U[f_1]$

(A_2) $U[f_2]$

and we prove :

(G) $U[f_1+f_2]$

From (A_1), using $f_1:\mathbb{R}\longrightarrow\mathbb{R}$, by the definition (D_1) we obtain :

$$(1)\quad \underset{\substack{\epsilon\in\mathbb{R}\\ \epsilon>0}}{\forall}\ \underset{\substack{\delta\in\mathbb{R}\\ \delta>0}}{\exists}\ \underset{\substack{x,y\in\mathbb{R}\\ \mathrm{Abs}[x-y]<\delta}}{\forall} \mathrm{Abs}[f_1[x]-f_1[y]]<\epsilon$$

From (A_2), using $f_2:\mathbb{R}\longrightarrow\mathbb{R}$, by the definition (D_1) we obtain :

$$(2)\quad \underset{\substack{\epsilon\in\mathbb{R}\\ \epsilon>0}}{\forall}\ \underset{\substack{\delta\in\mathbb{R}\\ \delta>0}}{\exists}\ \underset{\substack{x,y\in\mathbb{R}\\ \mathrm{Abs}[x-y]<\delta}}{\forall} \mathrm{Abs}[f_2[x]-f_2[y]]<\epsilon$$

Using $f_1,\ f_2:\mathbb{R}\longrightarrow\mathbb{R}$, by the definition (D_2) we obtain $(f_1+f_2):\mathbb{R}\longrightarrow\mathbb{R}$.

Using $(f_1+f_2):\mathbb{R}\longrightarrow\mathbb{R}$, by the definition (D_1), for proving (G) it suffices to prove:

$$(3)\quad \underset{\substack{\epsilon\in\mathbb{R}\\ \epsilon>0}}{\forall}\ \underset{\substack{\delta\in\mathbb{R}\\ \delta>0}}{\exists}\ \underset{\substack{x,y\in\mathbb{R}\\ \mathrm{Abs}[x-y]<\delta}}{\forall} \mathrm{Abs}\big[(f_1+f_2)[x]-(f_1+f_2)[y]\big]<\epsilon$$

Using $f_1,\ f_2:\mathbb{R}\longrightarrow\mathbb{R}$, $x, y\in\mathbb{R}$, by the definition (D_2), for proving (3) it suffices to prove:

$$(4)\quad \underset{\substack{\epsilon\in\mathbb{R}\\ \epsilon>0}}{\forall}\ \underset{\substack{\delta\in\mathbb{R}\\ \delta>0}}{\exists}\ \underset{\substack{x,y\in\mathbb{R}\\ \mathrm{Abs}[x-y]<\delta}}{\forall} \mathrm{Abs}\big[(f_1[x]+f_2[x])-(f_1[y]+f_2[y])\big]<\epsilon_0$$

For proving (4) we take $\epsilon_0 \in \mathbb{R}$ arbitrary but fixed, we assume :

(5) $\epsilon_0 > 0$

and we prove :

(6) $\underset{\substack{\delta\in\mathbb{R} \\ \delta>0}}{\exists} \; \underset{\substack{x,y\in\mathbb{R} \\ \text{Abs}[x-y]<\delta}}{\forall} \text{Abs}[(f_1[x] + f_2[x]) - (f_1[y] + f_2[y])] < \epsilon_0$

We consider :

$\epsilon_1 = \epsilon_0 / 2$

First we prove $\epsilon_1 \in \mathbb{R}$ and:

(7) $\epsilon_1 > 0$

These follow from $\epsilon_0 \in \mathbb{R}$, (5), and elementary properties of $\mathbb{R}$.

Using $\epsilon_1 \in \mathbb{R}$ and (7), we instantiate (1) and we obtain :

(8) $\underset{\substack{\delta\in\mathbb{R} \\ \delta>0}}{\exists} \; \underset{\substack{x,y\in\mathbb{R} \\ \text{Abs}[x-y]<\delta}}{\forall} \text{Abs}[f_1[x] - f_1[y]] < \epsilon_1$

Using $\epsilon_1 \in \mathbb{R}$ and (7), we instantiate (2) and we obtain :

(9) $\underset{\substack{\delta\in\mathbb{R} \\ \delta>0}}{\exists} \; \underset{\substack{x,y\in\mathbb{R} \\ \text{Abs}[x-y]<\delta}}{\forall} \text{Abs}[f_2[x] - f_2[y]] < \epsilon_1$

By (8) we can take $\delta_1 \in \mathbb{R}$, such that :

(10) $\delta_1 > 0 \wedge \underset{\substack{x,y\in\mathbb{R} \\ \text{Abs}[x-y]<\delta_1}}{\forall} \text{Abs}[f_1[x] - f_1[y]] < \epsilon_1$

By (9) we can take $\delta_2 \in \mathbb{R}$ such that :

(11) $\delta_2 > 0 \wedge \underset{\substack{x,y\in\mathbb{R} \\ \text{Abs}[x-y]<\delta_2}}{\forall} \text{Abs}[f_2[x] - f_2[y]] < \epsilon_1$

We consider :

$\delta_0 = \text{Min}[\delta_1, \delta_2]$

First we prove $\delta_0 \in \mathbb{R}$ and:

(12) $\delta_0 > 0$

These follow from δ_1, $\delta_2 \in \mathbb{R}$, (10.1), (11.1) and elementary properties of $\mathbb{R}$.

Using $\delta_0 \in \mathbb{R}$ and (12), in order to prove (6) it suffices to prove:

(13) $\underset{\substack{x,y\in\mathbb{R} \\ \text{Abs}[x-y]<\delta_0}}{\forall} \text{Abs}[(f_1[x] + f_2[x]) - (f_1[y] + f_2[y])] < \epsilon_0$

For proving (13) we take $x_0, y_0 \in \mathbb{R}$ arbitrary but fixed, we assume :

(14) $\text{Abs}[x_0 - y_0] < \delta_0$

and we prove :

(15) $\text{Abs}[(f_1[x_0] + f_2[x_0]) - (f_1[y_0] + f_2[y_0])] < \epsilon_0$

First we prove :

(16) $\text{Abs}[x_0 - y_0] < \delta_1 \wedge \text{Abs}[x_0 - y_0] < \delta_2$

This follows from $x_0, y_0, \delta_0, \delta_1, \delta_2 \in \mathbb{R}$, (14) and elementary properties of $\mathbb{R}$.

Using $x_0, y_0 \in \mathbb{R}$ and (16.1), from (10.2) we obtain:

(17) $\mathrm{Abs}[f_1[x_0] - f_1[y_0]] < \epsilon_1$

Using $x_0, y_0 \in \mathbb{R}$ and (16.2), from (11.2) we obtain:

(18) $\mathrm{Abs}[f_2[x_0] - f_2[y_0]] < \epsilon_1$

Using elementary properties of $\mathbb{R}$ we transform (15) into:

(19) $\mathrm{Abs}[(f_1[x_0] - f_1[y_0]) + (f_2[x_0] - f_2[y_0])] < \epsilon_0$

Using elementary properties of $\mathbb{R}$, from (17) and (18) we obtain:

$$\mathrm{Abs}[(f_1[x_0] - f_1[y_0]) + (f_2[x_0] - f_2[y_0])] \leq \quad (*\ \text{by Abs} +\ *)$$
$$\leq \mathrm{Abs}[f_1[x_0] - f_1[y_0]] + \mathrm{Abs}[f_2[x_0] - f_2[y_0]] < \quad (*\ \text{by (17), (18)}\ *)$$
$$< \epsilon_1 + \epsilon_1 = \epsilon_0 / 2 + \epsilon_0 / 2 = \epsilon_0$$

which proves the goal.

Proof discovery

Brown text constitutes comment and is not part of the proof.

[1] Inference rule "arbitrary but fixed" : For proving an universally quantified formula with type declarations and conditions, use arbitrary but fixed constants instead of the quantified variables, assume they satisfy the type declarations and the conditions, and prove the unquantified formula.

For proving (G) we take $f_1, f_2 : \mathbb{R} \longrightarrow \mathbb{R}$ arbitrary but fixed, we assume :

(A_1) $U[f_1]$

(A_2) $U[f_2]$

and we prove :

(G) $U[f_1 + f_2]$

Expanding the assumptions and the goal by definition. When instantiating the definition, check the type declaration and the conditions for the instantiating term.

From (A_1), using $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$, by the definition ($D_1$) we obtain :

(1) $\underset{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}}{\forall} \underset{\substack{\delta \in \mathbb{R} \\ \delta > 0}}{\exists} \underset{\substack{x, y \in \mathbb{R} \\ \mathrm{Abs}[x-y] < \delta}}{\forall} \mathrm{Abs}[f_1[x] - f_1[y]] < \epsilon$

From (A_2), using $f_2 : \mathbb{R} \longrightarrow \mathbb{R}$, by the definition ($D_1$) we obtain :

(2) $\underset{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}}{\forall} \underset{\substack{\delta \in \mathbb{R} \\ \delta > 0}}{\exists} \underset{\substack{x, y \in \mathbb{R} \\ \mathrm{Abs}[x-y] < \delta}}{\forall} \mathrm{Abs}[f_2[x] - f_2[y]] < \epsilon$

Using $f_1, f_2 : \mathbb{R} \longrightarrow \mathbb{R}$, by the definition ($D_2$) we obtain $(f_1 + f_2) : \mathbb{R} \longrightarrow \mathbb{R}$.

Using $(f_1 + f_2) : \mathbb{R} \longrightarrow \mathbb{R}$, by the definition ($D_1$) in order to prove (G) it suffices to prove:

(3) $\underset{\substack{\epsilon \in \mathbb{R} \\ \epsilon > 0}}{\forall} \underset{\substack{\delta \in \mathbb{R} \\ \delta > 0}}{\exists} \underset{\substack{x, y \in \mathbb{R} \\ \mathrm{Abs}[x-y] < \delta}}{\forall} \mathrm{Abs}[(f_1 + f_2)[x] - (f_1 + f_2)[y]] < \epsilon$

[2] Inference rule "rewriting by equality": A subterm which matches the LHS of a universally quantified equality is replaced by the instantiated RHS of the equality, after checking that the instantiat-

ing terms satisfy the type declarations and the conditions of the universal quantifiers. (The rule can be applied similarly for using the equality from right to left.)

Using $f_1,\ f_2 : \mathbb{R} \longrightarrow \mathbb{R}$, $x, y \in \mathbb{R}$, by (D_2), for proving (3) it suffices to prove:

(4) $\underset{\substack{\epsilon\in\mathbb{R}\\ \epsilon>0}}{\forall}\ \underset{\substack{\delta\in\mathbb{R}\\ \delta>0}}{\exists}\ \underset{\substack{x,y\in\mathbb{R}\\ \mathrm{Abs}[x-y]<\delta}}{\forall}\ \mathrm{Abs}\left[\left(f_1[x]+f_2[x]\right)-\left(f_1[y]+f_2[y]\right)\right]<\epsilon_0$

[3] Strategy “S-decomposition”: When the assumptions and the goal have identical alternating quantifiers, eliminate quantifiers in parallel step by step, using a block of inference rules for each quantifier[4], [7].

[4] “S-decomposition” block of inferences for the universal quantifier: When the assumptions and the goal are universally quantified, use first the rule “arbitrary but fixed”[1] for the goal, and then the rule “instantiation”[5] for the assumptions.

Inference rule “arbitrary but fixed” [1].

For proving (4) we take $\epsilon_0 \in \mathbb{R}$ arbitrary but fixed, we assume :

(5) $\epsilon_0 > 0$

and we prove :

(6) $\underset{\substack{\delta\in\mathbb{R}\\ \delta>0}}{\exists}\ \underset{\substack{x,y\in\mathbb{R}\\ \mathrm{Abs}[x-y]<\delta}}{\forall}\ \mathrm{Abs}\left[\left(f_1[x]+f_2[x]\right)-\left(f_1[y]+f_2[y]\right)\right]<\epsilon_0$

[5] Inference rule “instantiation”: When an assumption is universally quantified, instantiate it with an appropriate ground term, after checking that the term satisfies the type declaration and the condition. In this case we instantiate (1) and (2).

[6] Method: “meta-variables”: in order to find an appropriate term for the instantiation of an universal assumption[5] (or the witness term for an existential goal[10]), use a new meta-variable instead of the term (can be seen as a name for the unknown term) and continue the proof. Later in the proof find the appropriate term (“solve” the meta-variable), and check that the solution contains only constants which have been already present in the proof at the moment when the meta-variable was introduced. Also check that the value satisfies the type declaration and the condition of the quantifier. In this case we use the meta-variable ϵ_1 and the solution is found at step (20):

$\epsilon_1 = \epsilon_0 / 2$

First we prove $\epsilon_1 \in \mathbb{R}$ and:

(7) $\epsilon_1 > 0$

These follow from $\epsilon_0 \in \mathbb{R}$, (5), and elementary properties of $\mathbb{R}$.

Using $\epsilon_1 \in \mathbb{R}$ and (7), we instantiate (1) and we obtain :

(8) $\underset{\substack{\delta\in\mathbb{R}\\ \delta>0}}{\exists}\ \underset{\substack{x,y\in\mathbb{R}\\ \mathrm{Abs}[x-y]<\delta}}{\forall}\ \mathrm{Abs}[f_1[x]-f_1[y]]<\epsilon_1$

Using $\epsilon_1 \in \mathbb{R}$ and (7), we instantiate (2) and we obtain :

(9) $\underset{\substack{\delta\in\mathbb{R}\\ \delta>0}}{\exists}\ \underset{\substack{x,y\in\mathbb{R}\\ \mathrm{Abs}[x-y]<\delta}}{\forall}\ \mathrm{Abs}[f_2[x]-f_2[y]]<\epsilon_1$

[7] “S-decomposition” block of inferences for the existential quantifier: When the assumptions and the goal are existentially quantified, use first the rule “take such a”[8] for the assumptions, and then

the rule “witness”[10] for the goal.

[8] Inference rule “take such a”: When an assumption is existentially quantified, use new constants instead of the quantified variables, assume they satisfy the type declarations and the conditions, and use (as assumption) the unquantified formula.

By (8) we can take $\delta_1 \in \mathbb{R}$, such that :

(10) $\delta_1 > 0 \wedge \underset{\substack{x,y\in\mathbb{R} \\ \mathrm{Abs}[x-y]<\delta_1}}{\forall} \mathrm{Abs}[f_1[x] - f_1[y]] < \epsilon_1$

By (9) we can take $\delta_2 \in \mathbb{R}$ such that :

(11) $\delta_2 > 0 \wedge \underset{\substack{x,y\in\mathbb{R} \\ \mathrm{Abs}[x-y]<\delta_2}}{\forall} \mathrm{Abs}[f_2[x] - f_2[y]] < \epsilon_1$

[9] Inference rule “split assumed conjunction”: The conjuncts of an assumption can be used later in the proof as individual assumptions. In this case they will be used later as (10.1), (10.2), (11.1), and (11.2).

[10] Inference rule “witness”: When goal is existentially quantified, use an appropriate ground term as witness instead of the existential variable, checking that the term satisfies the type declaration and the condition, and then prove the unquantified formula. In this case the goal is (6).

Method: "meta-variables"[6]. We use the meta-variable δ_0 for the witness term. The solution is found at step (16):

$\delta_0 = \mathrm{Min}[\delta_1, \delta_2]$

First we prove $\delta_0 \in \mathbb{R}$ and:

(12) $\delta_0 > 0$

These follow from $\delta_1, \delta_2 \in \mathbb{R}$, (10.1), (11.1) and elementary properties of $\mathbb{R}$.

Using $\delta_0 \in \mathbb{R}$ and (12), in order to prove (6) it suffices to prove:

(13) $\underset{\substack{x,y\in\mathbb{R} \\ \mathrm{Abs}[x-y]<\delta_0}}{\forall} \mathrm{Abs}\left[(f_1[x] + f_2[x]) - (f_1[y] + f_2[y])\right] < \epsilon_0$

“S-decomposition” block of inferences for the universal quantifier [4].

Inference rule “arbitrary but fixed” [1].

For proving (13) we take $x_0, y_0 \in \mathbb{R}$ arbitrary but fixed, we assume :

(14) $\mathrm{Abs}[x_0 - y_0] < \delta_0$

and we prove :

(15) $\mathrm{Abs}\left[(f_1[x_0] + f_2[x_0]) - (f_1[y_0] + f_2[y_0])\right] < \epsilon_0$

Inference rule “instantiation” [5]: we instantiate (10.2) and (11.2).

[11] Method “equal arguments to arbitrary functions”: the appropriate instantiation term needed as an argument for an arbitrary function occurring in an assumption must equal a term which already occurs in the proof as the argument of this function occurring in the goal (because otherwise it will not be possible for the proof to succeed). In this case we choose x_0, y_0 because $f_1[x_0]$, $f_2[x_0]$, $f_1[y_0]$, $f_1[y_0]$ occur in the goal.

First we prove :

(16) $\text{Abs}[x_0 - y_0] < \delta_1 \wedge \text{Abs}[x_0 - y_0] < \delta_2$

[12] Method “replace equal ground terms by constants”: when the same ground term occurs in several places in the formulae composing the current proof situation, we replace it by a new constant. In this case the current proof situation consists in assumption (17) and goal (19). We replace $\text{Abs}[x_0 - y_0]$ by p and we transform the proof situation into:

(14') $p < \delta_0$

(16') $p < \delta_1 \wedge p < \delta_2$

[13] Method “solve meta-variable”: use for the meta-variable a value which makes a simple proof situation to succeed, then check the appropriate conditions — see [6]. In this case the solution is $\text{Min}[\delta_1, \delta_2]$. Now we go back to the proof step (12) and check that the constants δ_1, δ_2 are already present in the proof at this step, and that the solution satisfies the type declaration ($\delta \in \mathbb{R}$) and the condition ($\delta > 0$) of the main quantifier of goal (6).

In the proof presentation this sub-proof is hidden and we will just write as argument for (16):

This follows from $x_0, y_0, \delta_0, \delta_1, \delta_2 \in \mathbb{R}$, (14) and elementary properties of $\mathbb{R}$.

Inference rule "split assumed conjunction"[10]. Since (16) has been proved, we can use it now as an assumption, and split it into (16.1) and (16.2).

Using $x_0, y_0 \in \mathbb{R}$ and (16.1), from (10.2) we obtain:

(17) $\text{Abs}[f_1[x_0] - f_1[y_0]] < \epsilon_1$

Using $x_0, y_0 \in \mathbb{R}$ and (16.2), from (11.2) we obtain:

(18) $\text{Abs}[f_2[x_0] - f_2[y_0]] < \epsilon_1$

At this moment all quantifiers are eliminated by the “S-decomposition” strategy [3]. The proof situation contains only ground formulae: assumptions (17), (18) and goal (15).

Method “replace equal ground terms by constants” [12]. We replace $f_1[x_0]$, $f_2[x_0]$, $f_1[y_0]$, $f_2[y_0]$ by x_1, x_2, y_1, y_2, respectively. The new proof situation is:

(17') $\text{Abs}[x_1 - y_1] < \epsilon_1$

(18') $\text{Abs}[x_2 - y_2] < \epsilon_1$

(15') $\text{Abs}[(x_1 + x_2) - (y_1 + y_2)] < \epsilon_0$

[14] Method “use constants for terms with known behaviour”: replace certain ground terms occurring in the assumptions by constants, and then use equation solving, substitution, and computation in order to transform the corresponding subterms in the goal. In this case we replace $x_1 - y_1$ and $x_2 - y_2$ by a_1, a_2, respectively.

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In[*]:= Eqs = {a1 == x1 - y1, a2 == x2 - y2};
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In order to update the goal we eliminate y_1, y_2.

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In[*]:= Sols = First[Solve[Eqs, {y1, y2}]]
Out[*]= {y1 -> -a1 + x1, y2 -> -a2 + x2}
```

```
In[*]:= Expand[(x1 + x2) - (y1 + y2) /. Sols]
Out[*]= a1 + a2
```

The new proof situation is:

(17'') $Abs[a_1] < \epsilon_1$

(18'') $Abs[a_2] < \epsilon_1$

(15'') $Abs[a_1 + a_2] < \epsilon_0$

In the proof presentation this process is hidden, we just write the expression in (16'') by substituting back to the original terms:

Using elementary properties of ℝ we transform (15) into:

(19) $Abs[(f_1[x_0] - f_1[y_0]) + (f_2[x_0] - f_2[y_0])] < \epsilon_0$

[15] Method "rewrite goal by inequalities": in order to prove an inequality, replace subterms by bigger subterms.

$Abs[a_1 + a_2] \leq$ (* by Abs + *)

$Abs[a_1] + Abs[a_2] <$ (* by (17), (18) *)

$\epsilon_1 + \epsilon_1$

From the transformations above we have the new assumption:

(20) $Abs[a_1 + a_2] < 2 * \epsilon_1$

Method "replace equal ground terms by constants" [12]. We transform (20) and the current goal (15''):

(20') $c < 2 * \epsilon_1$

(15''') $c < \epsilon_0$

Method "solve meta-variable"[13]. In order to succeed in the proof situation above, we find ϵ_1:

$\epsilon_1 = \epsilon_0 / 2$

Now we go back to step (7) and check that the constant e_0 occurring in the solving term is already present in the proof, and that the type declaration and the conditions of the universal quantifiers in (1) and (2) are satisfied.

Now the proof is finished because the main goal was proven (by an appropriate choice of a solution for the meta-variable).

In the proof presentation this process is hidden, instead the following argument is given, which is produced by substituting in the expressions above the original ground terms:

Using elementary properties of ℝ, from (17) and (18) we obtain:

$Abs[(f_1[x_0] - f_1[y_0]) + (f_2[x_0] - f_2[y_0])] \leq$ (* by Abs + *)

$\leq Abs[f_1[x_0] - f_1[y_0]] + Abs[f_2[x_0] - f_2[y_0]] <$ (* by (17), (18) *)

$< \epsilon_1 + \epsilon_1 = \epsilon_0 / 2 + \epsilon_0 / 2 = \epsilon_0$

which proves the goal.