Introduction to Sequent Calculus

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Abstract

This is a tutorial introduction to sequent calculus, starting from a certain natural style of proving, and based on an example. We present this as a an exercise in constructing the abstract model "sequent calculus" of the of the real world method of proving logical statements in "natural style".

1 Introduction

The purpose of mathematical logic is to study the real–world[1] activity of proving mathematical statements and to construct and analyse abstract models of it.

In this presentation we analyse a concrete example of proof developed in the "natural style" which is used by mathematicians, we identify the essential aspects of the proof method, and we introduce sequent calculus as the mathematical model which can be constructed as an abstraction of it.

2 Proving in Natural Style

2.1 Proof Example

Let us consider the propositional formula:
(G0) $((A \Rightarrow C) \vee (B \Rightarrow C)) \Rightarrow ((A \wedge B) \Rightarrow C)$.

The proof this formula, by the intuitive method which is "built in" our brain, as the result of our mathematical training, may look as follows:

> [1] For proving (G0), we assume:
>
> (A1) $(A \Rightarrow C) \vee (B \Rightarrow C)$
>
> and we prove:
>
> (G1) $(A \wedge B) \Rightarrow C$.

[1] Mathematics operates on abstract models, however the activity of mathematicians takes place in the real world, which therefore also includes the proofs they develop.

[2] For proving (G1), we assume:

(A2) $A \wedge B$

and we prove:

(G2) C.

[3] From (A2) we obtain:

(A2.1) A

and

(A2.2) B.

[4] We prove (G2) by case distinction, using the disjunction (A1).

Case 1:

(A3) $A \Rightarrow C$.

[5] From (A2.1), by (A3), using modus ponens we obtain the goal (G2).

Case 2:

(A4) $B \Rightarrow C$.

[6] From (A2.2), by (A4), using modus ponens we obtain the goal (G2).

This style of proof is usually called *natural style*, because it is typically used by humans.

This example and the subsequent proof may look unusual and somewhat artificial, because in mathematical texts we do not really encounter theorems which look like (G0), neither do we find proofs which have all the steps presented in detail like in the proof above. However, the example captures the essence of the method, and it is simple enough to be presented and analysed in detail in a relatively small space. In mathematical texts the steps of the proofs are not different from the ones above, only the *presentation* of the proof is different, usually abbreviated by omitting the steps which are considered obvious for the intented audience. However for the purpose of constructing a mathematical model of this process, we need to exhibit every single detail.

Analysing of this proof we can identify the essential elements of the proof method: *proof situation, proof step, proof tree*, and *inference rule.*

2.2 Proof Situations

First, we notice that at each moment in the proof there is a certain "goal" which has to be proven, and a certain set of "assumptions" which can be used in the proof (forming a kind of "knowledge base" of propositions assumed to be true at that moment). For instance, at the very beginning of the proof, the goal is (G0), and there are no assumptions. After the first proof step [1], the goal is (G1) and there is only one assumption (A1). After the second proof step [2], the goal is (G2), and the assumptions are (A1) and (A2). Thus, at each moment in the course of the proof, one has a *proof situation*, composed from a set of assumptions and a goal, which may be represented as:

$$\begin{array}{|c|}\hline \text{goal} \\ \hline \text{assumption 1} \\ \text{assumption 2} \\ \dots \\ \hline \end{array}$$

For instance, the initial proof situation in the proof example is represented by:

$$\text{(a)} \begin{array}{|c|}\hline ((A \Rightarrow C) \vee (B \Rightarrow C)) \Rightarrow ((A \wedge B) \Rightarrow C) \\ \hline \\ \hline \end{array}$$

The proof situation in after step [1] is:

$$\text{(b)} \begin{array}{|c|}\hline (A \wedge B) \Rightarrow C \\ \hline (A \Rightarrow C) \vee (B \Rightarrow C) \\ \hline \end{array}$$

After step [2]:

$$\text{(c)} \begin{array}{|c|}\hline C \\ \hline A \wedge B \\ (A \Rightarrow C) \vee (B \Rightarrow C) \\ \hline \end{array}$$

After step [3]:

$$\text{(d)} \begin{array}{|c|}\hline C \\ \hline A \\ B \\ (A \Rightarrow C) \vee (B \Rightarrow C) \\ \hline \end{array}$$

After step [4] we have two proof situations:

$$\text{(e)} \begin{array}{|c|}\hline C \\ \hline A \\ B \\ A \Rightarrow C \\ \hline \end{array} \qquad \text{(f)} \begin{array}{|c|}\hline C \\ \hline A \\ B \\ B \Rightarrow C \\ \hline \end{array}$$

These two are modified by steps [5] and [6], respectively, into:

$$\text{(g)} \begin{array}{|c|}\hline C \\ \hline A \\ B \\ C \\ \hline \end{array} \qquad \text{(h)} \begin{array}{|c|}\hline C \\ \hline A \\ B \\ C \\ \hline \end{array}$$

The proof situations above have the property that the goal is among the assumptions, thus the proof stops successfully. We can call these proof situations *final*. There may be other kind of proof situations which we consider successfull, in particular the one where there are two contradictory assumptions. As we know from our experience, it may also happen that the proof stops because there are no possible ways to continue the proof, and in this case the proof does not succeed[2]. These are also final proof situations.

We can see that a proof may have several branches, and the whole proof succeeds if all the branches finish with successful final proof situations.

What we can also notice in the schematic representation of the proof situation is the removal of some assumptions during the proof, at least in the sense that they are replaced by other assumptions. From the methodological point of view[3], it is good to notice that by trying to represent in a more exact way the process of proving, we identify certain operations which are not clearly described in the text. This exhibits the power of the systematic scientific analysis, and the necessity of it. Indeed, in human produced proofs one never mentions explicitely the operations of removal or replacement of some assumptions. It simply happens that some assumptions are never used again in the proof, like in our proof the assumption $(A \Rightarrow C) \vee (B \Rightarrow C)$ is not used after step [4] and the assumptions $A \Rightarrow C$ and $B \Rightarrow C$ are not used after steps [5] and [6]. In our schematic representation these assumptions are removed after using them, and in this particular proof they are not necessary anymore, but is this a general rule which can allways be applied? Does it happen in every proof in similar situations that these kind of assumptions are not necessary anymore? The answer is "yes", however this is not obvious in this moment, and here we already have a glimpse of the usefulness of constructing the abstract model of the process and studying it mathematically: by this investigation we will be able, among other, to provide an answer to these questions[4].

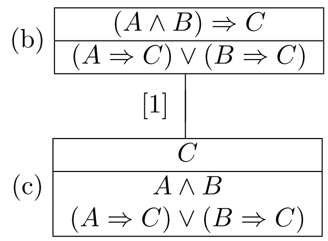
2.3 Proof Steps and Proof Tree

The proof proceeds in individual *proof steps*, each step consisting in creating one or more proof situations from the current proof situation. The new proof situation[s] may differ with respect to the goal, the set of assumptions, or both. For instance, step [2] creates one proof situation and modifies both the goal and the set of assumptions:

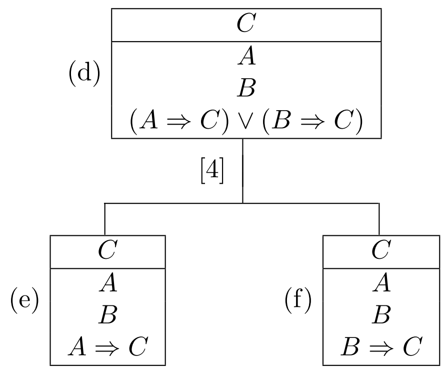
[2] We do not address here the situations when *we do not know* how to continue the proof.

[3] With respect to the methodology of analysing the proofs performed by humans in order to create an abstract model of the process.

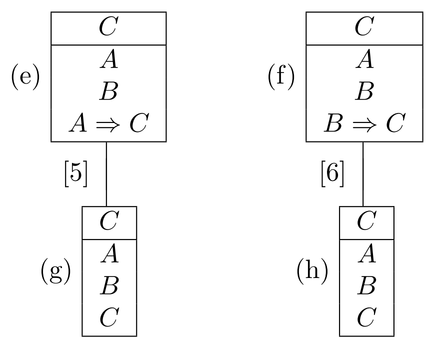
[4] The question about the removal of assumptions during proofs may appear insignificant for the proving practician, why should we care about them since we just do not use them anymore? Even from the theoretical point of view of classical logic the problem is not important, because classical logic is concerned with finding methods which make the proof *possible*, and not with the efficiency of these methods. However, when having in mind *automated reasoning*, where the efficiency of the methods is essential, it is quite important to remove unnecessary assumptions, because the mechanical prover has to take into account all the assumptions at every step for making a decision about the next operation to perform, and in fact the resource consumption depends exponentially on the number of assumptions.



Step [4] creates 2 proof situations and modifies only the set of assumptions:



Steps [5] and [6] are similar, each uses two assumptions and simplifies an assumption:



If we combine pictorially all proof steps, then we obtain the *proof tree* presented in Fig. 1.

Formally, a proof tree is a graph having as nodes the proof situations and as arcs the proof steps. The root is the initial proof situation and the leaves are the final proof situations.

2.4 Inference Rules

Each step of the proof follows a certain rule. For instance, steps [1] and [2] follow the rule:

(R1) In order to prove that a statement γ_1 implies a statement γ_2

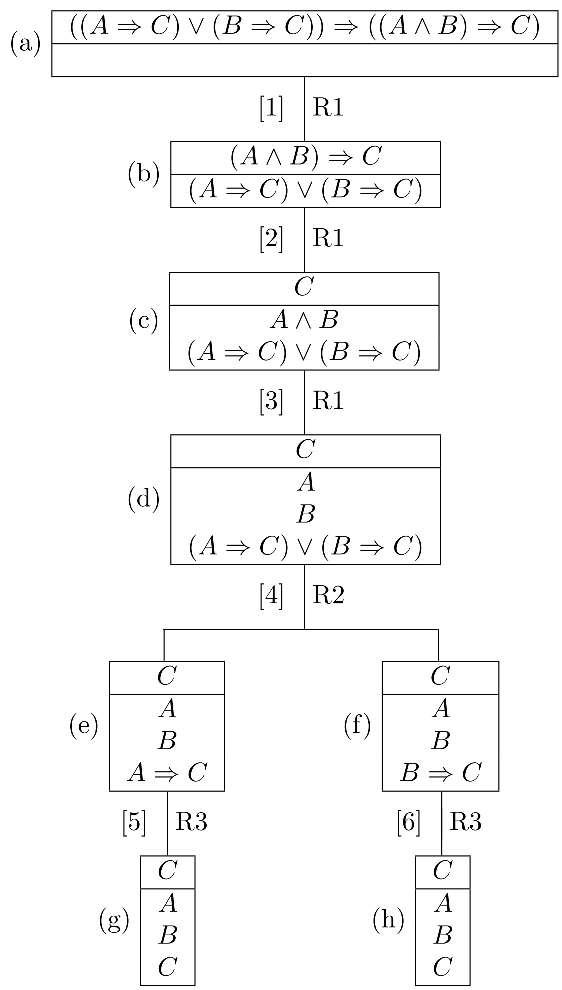


Figure 1: Proof tree.

assume γ_1 and prove γ_2.

In order to create a formal model of the inference rules, we need to represent proof situations in an abstract way, by emphasizing the elements which are important for the application of the inference rules.

For instance, a proof situation which is appropriate for the application of rule (R1) can be represented as:

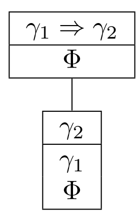
$$\begin{array}{|c|}\hline \gamma_1 \Rightarrow \gamma_2 \\ \hline \Phi \\ \hline \end{array}$$

where γ_1, γ_2 are individual formulae composing the goal and Φ is a (possibly empty) set of formulae — the assumptions. Likewise, the proof situation created by rule (R1) from this schematic proof situation may be represented as:

$$\begin{array}{|c|}\hline \gamma_2 \\ \hline \gamma_1 \\ \Phi \\ \hline \end{array}$$

which suggests that the goal now is the right hand side of the implication which was the goal of the previous proof situation, while the new set of assumptions consists from the old set of assumptions to which we add the left hand side of the previous goal.

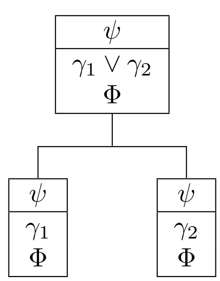
Thus inference rule (R1) may be represented as:



Step [4] follows the rule:

> (R2) If an assumption is a disjunction, create a proof situation for every disjunct, adding it to the current assumptions, and keep the same goal.

By the same principle, this inference rule may be represented as:



Steps [5] and [6] follow the rule:

(R3) If an assumption is an implication and the LHS[5] of it is also an assumption, then replace the implication by the RHS[6] of it.

We may represent this inference rule as:

$$\begin{array}{|c|}\hline \psi \\ \hline \gamma_1 \Rightarrow \gamma_2 \\ \gamma_1 \\ \Phi \\ \hline \end{array}$$

$$\Big|$$

$$\begin{array}{|c|}\hline \psi \\ \hline \gamma_2 \\ \gamma_1 \\ \Phi \\ \hline \end{array}$$

By inspecting the schematic proof tree of the proof example, we can see that the proof steps are *instances* of the inference rules.

Note also that the successfull proof situations where the goal is obtained as one of the assumptions can be represented as:

$$\begin{array}{|c|}\hline \gamma \\ \hline \gamma \\ \Phi \\ \hline \end{array}$$

and indeed the final nodes (the leaves) of the proof tree are instances of this schematic proof situation.

In some proofs we may have a successfull proof situation by "contradictory assumptions" (both a formula and its negation are in the set of assumptions), this can be represented as:

$$\begin{array}{|c|}\hline \psi \\ \hline \gamma \\ \neg\gamma \\ \Phi \\ \hline \end{array}$$

3 Sequent Calculus

3.1 Basic Elements

Now we construct the formal model for the elements identified above.

The idea of *proof situation* is modeled by the notion of *sequent.* A sequent is a pair $\langle \Phi, \Psi \rangle$, where Φ and Ψ are sets of formulae. The traditional notation used in mathematical logic for concrete sequents is:

$$\varphi_1, \ldots, \varphi_n \;\vdash\; \psi_1, \ldots, \psi_m,$$

[5] Left hand side.
[6] Right hand side.

where $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m$ are individual formulae. (When one of the sets is empty, then nothing is written on the corresponding side of $\vdash$.) We call the formulae on the left hand side *assumptions* and the ones on the right hand side *goals.* (Other names used in presentations are *premises/conclusions* and *antecedents/postcedents.*)

In this presentation[7], we define the semantics of sequents by considering that a sequent as above is the abbreviation of the formula

$$(\varphi_1 \wedge \ldots \wedge \varphi_n) \ \Rightarrow \ (\psi_1 \vee \ldots \vee \psi_m).$$

More abstractly, $\Phi \vdash \Psi$ is the abbreviation of $(\wedge\Phi) \Rightarrow (\vee\Psi)$, where $\wedge$ ($\vee$) is the conjunction (disjunction) of all the formulae in the respective sets. This also expresses the situations when one of the sets is empty: the empty set of assumptions (goals) represents the constant $\mathbb{T}(\mathbb{F})$[8].

Note that the abstract model is a little more general than the proof situation, because we may have *several goals*, while in the natural style proving we only have one goal at a time. Also, the model allows to have no goal at all (when the set of goals is empty). By the semantics defined above, this correspons to the proof situations when we are looking for a contradiction (one may also say that the goal is the truth constant $\mathbb{F}$, for instance when one applies the inference rule "proof by contradiction").

In order to represent schematically the inference rules, we represent abstract sequents as

$$\varphi_1, \varphi_2, \ldots, \Phi_1, \Phi_2, \ldots \vdash \psi_1, \psi_2, \ldots, \Psi_1, \Psi_2, \ldots,$$

where $\varphi_1, \varphi_2, \ldots, \psi_1, \psi_2, \ldots$ are either variables (at meta level) representing individual formulae, either logical constructions over such variables and constants, representing formulae which have a certain structure (like e. g. $\gamma_1 \Rightarrow \gamma_2$), while $\Phi_1, \Phi_2, \ldots, \Psi_1, \Psi_2, \ldots$ are sets of formulae. Each list of symbols is an abreviation for the respective set of formulae, thus the sequent represented is:

$$\{\varphi_1, \varphi_2, \ldots\} \cup \Phi_1 \cup \Phi_2 \cup \ldots \ \vdash \ \{\psi_1, \psi_2, \ldots\} \cup \Psi_1 \cup \Psi_2 \cup \ldots$$

In the formal model, the inference rules are represented by *sequent rules.* A sequent rule is a pair $\langle\{S_1, S_2, \ldots\}, S_0\rangle$ between a set of sequents and a sequent and is traditionally represented as:

$$\frac{S_1, S_2, \ldots}{S_0}$$

For the semantics of a sequent rule we adopt $(\wedge\{S_1, S_2, \ldots\}) \Rightarrow S_0$, and note that when the set is empty (in the traditional notation there will be nothing written above the line), then semantic of the sequent rule is $\mathbb{T} \Rightarrow S_0$, thus the validity of the sequent reduces to the validity of S_0 : such sequents are also called *axioms* and they model the successful terminal proof situations.

[7] There are different definitions of the semantics of sequents, we adopt here this particular one because it simplifies reasoning about sequents.

[8] *True* (*False*).

A *calculus* (in the sense of sequent calculus) is a collection of sequent rules, and constitutes a proof method for the validity of sequents.

With this notation, the inference rule (R1) is modeled as the sequent rule:

$$\frac{\Phi, \gamma_1 \ \vdash \ \gamma_2, \Psi}{\Phi \ \vdash \ \gamma_1 \Rightarrow \gamma_2, \Psi} \vdash\Rightarrow$$

and the inference rule (R2) is modeled as the sequent rule:

$$\frac{\Phi, \gamma_1 \ \vdash \ \Psi \qquad \Phi, \gamma_2 \ \vdash \ \Psi}{\Phi, \gamma_1 \vee \gamma_2 \ \vdash \ \Psi} \vee\vdash$$

In contrast to the natural–style inference rules, note the addtitional Ψ in the goals, these are the rest of the goal formulae, which are not changed by the application of the rule.

The successful situation (goal is assumed) in the example proof is modeled by the sequent rule:

$$\frac{}{\Phi, \gamma \ \vdash \ \gamma, \Psi} \text{a}$$

As for inference rules in natural–style proving, the sequent rules can be instantiated in order to realize inferences on sequents. Some inferences corresponding to the proof steps [2], [4], and after step [5] in the example proof, which are instances of the sequent rules "$\vdash\Rightarrow$", "$\vee\vdash$", and "a", respectively, are:

$$\frac{(A \Rightarrow C) \vee (B \Rightarrow C),\ A \wedge B \ \vdash \ C}{(A \Rightarrow C) \vee (B \Rightarrow C) \ \vdash \ (A \wedge B) \Rightarrow C} \vdash\Rightarrow$$

$$\frac{A \Rightarrow C,\ A \wedge B \ \vdash \ C \qquad B \Rightarrow C,\ A \wedge B \ \vdash \ C}{(A \Rightarrow C) \vee (B \Rightarrow C),\ A \wedge B \ \vdash \ C} \vee\vdash$$

$$\frac{}{A,\ B,\ C \ \vdash \ C} \text{a}$$

The natural–style proof trees are represented in the model by sequent proof trees, which are graphs of sequents, and whose arcs correspond to instances of sequent rules, as for example in:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{S_7} \quad \overline{S_9}}{S_5}}{S_4} \qquad \dfrac{\dfrac{\overline{S_8}}{S_6} \quad \overline{S_2}}{S_3}}{S_1}}{S_0}$$

In this example, the sequents S_2, S_7, S_8, and S_9 are axioms, and the tree constitutes a sequent calculus proof of the validity of the root node S_0.

A sequent proof for our example is:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, B, C \ \vdash \ C}\,\text{a}}{A, B, A \Rightarrow C \ \vdash \ C}\,\text{MP} \qquad \dfrac{\dfrac{}{A, B, C \ \vdash \ C}\,\text{a}}{A, B, B \Rightarrow C \ \vdash \ C}\,\text{MP}}{A, B, (A \Rightarrow C) \vee (B \Rightarrow C) \ \vdash \ C}\,\vee\vdash}{A \wedge B, (A \Rightarrow C) \vee (B \Rightarrow C) \ \vdash \ C}\,\wedge\vdash}{(A \Rightarrow C) \vee (B \Rightarrow C) \ \vdash \ (A \wedge B) \Rightarrow C}\,\vdash\Rightarrow}{\vdash \ ((A \Rightarrow C) \vee (B \Rightarrow C)) \Rightarrow ((A \wedge B) \Rightarrow C)}\,\vdash\Rightarrow
$$

Suggested exercise: Prove in natural style:

$$((A \wedge B) \Rightarrow C) \Rightarrow ((A \Rightarrow C) \vee (B \Rightarrow C)) \tag{1}$$

$$((A \Rightarrow C) \wedge (B \Rightarrow C)) \Leftrightarrow ((A \vee B) \Rightarrow C) \tag{2}$$

3.2 A Minimal Calculus: *Not–And*

We will construct now a simple calculus which operates on formulae containing only the logical connectives $\neg$ and $\wedge$ and we will call it "the *Not–And* calculus". The reason for starting with a small number of logical connectives (which are however able to express all the other) is that it will be easier to study the properties of this calculus, and then we will be able to extend it such that it operates on full propositional logic.

From the example presented above we can extract a rule for the axioms: a formula must be present both in the assumptions and in the goals.

Moreover we also had a conjunction in the assumptions: this was split into its conjuncts.

If we look at proofs where we have to prove a conjuction of formulae, we see that usually we prove each conjunct on a separate branch, so we can use this as a rule.

What about negation? In some proofs we may have experienced that when the goal is the negation of a formula, we proceed "by contradiction": we assume the positive version and we continue the proof until we obtain a contradiction. This suggests a rule for negative goal.

Assume we are in such a situation, namely we look for a contradiction, and one of our assumptions is negative. Then by proving the positive version of it we will obtain our contradition! This suggests a rule for negative assumption, and so we have the following tentative calculus:

Not–And	Assumptions	Goals
a	$\dfrac{}{\Phi, \gamma \ \vdash \ \gamma, \Psi}\,\text{a}$	
$\neg$	$\dfrac{\Phi \ \vdash \ \gamma, \Psi}{\Phi, \neg\gamma \ \vdash \ \Psi}\,\neg\vdash$	$\dfrac{\Phi, \gamma \ \vdash \ \Psi}{\Phi \ \vdash \ \neg\gamma, \Psi}\,\vdash\neg$
$\wedge$	$\dfrac{\Phi, \gamma_1, \gamma_2 \ \vdash \ \Psi}{\Phi, \gamma_1 \wedge \gamma_2 \ \vdash \ \Psi}\,\wedge\vdash$	$\dfrac{\Phi \ \vdash \ \gamma_1, \Psi \qquad \Phi \ \vdash \ \gamma_2, \Psi}{\Phi \ \vdash \ \gamma_1 \wedge \gamma_2, \Psi}\,\vdash\wedge$

Let us now investigate the properties of this calculus, in order to see if it is really solving the problem of proving in this restricted version of propositional logic.

For every mathematical method one may have two main legitimate questions:

(1) "correctness": if I apply the method, will the answer be correct?

(2) "completeness": can the method solve any problem of the respective type?

In the context of mathematical logic, a proof method is called "correct" (or "consistent") if whenever we receive the answer "proved" for a certain formula, we can be sure that the formula is valid. Dually, the method is called "complete" if whenever a formula is valid, the method will finish and will provide the answer "proved". One can see that together correctness and completeness is an equivalence statement: A formula is valid if and only if the method provides a proof – in our case a proof tree.

3.2.1 Correctness of the *Not–And* Calculus

We need to show that the existence of a proof tree for a sequent implies that the sequent is valid (that is, the disjunction of the goals is a logical consequence of the assumptions.)

Let us recall the semantics of sequent rules. In general a rule has the shape:

$$\frac{S_1, S_2, \ldots}{S_0}$$

and its semantics is $(\wedge\{S_1, S_2, \ldots\}) \Rightarrow S_0$, that is the rule corresponds to a valid formula iff S_0 is a semantical logical consequence of $S_1, S_2, \ldots$. We will call such a sequent rule *correct*. Remember also that axioms are those sequents for which the rule is $\mathbb{T} \Rightarrow S_0$, thus if the rule is correct, then the axiom is also valid. If all sequent rules are correct, then all sequents in any proof tree are logical consequences of the axioms, or actually of $\mathbb{T}$, thus all of them are valid, including the root sequent whose validity we want to prove. (We may say that the validity is transported in the tree from leaves to root.) Therefore, a sufficient condition for a sequent calculus to be correct is that *all sequent rules are correct.* This sufficient condition serves in general to prove consistency of a sequent calculus, and it is quite convenient because it splits the proof of consistency into individual independent proofs of correctness for each inference rule[9].

Let us now proceed in this way, namely prove that the sequent rules of the *Not-And* calculus are correct. For this purpose it is useful to look at the *normal form* of sequents. By equivalent transformations of the sequent

$$(\varphi_1 \wedge \ldots \wedge \varphi_n) \;\Rightarrow\; (\psi_1 \vee \ldots \vee \psi_m)$$

[9]In Mathematics, and in general in solving problems, it is allways convenient to split a problem in several independent smaller problems.

using:

$$\gamma_1 \Rightarrow \gamma_2 \ \equiv \ (\neg\gamma_1) \vee \gamma_2 \ \text{ and } \ \neg(\gamma_1 \wedge \ldots \wedge \gamma_n) \ \equiv \ \neg\gamma_1 \vee \ldots \vee \neg\gamma_n$$

we obtain:

$$\overline{\varphi}_1 \vee \ldots \vee \overline{\varphi}_n \vee \psi_1 \vee \ldots \vee \psi_m$$

(where overbar stands for negation). Thus we can allways think of a sequent as being a big disjunction between the negated assumptions and the positive goals. More abstractly, $\Phi \vdash \Psi$ represents the formula $(\wedge\Phi) \Rightarrow (\vee\Psi)$, which is equivalent to the formula $\vee(\overline{\Phi} \cup \Psi)$ (where $\overline{\Phi}$ is the set of all negated formulae from Φ).

Using this view we can now examine the inference rules of the calculus, which are either of the form $\dfrac{S_1}{S_0}$ or of the form $\dfrac{S_1 \qquad S_2}{S_0}$.

a: *Axiom*

$S_0 : \Phi, \gamma \ \vdash \ \gamma, \Psi \ \equiv \ \vee(\overline{\Phi \cup \{\gamma\}} \cup (\{\gamma\} \cup \Psi)) \ = \ \vee(\overline{\Phi} \cup \{\neg\gamma\} \cup \{\gamma\} \cup \Psi)$

The disjunction contains a pair of opposite formulae, therefore it is equivalent to $\mathbb{T}$.

$\neg\vdash$: *Negation in assumptions.*

$S_0 : \Phi, \neg\gamma \ \vdash \ \Psi \ \equiv \ \vee(\overline{\Phi \cup \{\neg\gamma\}} \cup \Psi) \ = \ \vee((\overline{\Phi} \cup \{\neg\neg\gamma\}) \cup \Psi)$

$S_1 : \Phi \ \vdash \ \gamma, \Psi \ \equiv \ \vee(\overline{\Phi} \cup (\{\gamma\} \cup \Psi))$

Thus not only that S_0 is a logical consequence of S_1, but they are even equivalent!

$\vdash\neg$: *Negation in goals.*

$S_0 : \Phi \ \vdash \ \neg\gamma, \Psi \ \equiv \ \vee(\overline{\Phi} \cup (\{\neg\gamma\} \cup \Psi))$

$S_1 : \Phi, \gamma \ \vdash \ \Psi \ \equiv \ \vee((\overline{\Phi} \cup \{\neg\gamma\}) \cup \Psi)$

Again we have equivalence.

$\wedge\vdash$: *Conjunction in assumptions.* Here it is more convenient to use the original form of the sequents.

$S_0 : \Phi, \gamma_1 \wedge \gamma_2 \ \vdash \ \Psi \ \equiv \ \wedge(\Phi \cup \{\gamma_1 \wedge \gamma_2\}) \Rightarrow \vee\Psi$

$S_1 : \Phi, \gamma_1, \gamma_2 \ \vdash \ \Psi \ \equiv \ \wedge(\Phi \cup \{\gamma_1, \gamma_2\}) \Rightarrow \vee\Psi$

The equivalence results from the associativity of conjunction.

$\vdash\wedge$: *Conjunction in goals.*

$S_0 : \Phi \ \vdash \ \gamma_1 \wedge \gamma_2, \Psi \ \equiv \ \vee(\overline{\Phi} \cup \{\gamma_1 \wedge \gamma_2\} \cup \Psi)$

$S_1 : \Phi \ \vdash \ \gamma_1, \Psi \ \equiv \ \vee(\overline{\Phi} \cup \{\gamma_1\} \cup \Psi)$

$S_2 : \Phi \ \vdash \ \gamma_2, \Psi \ \equiv \ \vee(\overline{\Phi} \cup \{\gamma_2\} \cup \Psi)$

The equivalence $S_0 \equiv S_1 \wedge S_2$ results from the distributivity of disjunction over conjunction: $\varphi \vee (\gamma_1 \wedge \gamma_2) \ \equiv \ (\varphi \vee \gamma_1) \wedge (\varphi \vee \gamma_2)$.

Thus for all inference rules of the calculus we have equivalence between the sequent below the line and the conjuction of the sequents above the line. We will call the rules which have this property *reversible*, because their opposite rules are also correct: $\dfrac{S_1}{S_0}$ can be transformed into the correct rule $\dfrac{S_0}{S_1}$ and $\dfrac{S_1 \qquad S_2}{S_0}$ generates two correct rules: $\dfrac{S_0}{S_1}$ and $\dfrac{S_0}{S_2}$. These "reversed" rules are in fact used in other versions of sequent calculus in classical logic, and they are called "elimination" rules (negation elimination, conjunction elimination), in contrast to the rules of our *Not–And* calculus, which are classically called "introduction" rules (negation introduction, conjunction introduction).

Note that the reversibility of the rules is not really necessary for the correctness of the calculus, where the semantic entailment from sequents above the line to the sequent below is sufficient, however this property will be useful when we address completeness.

3.2.2 Completeness of the *Not–And* Calculus

We want to prove that for any valid sequent there exists a proof tree by the calculus.

In contrast with the proof of correctness, this proof cannot be completely decomposed into some separate independent reasoning steps about the rules, we also need some reasoning about the whole calculus. Some independent reasoning steps about the rules which have been performed above are also useful for the completeness, because from the equivalences inferred before we will use the fact that the sequents above the line are logical consequences of the sequent below, thus in a proof tree all sequents, in particular *the leaves are logical consequences of the root* – **fact 1**.

In order to prove completeness we will also prove **fact 2**: *every sequent has an inference tree with atomic leaves* and **fact 3**: *atomic valid sequents are axioms.*

Fact 2. We want to prove that every valid sequent has a proof tree, however we can see that the calculus allows to construct a tree for any sequent, no matter whether it is valid or not, by just applying the rules in a bottom–up manner[10]. In fact, by inspecting the table of the rules, we can see that *there exist a rule for every logical connective in every position* (assumption or goals) thus if the sequent contains at least one logical connective, then at least one rule will apply. Therefore we can conclude that a *deduction tree*: a tree of sequents which, like in a proof tree, are connected by arcs corresponding to sequent rules, can be constructed for any sequent, even if it is not valid. Furthermore, we can see that *every rule removes the respective logical connective*, thus the inference tree can be extended until there are no more logical connectives in the leaves, we call the resulting sequents *atomic*. From these two aspects which can be detected by inspecting the table of the inference rules, we can conclude that for any sequent there exists at least one inference tree whose leaves are atomic.

[10] This constitutes an algorithmic aspect of this calculus, which will be discussed later

The only difference between such a tree and a proof tree is that in a proof tree the leaves must be axioms. For this we use the fact that the root is valid. Since the rules are reversible, the validity of the root sequent is transported bottom–up in the tree, thus the leaves (which are already atomic) of the inference tree are also valid. We only need to show now that they are axioms, and this is insured by the following fact.

Fact 3. Atomic valid sequents are axioms.

Proof. Let us consider an arbitrary but fixed sequent $\Phi \vdash \Psi$ which valid – that means it is satisfied by any interpretation, and atomic – that means $\Phi, \Psi \subset \mathcal{V}$[11]) Note that we do not speak about $\mathbb{T}$ and $\mathbb{F}$, they are not in the formulae handled by the *Not-And* sequent calculus. We want to prove that $\Phi \vdash \Psi$ is an axiom – that means it matches the first rule in the table of the calculus, which can be also expressed as $\Phi \cap \Psi \neq \emptyset$.

[1] In order to prove this, we assume $\Phi \cap \Psi = \emptyset$ and we try to derive a contradiction.

[2] Since we know that every interpretation satisfies $\Phi \vdash \Psi$, (which is the same as "no interpretation falsifies $\Phi \vdash \Psi$), in order to derive a contradiction we try to prove the opposite, namely that there exists an interpretation that falsifies $\Phi \vdash \Psi$.

[3] For proving this we construct such an interpretation.

[4] Interpretations assign values to the atoms in Φ, Ψ, thus we need to work with those atoms. Intuitively we may continue now by considering $\Phi = \{A_1, A_2, \ldots, A_n\}$, $\Psi = \{B_1, B_2, \ldots, B_m\}$, etc. However each of these two sets could in principle be also empty (we know that their intersection is empty!), a situation which is not properly expressed by the consideration above, so this could be logically incorrect. Therefore we consider cases about emptiness, using first Φ, then Ψ.

[5] Case $\Phi = \emptyset$, $\Psi = \emptyset$. In this case the sequent represents the formula $\wedge\emptyset \Rightarrow \vee\emptyset$, which is $\mathbb{T} \Rightarrow \mathbb{F}$, which is $\mathbb{F}$, therefore any interpretation falsifies the sequent.

[6] Case $\Phi = \emptyset$, $\Psi \neq \emptyset$. In this case the sequent represents the formula $\mathbb{T} \Rightarrow \vee\Psi$, which is equivalent to $\vee\Psi$, thus for falsifying it it suffices to assign $\mathbb{F}$ to every atom in Ψ.

[7] Case $\Phi \neq \emptyset$, $\Psi = \emptyset$. In this case the sequent represents the formula $\wedge\Phi \Rightarrow \mathbb{F}$, which is equivalent to $\neg \wedge \Phi$, thus for falsifying it it suffices to assign $\mathbb{T}$ to every atom in Φ.

[8] Case $\Phi \neq \emptyset$, $\Psi \neq \emptyset$. In this case we assign $\mathbb{T}$ to every atom in Φ and $\mathbb{F}$ to every atom in Ψ to obtain $\mathbb{T} \Rightarrow \mathbb{F}$ under the interpretation. Note that this

[11] This is the set of propositional variables.

is possible only because we have assumed that $\Phi \cap \Psi = \emptyset$ (otherwise there will be at least one atom both in Φ and in Ψ which must be assigned $\mathbb{T}$ and $\mathbb{F}$ in the same time!).

The proof is now finished because it was successfull on all branches. $\square$

Side Remark. As a matter of self reflection, let us look now at this proof from the point of view of our study. Step [1] is the practical application of the inference rule $\vdash \neg$ (negation in assumption), and step [2] (with some forcing) can be considered akin to $\neg \vdash$ (negation in goals). Step [3] is from predicate logic, the so called "witness", we will discuss it later. Step [4] corresponds to proof by cases, only that the disjunction is not actually among the current assumptions, however it allways holds, because is of the form $\gamma \vee \neg\gamma$, which can be constructed from every formula. Introducing such an assumption and using it later for proof by cases is logically correct, but not necessary in propositional logic – because a complete calculus can be constructed without using it, as we will see in the sequel. However, in predicate logic, or even in propositional logic for efficiency, we may use such a rule, which from the point of view of automation is relatively difficult to implement (one has to choose properly the formula for the disjunction, among many!). Steps [5] to [8] are just the branches of the case proof, and each proceeds in a specific way, in predicate logic, beyond the propositional proof rules which we are investigating now, but at least we can see that the proof has a tree structure, like in our model, and that each branch finishes successfully by obtaining the goal: the fact that the interpretation falsifies the formula, that means by our definition of axioms.

Summarizing the three facts which we have proven: we start with a valid sequent as root, an inference tree with atomic leaves can be constructed, but they are also valid, thus we have a proof tree. Therefor we may conclude that for any valid sequent there exists a sequent proof tree by this calculus.

Note also that atomic sequents are in fact clauses, where the atoms in the assumption set are the negative literals, and the ones in the goal set are the positive literals. Therefore, we can conclude that a this sequent calculus transforms a formula in CNF (Conjunctive Normal Form) – no matter whether it is valid or not. For valid formulae every atomic leaf of the tree has a at least a common variable among the assumptions and the goals, which corresponds to the fact that every clause of a valid formula in CNF has at least a pair of opposite literals (the proof of this is basically the same as the one above). It is quite interesting that sequent calculus and CNF reduction perform essentially the same operation, while their starting ideas are completely different: sequent calculus intents to model natural style proving, while CNF reduction intents to transform the formula by equivalence rewriting into a structurally simpler one.

3.2.3 The Algorithmic Aspect

I mentioned that in classical logic the emphasis was on studying proof systems in which the *existence* of the proof was insured, while the process of *finding* the proof was not addressed. The presentation style of the proofs in sequent calculus is typical for this approach: it looks like we start with the axioms and then by inferences we advance step by step until we obtain the goal. However this is quit unpractical, because it is very difficul to invent the appropriate axioms and the intermediary sequents such that we arrive at given goal. In our presentation we use a calculus which has an *algorithmic aspect*: by using the rules bottom–up, we can construct the proof of any valid sequent. This is illustrated by the proof of completeness, which is constructive: it shows how to construct the proof. This is a consequence of the fact that we are modeling a real world *method* of proving. Admittedly, the method we exhibit is nondeterministic, because at certain points in the proof there may be several rules which are applicable – however the completeness proofs insures us that the proof will succeed no matter the intermediate choices, and in fact (as we will see a little later) the final result is allways the same – in the automated reasoning literature this property is called *confluence.*

The proof algorithm given by this calculus is actually useful even when the root sequent is not valid. As we discussed above, the inference tree with atomic leaves can still be constructed, but not all the atomic leaves will be valid. (Indeed, this should be the case, because if they are all valid then the root sequent is also valid, according to the correctness statement!) Thus we have more than a proof method[12] here, but we have a decision procedure: the calculus decides whether a sequent is valid or not by constructing the inference tree and examining the atomic leaves. If all leaves $\Phi \vdash \Psi$ satisfy $\Phi \cap \Psi \neq \emptyset$, then the root sequent is valid, otherwise not. What can we expect from the atomic leaves which are not valid? The answer is in the proof of **fact 3** above: if the atomic sequent is not valid, then one can construct an interpretation which falsifies it, by assigning in a certain way the atoms present in the sequent. This means that from a "failed" proof we can easily obtain a so called "counterexample": a situation in which the statement does not hold. This is very important for practical purposes, because in the real life of a mathematician a problem is alive as long as the related proof does not work. As in programming, he has to "debug" the text of the statmenents involved, and the counterexamples are pointing to the "bugs". For instance, if the propositional formula we want to prove expresses the behaviour of a digital circuit, then the counterexamples are instances of the inputs for which the circuit does not behave correctly. We will look at some examples of failed proofs later, after we introduce the sequent rules for the other propositional connectives.

3.2.4 Reverse Engineering

In the presentation above we introduced the *Not–And* calculus as inspired by possible proof examples, and then we saw that it has certain properties which makes it correct and complete. What about proceeding in reverse: start from

[12]In mathematical logic we are usually happy if we find a method which succeeds to find a proof for any valid formula, deciding whether a formula is valid or not is impossible even for relatively simple logics, like first order predicate logic.

some desired properties and constructing the rules of the calculus?

In the natural–style proof examples we may notice that *removal of propositional connectives* is a basic principle in most of the inference steps. This is actually very natural, because it decomposes the formulae and the proofs (by branching) into simpler ones.

It is also very reasonable to try to transform proof situations into *equivalent* ones, or in equivalent conjunctions (by branching). This ensures correctness: we arrive at an equivalent formula which is obviously true. This also appears to support completeness: if we reduce the current goal to one that only *implies* it (that is: it is stronger), then we cannot be sure that the new one is provable, even if the old one was so.

By taking into account these principles one can replay the proofs of correctness of the rules in a kind of reverse order: take all 4 possible sequents having the negation or conjunction in assumptions or conclusions, and find the equivalent sequent equivalent such that the target logical connective is removed. In this way one easily finds the rules of the calculus. In what concerns the axioms, the proof of **fact 3** above shows that this is the only criterion which can ensure validity in general.

3.3 The Full Calculus

3.3.1 The Other Logical Connectives

In order to find out the rules for the other logica connectives, we can proceed by studying further proof examples, or by reverse engineering as mentioned above, and both methods would work. However, after having already studied the small calculus *Not–And*, the simplest way is to extend it by formal inferences, because this also ensures that the reversibility of the rules is preserved, and also that the target logical connective is removed.

Since each sequent is a formula, replacing a part of it by an equivalent one produces an equivalent sequent, thus we can allways replace an assumption or a goal by an equialent one, preserving reversibility – we will use this as a rule labeled $\equiv$. By using this and the fact that negation and conjunction can express by eqivalence other connectives, we can easily infer the new rules, by just applying the old ones.

Thus we look again at every logical connective in every position, for instance *disjunction in assumptions* and construct a sequent proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Phi, \gamma_1 \vdash \Psi}{\Phi \vdash \neg\gamma_1, \Psi}\vdash\neg \qquad \dfrac{\Phi, \gamma_2 \vdash \Psi}{\Phi \vdash \neg\gamma_2, \Psi}\vdash\neg}{\Phi \vdash \neg\gamma_1 \wedge \neg\gamma_2, \Psi}\vdash\wedge}{\Phi, \neg(\neg\gamma_1 \wedge \neg\gamma_2) \vdash \Psi}\neg\vdash}{\Phi, \gamma_1 \vee \gamma_2 \vdash \Psi}\equiv$$

This is the well–known rule “proof by cases” which we used before, and look how it is produced completely automatically by our calculus! This constitutes a confirmation of the usefulness of our model: some behaviour which is present in the real world is inferred in a purely formal manner by our model, and this pheonomenon will repeat for the other rules. The properties of the calculus insure that this rule is also reversible, and of course the logical connective is removed, because this is what the calculus does! It will be the same for all other logical connectives.

We should also note that this rule is the dual of the rule *conjunction in goals*. We could prove the correctness of this rule by reducing the sequents to their normal form, as we did before. Then again the equivalence will result from distributivity.

Let us proceed with *disjunction in the goal*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Phi \;\vdash\; \gamma_1, \gamma_2, \Psi}{\Phi, \neg\gamma_1 \;\vdash\; \gamma_2, \Psi}\,\neg\vdash}{\Phi, \neg\gamma_1, \neg\gamma_2 \;\vdash\; \Psi}\,\neg\vdash}{\Phi, \neg\gamma_1 \wedge \neg\gamma_2 \;\vdash\; \Psi}\,\wedge\vdash}{\Phi \;\vdash\; \neg(\neg\gamma_1 \wedge \neg\gamma_2), \Psi}\,\vdash\neg}{\Phi \;\vdash\; \gamma_1 \vee \gamma_2, \Psi}\,\equiv$$

We obtain the dual of the rule *conjunction in the assumptions*, which can be also proved by normal form and associativity of disjunction. Also, if we consider the rules for disjunction as known, we can obtain the rules for conjunction by applying this as an alternative calculus, in a similar way.

For finding out the rules for implication we can use the rules for disjunction, for instance for the *implication in assumptions*:

$$\dfrac{\dfrac{\dfrac{\Phi \;\vdash\; \gamma_1, \Psi}{\Phi, \neg\gamma_1 \;\vdash\; \Psi}\,\neg\vdash \qquad \Phi, \gamma_2 \;\vdash\; \Psi}{\Phi, \neg\gamma_1 \vee \gamma_2 \;\vdash\; \Psi}\,\vee\vdash}{\Phi, \gamma_1 \Rightarrow \gamma_2 \;\vdash\; \Psi}\,\equiv$$

Intuitively, the rule which we obtained says: "if an implication is in the assumptions, prove first the LHS, and then use the RHS". However, it is not exactly like this, because on the first branch an alternative goal is the old one, because we cannot be sure that the LHS of the implication is a logical consequence of the formulae in Φ.

The *implication in goals* is much simpler:

$$\dfrac{\dfrac{\dfrac{\Phi, \gamma_1 \;\vdash\; \gamma_2 \Psi}{\Phi \;\vdash\; \neg\gamma_1, \gamma_2, \Psi}\,\vdash\neg}{\Phi \;\vdash\; \neg\gamma_1 \vee \gamma_2, \Psi}\,\vdash\vee}{\Phi \;\vdash\; \gamma_1 \Rightarrow \gamma_2, \Psi}\,\equiv$$

We retrieve the natural inference "deduction rule" which we used two times in the first example.

Now let us see the logical equivalence, which we transform into two implications. *Equivalence in the assumptions*:

$$
\cfrac{\cfrac{\cfrac{\Phi, \gamma_1, \gamma_2 \;\vdash\; \gamma_1, \gamma_2, \Psi \qquad \cfrac{}{\Phi, \gamma_1 \;\vdash\; \gamma_1, \Psi}\,\text{a}}{\Phi, \gamma_2 \Rightarrow \gamma_1 \;\vdash\; \gamma_1, \Psi}\,\neg\vdash \qquad \cfrac{\cfrac{}{\Phi, \gamma_2 \;\vdash\; \gamma_2, \Psi}\,\text{a} \qquad \Phi, \gamma_1, \gamma_2 \;\vdash\; \Psi}{\Phi, \gamma_2, \gamma_2 \Rightarrow \gamma_1 \;\vdash\; \Psi}\,\Rightarrow\vdash}{\cfrac{\Phi, \gamma_1 \Rightarrow \gamma_2, \gamma_2 \Rightarrow \gamma_1 \;\vdash\; \Psi}{\Phi, (\gamma_1 \Rightarrow \gamma_2) \wedge (\gamma_2 \Rightarrow \gamma_1) \;\vdash\; \Psi}\,\wedge\vdash}\,\Rightarrow\vdash}{\Phi, \gamma_1 \Leftrightarrow \gamma_2 \;\vdash\; \Psi}\,\equiv
$$

From the four resulting sequents, we notice that two are axioms, thus they can be ignored (the success of the proof depends only on the other ones). Therefore we obtain a rule which intuitively reads: "try to prove any of the sides of the equivalence (or the old goal), and then use them both".

$$
\cfrac{\cfrac{\cfrac{\Phi, \gamma_1 \;\vdash\; \gamma_2, \Psi}{\Phi \;\vdash\; \gamma_1 \Rightarrow \gamma_2, \Psi}\,\vdash\Rightarrow \qquad \cfrac{\Phi, \gamma_2 \;\vdash\; \gamma_1, \Psi}{\Phi \;\vdash\; \gamma_2 \Rightarrow \gamma_1, \Psi}\,\vdash\Rightarrow}{\Phi \;\vdash\; (\gamma_1 \Rightarrow \gamma_2) \wedge (\gamma_2 \Rightarrow \gamma_1), \Psi}\,\wedge\vdash}{\Phi \;\vdash\; \gamma_1 \Leftrightarrow \gamma_2, \Psi}\,\equiv
$$

Thus we obtain the known natural rule for proving an equivalence: *Equivalence in the goal* is easier and more intuitive: "first assume the LHS and prove the RHS, then assume the RHS and prove the LHS".

The new rules for these three logical connectives are presented in the table below:

	Assumptions	Goals
$\vee$	$\dfrac{\Phi\, \gamma_1 \vdash \;\Psi \qquad \Phi, \gamma_2 \;\vdash\; \Psi}{\Phi, \gamma_1 \vee \gamma_2 \;\vdash\; \Psi}\,\vee\vdash$	$\dfrac{\Phi \;\vdash\; \gamma_1, \gamma_2, \Psi}{\Phi \;\vdash\; \gamma_1 \vee \gamma_2, \Psi}\,\vdash\vee$
$\Rightarrow$	$\dfrac{\Phi \;\vdash\; \gamma_1, \Psi \qquad \Phi, \gamma_2 \;\vdash\; \Psi}{\Phi, \gamma_1 \Rightarrow \gamma_2 \;\vdash\; \Psi}\,\Rightarrow\vdash$	$\dfrac{\Phi, \gamma_1 \;\vdash\; \gamma_2, \Psi}{\Phi \;\vdash\; \gamma_1 \Rightarrow \gamma_2, \Psi}\,\vdash\Rightarrow$
$\Leftrightarrow$	$\dfrac{\Phi \;\vdash\; \gamma_1, \gamma_2, \Psi \qquad \Phi, \gamma_1, \gamma_2 \;\vdash\; \Psi}{\Phi, \gamma_1 \Leftrightarrow \gamma_2 \;\vdash\; \Psi}\,\Leftrightarrow\vdash$	$\dfrac{\Phi, \gamma_1 \;\vdash\; \gamma_2, \Psi \qquad \Phi, \gamma_2 \;\vdash\; \gamma_1, \Psi}{\Phi \;\vdash\; \gamma_1 \Leftrightarrow \gamma_2, \Psi}\,\vdash\Leftrightarrow$

There is something missing in our rules: the truth constants $\mathbb{T}$ and $\mathbb{F}$.

Before going to those let us remember that the conjunction and disjunction can be actually applied to sets. For a conjunctions and disjunctions with many arguments the obvious generalization of the rules are:

	Assumptions	Goals
$\wedge$	$\dfrac{\Phi, \gamma_1, \ldots, \gamma_n \;\vdash\; \Psi}{\Phi, \gamma_1 \wedge \ldots \wedge \gamma_n \;\vdash\; \Psi}\,\wedge\vdash$	$\dfrac{\Phi \;\vdash\; \gamma_1, \Psi \qquad \ldots \qquad \Phi \;\vdash\; \gamma_n, \Psi}{\Phi \;\vdash\; \gamma_1 \wedge \ldots \wedge \gamma_n, \Psi}\,\vdash\wedge$
$\vee$	$\dfrac{\Phi\, \gamma_1 \vdash \;\Psi \qquad \ldots \qquad \Phi, \gamma_n \;\vdash\; \Psi}{\Phi, \gamma_1 \vee \ldots \vee \gamma_n \vdash \;\Psi}\,\vee\vdash$	$\dfrac{\Phi \;\vdash\; \gamma_1, , \ldots, \gamma_n, \Psi}{\Phi \;\vdash\; \gamma_1 \vee \ldots \vee \gamma_2, \Psi}\,\vdash\vee$

As usual with the "..." notation, these rules do not express situations when the number of arguments is small, therefore it is better to present these rules as:

	Assumptions	Goals
$\wedge$	$\dfrac{\Phi, \Gamma \ \vdash \ \Psi}{\Phi, \wedge\Gamma \ \vdash \ \Psi} \wedge\vdash$	$\dfrac{\{\Phi \ \vdash \ \gamma, \Psi \mid \gamma \in \Gamma\}}{\Phi \ \vdash \ \wedge\Gamma, \Psi} \vdash\wedge$
$\vee$	$\dfrac{\{\Phi, \gamma \ \vdash \ \Psi \mid \gamma \in \Gamma\}}{\Phi, \vee\Gamma \ \vdash \ \Psi} \vee\vdash$	$\dfrac{\Phi \ \vdash \ \Gamma, \Psi}{\Phi \ \vdash \ \vee\Gamma, \Psi} \vdash\vee$

These rules are final for the general situations involving conjunctions and disjunctions, and their correctness can be proven using the generalizations of the associativity and of the distributivity for the case of sets.

To do: proof using the generalizations of the associativity and of the distributivity for the case of sets

	Assumptions	Goals
$\wedge\emptyset$	$\dfrac{\Phi, \emptyset \ \vdash \ \Psi}{\Phi, \wedge\emptyset \ \vdash \ \Psi} \wedge\vdash$	$\dfrac{\{\Phi \ \vdash \ \gamma, \Psi \mid \gamma \in \emptyset\}}{\Phi \ \vdash \ \wedge\emptyset, \Psi} \vdash\wedge$
$\vee\emptyset$	$\dfrac{\{\Phi, \gamma \ \vdash \ \Psi \mid \gamma \in \emptyset\}}{\Phi, \vee\emptyset \ \vdash \ \Psi} \vee\vdash$	$\dfrac{\Phi \ \vdash \ \emptyset, \Psi}{\Phi \ \vdash \ \vee\emptyset, \Psi} \vdash\vee$

Now we can easily infer the rules for the truth constants, by just using the emptyset:

	Assumptions	Goals
$\mathbb{T}$	$\dfrac{\Phi \vdash \ \Psi}{\Phi, \mathbb{T} \vdash \ \Psi} \mathrm{T}\vdash$	$\dfrac{}{\Phi \ \vdash \ \mathbb{T}, \Psi} \vdash\mathrm{T}$
$\mathbb{F}$	$\dfrac{}{\Phi, \mathbb{F} \ \vdash \ \Psi} \mathrm{F}\vdash$	$\dfrac{\Phi \ \vdash \ \Psi}{\Phi \ \vdash \ \mathbb{F}, \Psi} \vdash\mathrm{F}$

By the semantics of the conjunction and disjunction applied to sets, as well as from the definition of the set quantifier, the rules become:

Therefore we have finally the following table for a full sequent calculus in propositional logic:

Full	Assumptions	Goal
a	$\dfrac{}{\Phi, \gamma \ \vdash \ \gamma, \Psi}\ \mathrm{a}$	
$\mathbb{T}$	$\dfrac{\Phi \vdash \ \Psi}{\Phi, \mathbb{T} \vdash \ \Psi}\ \mathrm{T}\vdash$	$\dfrac{}{\Phi \ \vdash \ \mathbb{T}, \Psi}\ \vdash\mathrm{T}$
$\mathbb{F}$	$\dfrac{}{\Phi, \mathbb{F} \ \vdash \ \Psi}\ \mathrm{F}\vdash$	$\dfrac{\Phi \ \vdash \ \Psi}{\Phi \ \vdash \ \mathbb{F}, \Psi}\ \vdash\mathrm{F}$
$\neg$	$\dfrac{\Phi \ \vdash \ \gamma, \Psi}{\Phi, \neg\gamma \ \vdash \ \Psi}\ \neg\vdash$	$\dfrac{\Phi, \gamma \ \vdash \ \Psi}{\Phi \ \vdash \ \neg\gamma, \Psi}\ \vdash\neg$
$\wedge$	$\dfrac{\Phi, \Gamma \ \vdash \ \Psi}{\Phi, \wedge\Gamma \ \vdash \ \Psi}\ \wedge\vdash$	$\dfrac{\{\Phi \ \vdash \ \gamma, \Psi \mid \gamma \in \Gamma\}}{\Phi \ \vdash \ \wedge\Gamma, \Psi}\ \vdash\wedge$
$\vee$	$\dfrac{\{\Phi, \gamma \ \vdash \ \Psi \mid \gamma \in \Gamma\}}{\Phi, \vee\Gamma \ \vdash \ \Psi}\ \vee\vdash$	$\dfrac{\Phi \ \vdash \ \Gamma, \Psi}{\Phi \ \vdash \ \vee\Gamma, \Psi}\ \vdash\vee$
$\Rightarrow$	$\dfrac{\Phi \ \vdash \ \gamma_1, \Psi \qquad \Phi, \gamma_2 \ \vdash \ \Psi}{\Phi, \gamma_1 \Rightarrow \gamma_2 \ \vdash \ \Psi}\ \Rightarrow\vdash$	$\dfrac{\Phi, \gamma_1 \ \vdash \ \gamma_2, \Psi}{\Phi \ \vdash \ \gamma_1 \Rightarrow \gamma_2, \Psi}\ \vdash\Rightarrow$
$\Leftrightarrow$	$\dfrac{\Phi \ \vdash \ \gamma_1, \gamma_2, \Psi \qquad \Phi, \gamma_1, \gamma_2 \ \vdash \ \Psi}{\Phi, \gamma_1 \Leftrightarrow \gamma_2 \ \vdash \ \Psi}\ \Leftrightarrow\vdash$	$\dfrac{\Phi, \gamma_1 \ \vdash \ \gamma_2, \Psi \qquad \Phi, \gamma_2 \ \vdash \ \gamma_1, \Psi}{\Phi \ \vdash \ \gamma_1 \Leftrightarrow \gamma_2, \Psi}\ \vdash\Leftrightarrow$

Suggested exercise: Prove by sequent calculus, and in case the proof fails find the counterexample.

$$((A \wedge B) \Rightarrow C) \Rightarrow ((A \Rightarrow C) \vee (B \Rightarrow C)) \tag{3}$$

$$((A \Rightarrow C) \wedge (B \Rightarrow C)) \Leftrightarrow ((A \vee B) \Rightarrow C) \tag{4}$$

$$((A \Rightarrow B) \wedge (B \Rightarrow C)) \Leftrightarrow ((A \vee B) \Rightarrow C) \tag{5}$$

3.3.2 Special Rules

This calculus is correct and complete for full propositional logic, however we can add some rules to make it more efficient. For instance, in our first example we used the rule “Modus Ponens”. There are many such rules which are not actually necessary for the calculus, because the proofs will work anyway, however they can make the proofs smaller, which is an important goal in automated reasoning.

These rules simplify the proof either by removing a part of a formula, a formula, or a branch of the proof. When using a rule which removes a branch of the proof, the resulting set of atomic sequents will not mirror anymore to the CNF of the root sequent, because some clauses will be missing.

We present these rules in three groups: resolution, subsumption, and implication related.

Resolution Rules. Let us now look at two rules which are similar to unit resolution for *disjunction in assumptions:*

$$\dfrac{\dfrac{\dfrac{}{\Phi, \gamma_1 \ \vdash \ \gamma_1, \Psi}\,\mathrm{a}}{\Phi, \gamma_1, \neg\gamma_1 \ \vdash \ \Psi}\,\neg\vdash \qquad \dfrac{}{\Phi, \gamma_1, \gamma_2 \ \vdash \ \Psi}\,\mathrm{a}}{\Phi, \gamma_1, (\neg\gamma_1) \vee \gamma_2 \ \vdash \ \Psi}\,\vee\vdash$$

$$\dfrac{\dfrac{\dfrac{}{\Phi, \gamma_1 \ \vdash \ \gamma_1, \Psi}\,\mathrm{a}}{\Phi, \neg\gamma_1, \gamma_1 \ \vdash \ \Psi}\,\neg\vdash \qquad \dfrac{}{\Phi, \neg\gamma_1, \gamma_2 \ \vdash \ \Psi}\,\mathrm{a}}{\Phi, \neg\gamma_1, \gamma_1 \vee \gamma_2 \ \vdash \ \Psi}\,\vee\vdash$$

In both cases the axiom can be deleted, thus, similar to unit resolution, a formula deletes its opposite from a disjunction. These rules can also be expressed in general way for disjunctive sets:

$$\dfrac{\Phi, \gamma, \vee\Gamma \ \vdash \ \Psi}{\Phi, \gamma, \vee(\{\neg\gamma\} \cup \Gamma) \ \vdash \ \Psi}\,\mathrm{R}\vdash \qquad \dfrac{\Phi, \neg\gamma, \vee\Gamma \ \vdash \ \Psi}{\Phi, \neg\gamma, \vee(\{\gamma\} \cup \Gamma) \ \vdash \ \Psi}\,\mathrm{R}\vdash$$

Because of the rules for negation, we know that a formula in the goals behaves like its negation in the assumptions. Thus we can discover another resolution rule for assumed disjunction, in which the simple formula is in the goals: A goal removes its copy from a disjunction in the assumptions.

$$\dfrac{\Phi, \gamma_2 \ \vdash \ \gamma_1, \Psi}{\Phi, \gamma_1 \vee \gamma_2 \ \vdash \ \gamma_1, \Psi}\,\mathrm{R}\vdash \qquad \dfrac{\Phi, \vee\Gamma \ \vdash \ \gamma, \Psi}{\Phi, \vee(\{\gamma\} \cup \Gamma) \ \vdash \ \gamma, \Psi}\,\mathrm{R}\vdash$$

We noticed a certain duality in the rules for conjunction and disjunction: they are very similar when you change the position in the sequent between assumptions and goals. As we discovered some interesting rules for disjunctions in the assumptions, we can now study the behavior of conjunctions in the goal.

$$\dfrac{\dfrac{\dfrac{}{\Phi, \gamma_1 \ \vdash \ \gamma_1, \Psi}\,\mathrm{a}}{\Phi \ \vdash \ \gamma_1, \neg\gamma_1, \Psi}\,\neg\vdash \qquad \dfrac{}{\Phi \ \vdash \ \gamma_1, \gamma_2, \Psi}\,\mathrm{a}}{\Phi \ \vdash \ \gamma_1, (\neg\gamma_1) \wedge \gamma_2, \Psi}\,\vdash\wedge$$

$$\dfrac{\dfrac{\dfrac{}{\Phi, \gamma_1 \ \vdash \ \gamma_1, \Psi}\,\mathrm{a}}{\Phi \ \vdash \ \neg\gamma_1, \gamma_1, \Psi}\,\neg\vdash \qquad \Phi \ \vdash \ \neg\gamma_1, \gamma_2, \Psi}{\Phi \ \vdash \ \neg\gamma_1, \gamma_1 \wedge \gamma_2, \Psi}\,\vdash\wedge$$

We see that we obtain the perfect duals of the rules for disjunction in the assumptions: a formula in the goal deletes its occurrence in a conjunction in the goal. The generalizations are:

$$\dfrac{\Phi \ \vdash \ \gamma, \wedge\Gamma, \Psi}{\Phi \ \vdash \ \gamma, \wedge(\{\neg\gamma\} \cup \Gamma), \Psi}\,\vdash\mathrm{R} \qquad \dfrac{\Phi \ \vdash \ \neg\gamma, \wedge\Gamma, \Psi}{\Phi \ \vdash \ \neg\gamma, \wedge(\{\gamma\} \cup \Gamma), \Psi}\,\vdash\mathrm{R}$$

Furthermore we can also find here the rule by which a formula in the assumptions removes its copy from a conjunction in the goals, which is very natural.

$$\dfrac{\Phi, \gamma_1 \ \vdash \ \gamma_2, \Psi}{\Phi, \gamma_1 \ \vdash \ \gamma_1 \wedge \gamma_2, \Psi}\,\vdash\mathrm{R} \qquad \dfrac{\Phi, \gamma \ \vdash \ \wedge\Gamma, \Psi}{\Phi, \gamma \ \vdash \ \wedge(\{\gamma\} \cup \Gamma), \Psi}\,\vdash\mathrm{R}$$

Subsumption Rules. In the resolution proof method we also used subsumption: a clause containing another one can be removed. Let us see what happens when a disjunct of another clause is among the assumptions.

$$\frac{\Phi, \gamma_1 \ \vdash \ \Psi \qquad \Phi, \gamma_1, \gamma_2 \ \vdash \ \Psi}{\Phi, \gamma_1, \gamma_1 \vee \gamma_2 \ \vdash \ \Psi} \vee\vdash$$

On the left branch the case γ_1 is used, but this is already an assumption, so we do not write it twice. If the proof succeeds on the left hand side branch, then it will certainly succeed also on the right hand side, because we have an additional assumption. However if the proof does not succeed on the left hand side, then it does not matter what happens on the right hand side, because the whole proof will fail anyway. More formally, we can use the fact that the two sequents above the line form a conjunction. If we express the two sequents as disjunctions then we have:

$$(\vee(\overline{\Phi} \cup \{\overline{\gamma}_1\} \cup \Psi)) \wedge (\vee(\overline{\Phi} \cup \{\overline{\gamma}_1, \overline{\gamma}_2\} \cup \Psi)),$$

which is equivalent to the first conjunct by the absorption rule: $A \wedge (A \vee B) \equiv A$. Alternatively, we can see the sequents as implications, and then their conjunct is of the form $(P \Rightarrow R) \wedge ((P \wedge Q) \Rightarrow R)$, which by (2) becomes $(P \vee (P \wedge Q)) \Rightarrow R$, whose LHS reduces again by the other absorption rule: $A \vee (A \wedge B) \equiv A$. Therefore, we can ignore the right hand side branch, thus, in general, the rule says that a disjunction removes the disjunction which includes it, and in the particular case of unit disjunctions, a formula removes the disjunction which contains it.

$$\frac{\Phi, \vee\Gamma_1 \ \vdash \ \Psi}{\Phi, \vee\Gamma_1, \vee(\Gamma_1 \cup \Gamma_2) \ \vdash \ \Psi} \mathsf{S}\vdash \qquad \frac{\Phi, \gamma \ \vdash \ \Psi}{\Phi, \gamma, \vee(\{\gamma\} \cup \Gamma) \ \vdash \ \Psi} \mathsf{S}\vdash$$

As in the case of resolution, we can now study the interaction between an assumed disjunction and a formula from the goal which is opposite to a disjunct. The most practically useful rule appears to be the one which removes the assumed disjunction when in contains the opposite version of a goal:

$$\frac{\Phi \ \vdash \ \gamma, \Psi}{\Phi, \vee(\{\neg\gamma\} \cup \Gamma) \ \vdash \ \gamma, \Psi} \mathsf{S}\vdash \qquad \frac{\Phi \ \vdash \ \neg\gamma, \Psi}{\Phi, \vee(\{\gamma\} \cup \Gamma) \ \vdash \ \neg\gamma, \Psi} \mathsf{S}\vdash$$

However this of course can also be specified in a more general way, by using overbar for expressing the opposite of a formula and the set of opposite formulae of a set:

$$\frac{\Phi \ \vdash \ \overline{\gamma}, \Psi}{\Phi, \vee(\{\gamma\} \cup \Gamma) \ \vdash \ \overline{\gamma}, \Psi} \mathsf{S}\vdash \qquad \frac{\Phi \ \vdash \ \vee\Gamma_1, \Psi}{\Phi, \vee(\Gamma_1 \cup \Gamma_2) \ \vdash \ \vee\overline{\Gamma}_1, \Psi} \mathsf{S}\vdash$$

We apply the rule for conjunction in the goal to the dual sequent:

$$\frac{\Phi \ \vdash \ \gamma_1, \Psi \qquad \Phi \ \vdash \ \gamma_1, \gamma_2, \Psi}{\Phi \ \vdash \ \gamma_1, \gamma_1 \wedge \gamma_2, \Psi} \vdash\wedge$$

On the left branch the goal γ_1 is used, but this is already a goal, so we do not write it twice. The right hand side branch is easier, therefore we can ignore it. Formally, the conjunction of the two sequents is:

$$(\vee(\overline{\Phi} \cup \{\gamma_1\} \cup \Psi)) \wedge (\vee(\overline{\Phi} \cup \{\gamma_1, \gamma_2\} \cup \Psi)),$$

which again reduces by absorption. The rule says that a conjunction removes the conjunction which includes it, and in the particular case of unit conjunctions, a formula removes the conjunction which contains it.

$$\frac{\Phi \ \vdash\ \wedge\Gamma_1, \Psi}{\Phi \ \vdash\ \wedge\Gamma_1, \wedge(\Gamma_1 \cup \Gamma_2), \Psi}\vdash\mathrm{S} \qquad\qquad \frac{\Phi, \gamma \ \vdash\ \Psi}{\Phi \ \vdash\ \gamma, \wedge(\{\gamma\} \cup \Gamma), \Psi}\vdash\mathrm{S}$$

The next dual rule removes the goal conjunction when in contains the opposite version of an assumption:

$$\frac{\Phi, \gamma \ \vdash\ \Psi}{\Phi, \gamma \ \vdash\ \wedge(\{\neg\gamma\} \cup \Gamma), \Psi}\vdash\mathrm{S} \qquad\qquad \frac{\Phi, \neg\gamma \ \vdash\ \Psi}{\Phi, \neg\gamma \ \vdash\ \wedge(\{\gamma\} \cup \Gamma), \Psi}\vdash\mathrm{S}$$

The general rules use overbar for expressing the opposite:

$$\frac{\Phi, \overline{\gamma} \ \vdash\ \Psi}{\Phi, \overline{\gamma} \ \vdash\ \wedge(\{\gamma\} \cup \Gamma), \Psi}\vdash\mathrm{S} \qquad\qquad \frac{\Phi, \wedge\overline{\Gamma}_1 \ \vdash\ \Psi}{\Phi, \wedge\overline{\Gamma}_1 \ \vdash\ \wedge(\Gamma_1 \cup \Gamma_2), \Psi}\vdash\mathrm{S}$$

Implication Rules. The rule "Modus Ponens": "when we know an assumption and an implication having it as LHS, then we can replace the implication by its RHS" – where we actually do not know whether it is safe to delete the old implication.

Let us investigate this situation formally, by applying the calculus:

$$\frac{\dfrac{}{\Phi, \gamma_1 \ \vdash\ \gamma_1, \Psi}\mathrm{a} \qquad \Phi, \gamma_1, \gamma_2 \ \vdash\ \Psi}{\Phi, \gamma_1, \gamma_1 \Rightarrow \gamma_2 \ \vdash\ \Psi}\Rightarrow\vdash$$

The first axiomatic sequent can be ignored, so what remains is the rule which we already applied intuitively. Now we have a formal argument about the deletion of the old assumption: the proof does not need it anymore.

Similarly we can investigate the "Modus Tollens" rule:

$$\frac{\dfrac{\Phi, \neg\gamma_1, \neg\gamma_2 \ \vdash\ \Psi}{\Phi, \neg\gamma_2 \ \vdash\ \gamma_1, \Psi}\vdash\neg \qquad \dfrac{\dfrac{}{\Phi, \gamma_2 \ \vdash\ \gamma_2, \Psi}\mathrm{a}}{\Phi, \gamma_2, \neg\gamma_2 \ \vdash\ \Psi}\neg\vdash}{\Phi, \gamma_1 \Rightarrow \gamma_2, \neg\gamma_2 \ \vdash\ \Psi}\Rightarrow\vdash$$

Again by deleting the branch which ends with an axiomatic sequent, we get a rule which replaces the implication by the negated LHS of it.

In resolution calculus we saw that both "Modus Ponens" and "Modus Tollens" are special cases of resolution. Special cases of subsumption apply to implication: this can be removed when its right hand side or the negation of its left hand side is present among the assumptions.

$$\frac{\Phi, \gamma_2 \ \vdash\ \gamma_1, \Psi \qquad \Phi, \gamma_2 \ \vdash\ \Psi}{\Phi, \gamma_2, \gamma_1 \Rightarrow \gamma_2 \ \vdash\ \Psi}\Rightarrow\vdash$$

$$\dfrac{\dfrac{\Phi \vdash \gamma_1, \Psi \qquad \Phi, \gamma_2 \vdash \gamma_1, \Psi}{\Phi, \gamma_1 \Rightarrow \gamma_2 \vdash \gamma_1, \Psi} \Rightarrow\vdash}{\Phi, \neg\gamma_1, \gamma_1 \Rightarrow \gamma_2 \vdash \Psi} \neg\vdash$$

In the first case the left hand side branch is easier because it has an additional assumption, thus it can be removed. In the second case the right hand side has more assumptions, thus it is easier and can be removed.

Other rules: suggested exercise 04-28-1

unique labels and table for all special rules

Examples

3.3.3 Unique goal sequent calculus

In natural style we actually use only one goal at a time. We define now a sequent calculus whose set of goals always contains one formula. There will be no empty set on the right hand side of the sequent, this is simulated by the goal $\mathbb{F}$.

UG	Assumptions	Goal
a	$\dfrac{}{\Phi, \gamma \vdash \gamma}\,\mathrm{a}$	$\dfrac{}{\Phi, \gamma, \neg\gamma \vdash \psi}\,\mathrm{a}$
$\mathbb{T}$	$\dfrac{\Phi \vdash \psi}{\Phi, \mathbb{T} \vdash \psi}\,\mathrm{T}\vdash$	$\dfrac{}{\Phi \vdash \mathbb{T}}\,\vdash\mathrm{T}$
$\mathbb{F}$	$\dfrac{}{\Phi, \mathbb{F} \vdash \psi}\,\mathrm{F}\vdash$	
$\neg$	$\dfrac{\dfrac{\Phi \vdash \gamma}{\Phi, \neg\gamma \vdash \mathbb{F}} \qquad \Phi, (\neg\gamma)_{\neg\downarrow} \vdash \psi}{\Phi, \neg\gamma \vdash \psi}\,\neg\vdash$	$\dfrac{\Phi, \gamma \vdash \mathbb{F}}{\Phi \vdash \neg\gamma}\,\vdash\neg$
$\wedge$	$\dfrac{\Phi, \Gamma \vdash \psi}{\Phi, \wedge\Gamma \vdash \psi}\,\wedge\vdash$	$\dfrac{\{\Phi \vdash \gamma \mid \gamma \in \Gamma\}}{\Phi \vdash \wedge\Gamma}\,\vdash\wedge$
$\vee$	$\dfrac{\{\Phi, \gamma \vdash \psi \mid \gamma \in \Gamma\}}{\Phi, \vee\Gamma \vdash \psi}\,\vee\vdash$	$\dfrac{\Phi \cup \overline{\Gamma} \vdash \gamma}{\Phi \vdash \vee(\{\gamma\} \cup \Gamma)}\,\vdash\vee$
$\Rightarrow$	$\dfrac{\Phi, \overline{\psi} \vdash \gamma_1 \qquad \Phi, \gamma_2 \vdash \psi}{\Phi, \gamma_1 \Rightarrow \gamma_2 \vdash \psi}\,\Rightarrow\vdash$	$\dfrac{\Phi, \gamma_1 \vdash \gamma_2}{\Phi \vdash \gamma_1 \Rightarrow \gamma_2}\,\vdash\Rightarrow$
$\Leftrightarrow$	$\dfrac{\Phi, \overline{\gamma}_1, \overline{\gamma}_2 \vdash \psi \qquad \Phi, \gamma_1, \gamma_2 \vdash \psi}{\Phi, \gamma_1 \Leftrightarrow \gamma_2 \vdash \psi}\,\Leftrightarrow\vdash$	$\dfrac{\Phi, \gamma_1 \vdash \gamma_2 \qquad \Phi, \gamma_2 \vdash \gamma_1}{\Phi \vdash \gamma_1 \Leftrightarrow \gamma_2}\,\vdash\Leftrightarrow$

Explanation of the transformation of the old rules into the new ones

Intuition about each rule

Example 1:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, B, C \ \vdash \ C}\,\text{a}}{A, B, A \Rightarrow C \ \vdash \ C}\,\text{MP} \qquad \dfrac{\dfrac{}{A, B, C \ \vdash \ C}\,\text{a}}{A, B, B \Rightarrow C \ \vdash \ C}\,\text{MP}}{A, B, (A \Rightarrow C) \vee (B \Rightarrow C) \ \vdash \ C}\,\vee\vdash}{A \wedge B, (A \Rightarrow C) \vee (B \Rightarrow C) \ \vdash \ C}\,\wedge\vdash}{(A \Rightarrow C) \vee (B \Rightarrow C) \ \vdash \ (A \wedge B) \Rightarrow C}\,\vdash\Rightarrow}{\vdash \ ((A \Rightarrow C) \vee (B \Rightarrow C)) \Rightarrow ((A \wedge B) \Rightarrow C)}\,\vdash\Rightarrow
$$

Example 2:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{? \ \vdash \ ?}{(A \wedge B) \Rightarrow C, B, \neg C, A \ \vdash \ C}\,?}{(A \wedge B) \Rightarrow C, B, \neg C \ \vdash \ A \Rightarrow C}\,\wedge\vdash}{(A \wedge B) \Rightarrow C, B \wedge \neg C \ \vdash \ A \Rightarrow C}\,\neg\vdash}{(A \wedge B) \Rightarrow C, \neg(B \Rightarrow C) \ \vdash \ A \Rightarrow C}\,\neg\vdash}{(A \wedge B) \Rightarrow C \ \vdash \ (A \Rightarrow C) \vee (B \Rightarrow C)}\,\vdash\vee}{\vdash \ ((A \wedge B) \Rightarrow C) \Rightarrow ((A \Rightarrow C) \vee (B \Rightarrow C))}\,\vdash\Rightarrow
$$

Finish the proof: suggested exercise 04-28-2
more examples

3.3.4 Sequent calculus with unit propagation

remind unit propagation in DPLL

proof by cases, var true or false

$$
\dfrac{\Phi_{A\to\mathbb{T}} \ \vdash \ \Psi_{A\to\mathbb{T}} \qquad \Phi_{A\to\mathbb{F}} \ \vdash \ \Psi_{A\to\mathbb{F}}}{\Phi \ \vdash \ \Psi}
$$

$$
\dfrac{\Phi_{A\to\mathbb{T}}, \mathbb{T} \ \vdash \ \Psi_{A\to\mathbb{T}} \qquad \Phi_{A\to\mathbb{F}}, \mathbb{F} \ \vdash \ \Psi_{A\to\mathbb{F}}}{\Phi, A \ \vdash \ \Psi} \quad \dfrac{\Phi_{A\to\mathbb{T}} \ \vdash \ \Psi_{A\to\mathbb{T}}}{\Phi, A \ \vdash \ \Psi}
$$

$$
\dfrac{\Phi_{A\to\mathbb{T}} \ \vdash \ \mathbb{T}, \Psi_{A\to\mathbb{T}} \qquad \Phi_{A\to\mathbb{F}} \ \vdash \ \mathbb{F}, \Psi_{A\to\mathbb{F}}}{\Phi \ \vdash \ A, \Psi} \quad \dfrac{\Phi_{A\to\mathbb{F}} \ \vdash \ \Psi_{A\to\mathbb{F}}}{\Phi \ \vdash \ A, \Psi}
$$

Negative literal

$$
\dfrac{\Phi_{A\to\mathbb{T}}, \mathbb{F} \ \vdash \ \Psi_{A\to\mathbb{T}} \qquad \Phi_{A\to\mathbb{F}}, \mathbb{T} \ \vdash \ \Psi_{A\to\mathbb{F}}}{\Phi, \overline{A} \ \vdash \ \Psi} \quad \dfrac{\Phi_{A\to\mathbb{F}} \ \vdash \ \Psi_{A\to\mathbb{F}}}{\Phi, \overline{A} \ \vdash \ \Psi}
$$

$$
\dfrac{\Phi_{A\to\mathbb{T}} \ \vdash \ \Psi_{A\to\mathbb{T}}, \mathbb{F} \qquad \Phi_{A\to\mathbb{F}} \ \vdash \ \Psi_{A\to\mathbb{F}}, \mathbb{T}}{\Phi \ \vdash \ \Psi, \overline{A}} \quad \dfrac{\Phi_{A\to\mathbb{T}} \ \vdash \ \Psi_{A\to\mathbb{T}}}{\Phi \ \vdash \ \Psi, \overline{A}}
$$

Example:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathbb{F} \ \vdash}\,\mathrm{a}}{(\neg\mathbb{T}) \lor (\neg\mathbb{T}) \ \vdash}\,\text{Simplify}}{A, B, (\neg A) \lor (\neg B) \ \vdash}\,A \to \mathbb{T}, B \to \mathbb{T}}{A \land B, (\neg A) \lor (\neg B) \ \vdash}\,\land\vdash}{A \land B, (A \Rightarrow \mathbb{F}) \lor (B \Rightarrow \mathbb{F}) \ \vdash}\,\text{Simplify}}{A \land B, (A \Rightarrow C) \lor (B \Rightarrow C) \ \vdash \ C}\,C \to \mathbb{F}}{(A \Rightarrow C) \lor (B \Rightarrow C) \ \vdash \ (A \land B) \Rightarrow C}\,\vdash\Rightarrow}{\vdash \ ((A \Rightarrow C) \lor (B \Rightarrow C)) \Rightarrow ((A \land B) \Rightarrow C)}\,\vdash\Rightarrow
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg B \ \vdash \ \neg B}\,\mathrm{a}}{\neg(\mathbb{T} \land B) \ \vdash \ \neg B}\,\text{Simplify}}{\neg(A \land B), A \ \vdash \ \neg B}\,A \to \mathbb{T}}{(A \land B) \Rightarrow \mathbb{F}, A \ \vdash \ B \Rightarrow \mathbb{F}}\,\text{Simplify}}{(A \land B) \Rightarrow C, A \ \vdash \ C, B \Rightarrow C}\,C \to \mathbb{F}}{(A \land B) \Rightarrow C \ \vdash \ A \Rightarrow C, B \Rightarrow C}\,\vdash\Rightarrow}{(A \land B) \Rightarrow C \ \vdash \ (A \Rightarrow C) \lor (B \Rightarrow C)}\,\vdash\lor}{\vdash \ ((A \land B) \Rightarrow C) \Rightarrow ((A \Rightarrow C) \lor (B \Rightarrow C))}\,\vdash\Rightarrow
$$

3.3.5 Implementation in Theorema

Implementation in Theorema.

3.3.6 Sequent calculus in first order logic

Quantifiers.