

# First Order Logic

## *Exercises*

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**Example 1** (Semantics) Let

$$F_1 : \iff \forall_x \forall_y x \leq y$$

$$I : \begin{cases} D = \{0, 1\} \\ \leq_I \rightarrow \leq_{\mathbb{Z}} \end{cases}$$

**Solution.**  $\langle F_1 \rangle_I = \mathbb{T}$  iff for each  $d \in D$ :

$$\left\langle \forall_y x \leq y \right\rangle_{\sigma \uplus \{x \rightarrow d\}}^I = \mathbb{T}$$

- **Case  $x \rightarrow 0$ .** We have

$$\left\langle \forall_y x \leq y \right\rangle_{\{x \rightarrow 0\}}^I = \mathbb{T} \text{ iff for each } d \in D : \langle x \leq y \rangle_{\{x \rightarrow 0\} \uplus \{y \rightarrow d\}}^I = \mathbb{T}$$

- **Case  $y \rightarrow 0$ .** We have

$$\begin{aligned} & \langle x \leq y \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I \\ \rightsquigarrow & \leq_{\mathbb{Z}} \left[ \langle x \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I, \langle y \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I \right] \\ \rightsquigarrow & \leq_{\mathbb{Z}} [0, 0] \\ \rightsquigarrow & \mathbb{T} \end{aligned}$$

- **Case  $y \rightarrow 1$ .** We have

$$\begin{aligned} & \langle x \leq y \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I \\ \rightsquigarrow & \leq_{\mathbb{Z}} \left[ \langle x \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I, \langle y \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I \right] \\ \rightsquigarrow & \leq_{\mathbb{Z}} [0, 1] \\ \rightsquigarrow & \mathbb{T} \end{aligned}$$

- **Case  $x \rightarrow 1$ .** We have

$$\left\langle \forall_y x \leq y \right\rangle_{\{x \rightarrow 1\}}^I = \mathbb{T} \text{ iff for each } d \in D : \langle x \leq y \rangle_{\{x \rightarrow 1\} \uplus \{y \rightarrow d\}}^I = \mathbb{T}$$

– Case  $y \rightarrow 0$ . We have

$$\begin{aligned}
& \langle x \leq y \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I \\
\rightsquigarrow & \leq_{\mathbb{Z}} \left[ \langle x \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I, \langle y \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I \right] \\
\rightsquigarrow & \leq_{\mathbb{Z}} [1, 0] \\
\rightsquigarrow & \mathbb{F}
\end{aligned}$$

Hence  $\langle F_1 \rangle_I = \mathbb{F}$ . ◀

**Example 2 (Semantics)** Let

$$\begin{aligned}
F_2 & : \iff \forall x \exists y x + y > c \\
I & : \begin{cases} D = \{0, 1\} \\ c_I = 0 \\ +_I \rightarrow +_{\mathbb{Z}} \\ >_I \rightarrow >_{\mathbb{Z}} \end{cases}
\end{aligned}$$

**Solution.**  $\langle F_2 \rangle_I = \mathbb{T}$  iff for each  $d \in D$ :

$$\left\langle \exists_y x + y > c \right\rangle_{\sigma \Psi \{x \rightarrow d\}}^I = \mathbb{T}$$

• Case  $x \rightarrow 0$ . We have

$$\left\langle \exists_y x + y > c \right\rangle_{\{x \rightarrow 0\}}^I = \mathbb{T} \text{ iff for some } d \in D : \langle x + y > c \rangle_{\{x \rightarrow 0\} \uplus \{y \rightarrow d\}}^I = \mathbb{T}$$

– Case  $y \rightarrow 0$ . We have

$$\begin{aligned}
& \langle x + y > c \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I \\
\rightsquigarrow & >_{\mathbb{Z}} \left[ \langle x + y \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I, \langle c \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I \right] \\
\rightsquigarrow & >_{\mathbb{Z}} \left[ +_{\mathbb{Z}} \left[ \langle x \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I, \langle y \rangle_{\{x \rightarrow 0, y \rightarrow 0\}}^I \right], 0 \right] \\
\rightsquigarrow & >_{\mathbb{Z}} [+_{\mathbb{Z}} [0, 0], 0] \\
\rightsquigarrow & >_{\mathbb{Z}} [0, 0] \\
\rightsquigarrow & \mathbb{F}
\end{aligned}$$

– Case  $y \rightarrow 1$ . We have

$$\begin{aligned}
& \langle x + y > c \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I \\
\rightsquigarrow & >_{\mathbb{Z}} \left[ \langle x + y \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I, \langle c \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I \right] \\
\rightsquigarrow & >_{\mathbb{Z}} \left[ +_{\mathbb{Z}} \left[ \langle x \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I, \langle y \rangle_{\{x \rightarrow 0, y \rightarrow 1\}}^I \right], 0 \right] \\
\rightsquigarrow & >_{\mathbb{Z}} [+_{\mathbb{Z}} [0, 1], 0] \\
\rightsquigarrow & >_{\mathbb{Z}} [1, 0] \\
\rightsquigarrow & \mathbb{T}
\end{aligned}$$

- **Case  $x \rightarrow 1$ .** We have

$$\left\langle \exists_y x + y > c \right\rangle_{\{x \rightarrow 1\}}^I = \mathbb{T} \text{ iff for some } d \in D : \langle x + y > c \rangle_{\{x \rightarrow 1\} \uplus \{y \rightarrow d\}}^I = \mathbb{T}$$

- **Case  $y \rightarrow 0$ .** We have

$$\begin{aligned} & \langle x + y > c \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I \\ \rightsquigarrow & >_{\mathbb{Z}} \left[ \langle x + y \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I, \langle c \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I \right] \\ \rightsquigarrow & >_{\mathbb{Z}} \left[ +_{\mathbb{Z}} \left[ \langle x \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I, \langle y \rangle_{\{x \rightarrow 1, y \rightarrow 0\}}^I \right], 0 \right] \\ \rightsquigarrow & >_{\mathbb{Z}} [+_{\mathbb{Z}} [1, 0], 0] \\ \rightsquigarrow & >_{\mathbb{Z}} [1, 0] \\ \rightsquigarrow & \mathbb{T} \end{aligned}$$

Hence  $\langle F_2 \rangle_I = \mathbb{T}$ . ◀

**Example 3 (Semantics)** Let

$$F_3 : \iff \forall_x (P[x] \implies Q[f[x], a])$$

$$I : \begin{cases} D = \{1, 2\} \\ a_I = 1 \\ f_I : D \rightarrow D \\ P_I : D \rightarrow \{\mathbb{T}, \mathbb{F}\} \\ Q_I : D^2 \rightarrow \{\mathbb{T}, \mathbb{F}\} \end{cases} \begin{cases} \begin{cases} f_I[1] = 1 \\ f_I[2] = 1 \end{cases} \\ \begin{cases} P_I[1] = \mathbb{T} \\ P_I[2] = \mathbb{F} \end{cases} \\ \begin{cases} Q_I[1, 1] = \mathbb{T} & Q_I[1, 2] = \mathbb{F} \\ Q_I[2, 1] = \mathbb{F} & Q_I[2, 2] = \mathbb{T} \end{cases} \end{cases}$$

**Solution.**  $\langle F_3 \rangle_I = \mathbb{T}$  iff for each  $d \in D$ :

$$\langle P[x] \implies Q[f[x], a] \rangle_{\sigma \uplus \{x \rightarrow d\}}^I = \mathbb{T}$$

- **Case  $x \rightarrow 1$ .** We have

$$\begin{aligned} & \langle P[x] \implies Q[f[x], a] \rangle_{\{x \rightarrow 1\}}^I \\ \rightsquigarrow & \mathcal{B} \implies \left[ \langle P[x] \rangle_{\{x \rightarrow 1\}}^I, \langle Q[f[x], a] \rangle_{\{x \rightarrow 1\}}^I \right] \\ \rightsquigarrow & \mathcal{B} \implies \left[ P_I[\langle [x] \rangle_{\{x \rightarrow 1\}}^I], Q_I[f_I[\langle [x] \rangle_{\{x \rightarrow 1\}}^I], \langle a \rangle_{\{x \rightarrow 1\}}^I] \right] \\ \rightsquigarrow & \mathcal{B} \implies [P_I[1], Q_I[f_I[1], 1]] \\ \rightsquigarrow & \mathcal{B} \implies [P_I[1], Q_I[1, 1]] \\ \rightsquigarrow & \mathcal{B} \implies [\mathbb{T}, \mathbb{T}] \\ \rightsquigarrow & \mathbb{T} \end{aligned}$$

- Case  $x \rightarrow 2$ . We have

$$\begin{aligned}
& \langle P[x] \Rightarrow Q[f[x], a] \rangle_{\{x \rightarrow 2\}}^I \\
\rightsquigarrow & \mathcal{B} \Rightarrow \left[ \langle P[x] \rangle_{\{x \rightarrow 2\}}^I, \langle Q[f[x], a] \rangle_{\{x \rightarrow 2\}}^I \right] \\
\rightsquigarrow & \mathcal{B} \Rightarrow \left[ P_I[\langle [x] \rangle_{\{x \rightarrow 2\}}^I], Q_I[f_I[\langle x \rangle_{\{x \rightarrow 2\}}^I], \langle a \rangle_{\{x \rightarrow 2\}}^I] \right] \\
\rightsquigarrow & \mathcal{B} \Rightarrow [P_I[2], Q_I[f_I[2], 1]] \\
\rightsquigarrow & \mathcal{B} \Rightarrow [P_I[2], Q_I[1, 1]] \\
\rightsquigarrow & \mathcal{B} \Rightarrow [\mathbb{F}, \mathbb{T}] \\
\rightsquigarrow & \mathbb{T}
\end{aligned}$$

Hence  $\langle F_1 \rangle_I = \mathbb{T}$ . ◀

**Example 4** ((Un)Satisfiability & (In)Validity) Prove that the formula

$$\forall_x P[x] \wedge \exists_y \neg P[y]$$

is inconsistent by definition.

**Solution.** We have to prove that the formula is  $\mathbb{F}$  for all interpretations  $I$ . Let  $I = (D_I, P_I)$  be an arbitrary interpretation. Then

$$\begin{aligned}
& \left\langle \forall_x P[x] \wedge \exists_y \neg P[y] \right\rangle_I = \mathbb{F} \\
\rightsquigarrow & \mathcal{B}_\wedge \left[ \left\langle \forall_x P[x] \right\rangle_I, \left\langle \exists_y \neg P[y] \right\rangle_I \right] = \mathbb{F}
\end{aligned}$$

- Case  $\left\langle \forall_x P[x] \right\rangle_I = \mathbb{T}$  and  $\left\langle \exists_y \neg P[y] \right\rangle_I = \mathbb{F}$ . We have
  - For each  $x \in D_I$ ,  $\langle P[x] \rangle_I = \mathbb{T}$  and
  - For some  $y \in D_I$ ,  $\langle \neg P[y] \rangle_I = \mathbb{F}$  and further for some  $y \in D_I$ ,  $\mathcal{B}_\neg[\langle P[y] \rangle_I] = \mathbb{F}$  We obtain that for some  $y \in D_I$ ,  $\langle P[y] \rangle_I = \mathbb{T}$
- Case  $\left\langle \forall_x P[x] \right\rangle_I = \mathbb{F}$  and  $\left\langle \exists_y \neg P[y] \right\rangle_I = \mathbb{T}$ . Similarly. ◀

**Example 5** (CNF) Prove the following by bringing the formulae into CNF

$$\left( \forall_x P[x] \right) \Rightarrow Q \equiv \exists_x (P[x] \Rightarrow Q).$$

**Solution.** We have

$$\left( \forall_x P[x] \right) \Rightarrow Q \equiv \neg \left( \forall_x P[x] \right) \vee Q \equiv \left( \exists_x \neg P[x] \right) \vee Q \equiv \exists_x (\neg P[x] \vee Q)$$

Further we have

$$\exists_x (P[x] \Rightarrow Q) \equiv \exists_x (\neg P[x] \vee Q)$$
◀

**Example 6** (Skolem Standard Form) Bring the following formula into Skolem Standard Form

$$\forall_{x,y,z} \exists ((\neg P[x,y] \wedge Q[x,z]) \vee R[x,y,z])$$

**Solution.**

$$\begin{aligned} & \forall_{x,y,z} \exists ((\neg P[x,y] \wedge Q[x,z]) \vee R[x,y,z]) \\ \iff & \forall_{x,y,z} ((\neg P[x,y] \vee R[x,y,z]) \wedge (Q[x,z] \vee R[x,y,z])) \\ \rightsquigarrow & \forall_x ((\neg P[x, f[x]] \vee R[x, f[x], g[x]]) \wedge (Q(x, g[x]) \vee R[x, f[x], g[x]])) \end{aligned}$$

**Example 7** (Skolem Standard Form) Bring the following formula into Skolem Standard Form

$$\forall_{x,y} \left( \exists_z P[x,z] \wedge P[y,z] \right) \Rightarrow \exists_u Q[x,y,u]$$

**Solution.**

$$\begin{aligned} & \forall_{x,y} \left( \exists_z P[x,z] \wedge P[y,z] \right) \Rightarrow \exists_u Q[x,y,u] \\ \iff & \forall_{x,y} \neg \left( \exists_z P[x,z] \wedge P[y,z] \right) \vee \exists_u Q[x,y,u] \\ \iff & \forall_{x,y} \left( \forall_z \neg P[x,z] \vee \neg P[y,z] \right) \vee \exists_u Q[x,y,u] \\ \iff & \forall_{x,y,z} (\neg P[x,z] \vee \neg P[y,z]) \vee \exists_u Q[x,y,u] \\ \iff & \forall_{x,y,z} \exists_u (\neg P[x,z] \vee \neg P[y,z]) \vee Q[x,y,u] \\ \rightsquigarrow & \forall_{x,y,z} \neg P[x,z] \vee \neg P[y,z] \vee Q[x,y, f[x,y,z]] \end{aligned}$$

**Example 8** Prove that the formula

$$\forall_x P[x] \Rightarrow \exists_y P[y]$$

is valid by equivalent transformations.

**Solution.** We assume that the formula is invalid and derive a contradiction. Hence, it exists an interpretation  $I$  under which the formula is false. That is

$$\begin{aligned} & \left\langle \forall_x P[x] \right\rangle^I = \mathbb{T} \\ \wedge & \\ & \left\langle \exists_y P[y] \right\rangle^I = \mathbb{F} \rightsquigarrow \left\langle \forall_y \neg P[y] \right\rangle^I = \mathbb{T} \end{aligned}$$

From the above we obtain that

$$\left\langle \forall_x P[x] \right\rangle^I = \left\langle \forall_y \neg P[y] \right\rangle^I \text{ which is a contradiction}$$

**Example 9** ((Un)Satisfiability & (In)Validity) Prove that the formula

$$\forall_x P[x] \wedge \exists_y \neg P[y]$$

is inconsistent by equivalent transformations.

**Solution.** We have

$$\forall_x P[x] \wedge \exists_y \neg P[y] \equiv \forall_x P[x] \wedge \neg \left( \forall_y P[y] \right) \equiv \forall_x P[x] \wedge \neg \left( \forall_x P[x] \right) \equiv \mathbb{F}$$

**Example 10** (Clausification) Transform the formulae  $F_1, F_2, F_3, F_4$ , and  $\neg G$  into a set of clauses, where

$$F_1 : \forall_{x,y} \exists_z P[x, y, z]$$

$$F_2 : \begin{aligned} & \forall_{x,y,z,u,v,w} (P[x, y, u] \wedge P[y, z, v] \wedge P[u, z, w] \Rightarrow P[x, v, w]) \\ & \wedge \\ & \forall_{x,y,z,u,v,w} (P[x, y, u] \wedge P[y, z, v] \wedge P[x, v, w] \Rightarrow P[u, z, w]) \end{aligned}$$

$$F_3 : \forall_x P[x, e, x] \wedge \forall_x P[e, x, x]$$

$$F_4 : \forall_x P[x, i[x], e] \wedge \forall_x P[i[x], x, e]$$

$$G : \left( \forall_x P[x, x, e] \right) \Rightarrow \left( \forall_{u,v,w} (P[u, v, w] \Rightarrow P[v, u, w]) \right)$$

**Solution.**  $F_1, F_2, F_3, F_4$  can almost immediately transformed into clauses. We have

$$\begin{aligned} & P[x, y, f[x, y]] \\ & \neg P[x, y, u] \vee \neg P[y, z, v] \vee \neg P[u, z, w] \vee P[x, v, w] \\ & \neg P[x, y, u] \vee \neg P[y, z, v] \vee \neg P[x, v, w] \vee P[u, z, w] \\ & P[x, e, x] \\ & P[e, x, x] \\ & P[x, i[x], e] \\ & P[i[x], x, e] \end{aligned}$$

We transform  $\neg G$  into standard form

$$\begin{aligned} & \neg \left( \left( \forall_x P[x, x, e] \right) \Rightarrow \left( \forall_{u,v,w} (P[u, v, w] \Rightarrow P[v, u, w]) \right) \right) \\ \iff & \neg \left( \neg \left( \forall_x P[x, x, e] \right) \vee \left( \forall_{u,v,w} (\neg P[u, v, w] \vee P[v, u, w]) \right) \right) \\ \iff & \left( \forall_x P[x, x, e] \right) \wedge \left( \exists_{u,v,w} (P[u, v, w] \wedge \neg P[v, u, w]) \right) \\ \rightsquigarrow & \forall_x P[x, x, e] \wedge P[a, b, c] \wedge \neg P[b, a, c] \end{aligned}$$

which gives the following clauses

$$\begin{aligned} & P[x, x, e] \\ & P[a, b, c] \\ & \neg P[b, a, c] \end{aligned}$$