

We prove:

$$(A \vee B) \Rightarrow C \Leftrightarrow ((A \Rightarrow C) \wedge (B \Rightarrow C)) \quad (\text{abc1})$$

with no assumptions.

Proof by sequent calculus with unit propagation:

$$\{ \{ (A \vee B) \Rightarrow C \Leftrightarrow ((A \Rightarrow C) \wedge (B \Rightarrow C)), "abc1" \} \}. \quad (\text{G\#0})$$

Equivalence goal (abc1) is proven in both directions.

☑ {1} Direction from left to right: Assume:

$$(A \vee B) \Rightarrow C, \quad (\text{A\#1})$$

and prove:

$$(A \Rightarrow C) \wedge (B \Rightarrow C), \quad (\text{G\#2})$$

$$\{ \{ (A \Rightarrow C) \wedge (B \Rightarrow C), "G\#2" \} \}. \quad (\text{G\#3})$$

Conjunctive goal (G#2) is decomposed on different proof branches.

☑ {1, 1} Proof of (G#2.1):

$$A \Rightarrow C. \quad (\text{G\#2.1})$$

$$\{ \{ A \Rightarrow C, "G\#2.1" \} \}. \quad (\text{G\#9})$$

Implicative goal (G#2.1) is split. Assume:

$$A, \quad (\text{A\#15})$$

and prove:

$$C, \quad (\text{G\#16})$$

$$\{ \{ C, "G\#16" \} \}. \quad (\text{G\#17})$$

Implicative assumption (A#1) is used for Modus Ponens.

☑ {1, 1, 1} First the premise of (A#1) is proven:

$$A \vee B. \quad (\text{G\#25})$$

$$\{ \{ A \vee B, "G\#25" \}, \{ C, "G\#16" \} \}. \quad (\text{G\#26})$$

Disjunctive goal (G#25) is split:

$$A, \quad (\text{G\#25.1})$$

$$B. \quad (\text{G\#25.2})$$

$$\{ \{ A, "G\#25.1" \}, \{ B, "G\#25.2" \}, \{ C, "G\#16" \} \}. \quad (\text{G\#27})$$

Assumed (A#15) goal (G#25.1): success.

☑ {1, 1, 2} Now the conclusion of (A#1) is assumed:

$$C. \quad (\text{A\#24})$$

Assumed (A#24) goal (G#16): success.

☑ {1, 2} Proof of (G#2.2):

$$B \Rightarrow C. \quad (\text{G\#2.2})$$

$\{ \{ B \Rightarrow C, "G\#2.2" \} \} .$ (G#10)

Implicative goal (G#2.2) is split. Assume:

$B,$ (A#28)

and prove:

$C,$ (G#29)

$\{ \{ C, "G\#29" \} \} .$ (G#30)

Implicative assumption (A#1) is used for Modus Ponens.

☑ {1, 2, 1} First the premise of (A#1) is proven:

$A \vee B.$ (G#38)

$\{ \{ A \vee B, "G\#38" \}, \{ C, "G\#29" \} \} .$ (G#39)

Disjunctive goal (G#38) is split:

$A,$ (G#38.1)

$B.$ (G#38.2)

$\{ \{ A, "G\#38.1" \}, \{ B, "G\#38.2" \}, \{ C, "G\#29" \} \} .$ (G#40)

Assumed (A#28) goal (G#38.2): success.

☑ {1, 2, 2} Now the conclusion of (A#1) is assumed:

$C.$ (A#37)

Assumed (A#37) goal (G#29): success.

☑ {2} Direction from right to left: Assume:

$(A \Rightarrow C) \wedge (B \Rightarrow C),$ (A#4)

and prove:

$(A \vee B) \Rightarrow C,$ (G#5)

$\{ \{ (A \vee B) \Rightarrow C, "G\#5" \} \} .$ (G#6)

Assumed conjunction (A#4) is split into:

$A \Rightarrow C,$ (A#4.1)

$B \Rightarrow C.$ (A#4.2)

Implicative goal (G#5) is split. Assume:

$A \vee B,$ (A#49)

and prove:

$C,$ (G#50)

$\{ \{ C, "G\#50" \} \} .$ (G#51)

Disjunctive assumption (A#49) is used for proof by cases.

☑ {2, 1} Case (A#49.1):

$A.$ (A#49.1)

Implicative assumption (A#4.1) is used for Modus Ponens.

- ☑ {2, 1, 1} First the premise of (A#4.1) is proven:

A.

(G#70)

{A, "G#70"}, {C, "G#50"}.

(G#71)

Assumed (A#49.1) goal (G#70): success.

- ☑ {2, 1, 2} Now the conclusion of (A#4.1) is assumed:

C.

(A#69)

Assumed (A#69) goal (G#50): success.

- ☑ {2, 2} Case (A#49.2):

B.

(A#49.2)

Implicative assumption (A#4.1) is used for Modus Ponens.

- ☑ {2, 2, 1} First the premise of (A#4.1) is proven:

A.

(G#73)

{A, "G#73"}, {C, "G#50"}.

(G#74)

Implicative assumption (A#4.2) is used for Modus Ponens.

- ☑ {2, 2, 1, 1} First the premise of (A#4.2) is proven:

B.

(G#76)

{B, "G#76"}, {A, "G#73"}, {C, "G#50"}.

(G#77)

Assumed (A#49.2) goal (G#76): success.

- ☑ {2, 2, 1, 2} Now the conclusion of (A#4.2) is assumed:

C.

(A#75)

Assumed (A#75) goal (G#50): success.

- ☑ {2, 2, 2} Now the conclusion of (A#4.1) is assumed:

C.

(A#72)

Assumed (A#72) goal (G#50): success.