First Order Predicate Logic
5. Sequent Calculus

June 23, 2020

Sequent Calculus: Inference Rules

In addition to the already known sequent calculus rules from propositional logic, we only need to add the rules for quantified formulae:

	Assumptions	Goals
$\forall$	instantiation $\dfrac{\Phi, \underset{x}{\forall}\gamma, \gamma_{x\to t} \ \vdash \ \Psi}{\Phi, \underset{x}{\forall}\gamma \ \vdash \ \Psi} \forall\vdash$	arbitrary but fixed $\dfrac{\Phi \ \vdash \ \gamma_{x\to a}, \Psi}{\Phi \ \vdash \ \underset{x}{\forall}\gamma, \Psi} \vdash\forall$
$\exists$	take such a ... $\dfrac{\Phi, \gamma_{x\to a} \ \vdash \ \Psi}{\Phi, \underset{x}{\exists}\gamma \ \vdash \ \Psi} \exists\vdash$	witness $\dfrac{\Phi \ \vdash \ \gamma_{x\to t}, \underset{x}{\exists}\gamma, \Psi}{\Phi \ \vdash \ \underset{x}{\exists}\gamma, \Psi} \vdash\exists$

a: "new" constant (*a* does not occur in γ, Φ, Ψ)
(Skolem constant)

t: "appropriate" ground term (for any ground term the rule is correct, but only some terms will be appropriate for a successful proof)

Sequent Calculus: Correctness $\forall$

Instantiation

$$\boxed{\dfrac{\Phi, \underset{x}{\forall}\gamma, \gamma_{x\to t} \;\vdash\; \Psi}{\Phi, \underset{x}{\forall}\gamma \;\vdash\; \Psi}\ \forall\vdash}$$

We know $\underset{x}{\forall}\gamma \models \gamma_{x\to t}$ for any ground term t.

Prove: $A \Rightarrow B \;\models\; ((A \wedge C) \Rightarrow D) \Leftrightarrow ((A \wedge B \wedge C) \Rightarrow D)$

Arbitrary but fixed

$$\boxed{\dfrac{\Phi \;\vdash\; \gamma_{x\to a}, \Psi}{\Phi \;\vdash\; \underset{x}{\forall}\gamma, \Psi}\ \vdash\forall}$$

$$\varphi \Rightarrow \left(\left(\underset{x}{\forall}\gamma\right) \vee \psi\right) \;\equiv\; (\neg\varphi) \vee \left(\left(\underset{a}{\forall}\gamma_{x\to a}\right) \vee \psi\right)$$
$$\equiv\; \underset{a}{\forall}((\neg\varphi) \vee (\gamma_{x\to a} \vee \psi)) \;\equiv\; \underset{a}{\forall}(\varphi \Rightarrow (\gamma_{x\to a} \vee \psi)) \;\underset{\text{proof}}{\longrightarrow}\; \underset{a}{\forall}(\mathbb{T}) \;\equiv\; \mathbb{T}$$

Sequent Calculus: Correctness $\exists$

Take such a ...

$$\boxed{\dfrac{\Phi, \gamma_{x\to a} \;\vdash\; \Psi}{\Phi, \underset{x}{\exists}\gamma \;\vdash\; \Psi}\; \exists\vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Phi, \gamma_{x\to a} \;\vdash\; \Psi}{\Phi \;\vdash\; \neg\gamma_{x\to a}, \Psi}\;\vdash\neg}{\Phi \;\vdash\; \underset{x}{\forall}\neg\gamma, \Psi}\;\vdash\forall}{\Phi, \neg\underset{x}{\forall}\neg\gamma \;\vdash\; \Psi}\;\neg\vdash}{\Phi, \underset{x}{\exists}\gamma \;\vdash\; \Psi}\;\equiv$$

Witness

$$\boxed{\dfrac{\Phi \;\vdash\; \gamma_{x\to t}, \underset{x}{\exists}\gamma, \Psi}{\Phi \;\vdash\; \underset{x}{\exists}\gamma, \Psi}\; \vdash\exists}$$

Exercise 1

Sequent Calculus: Example 1

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Q, P[a], \underset{x}{\forall} P[x] \;\vdash\; Q}\ \text{a}}{P[a] \Rightarrow Q, P[a], \underset{x}{\forall} P[x] \;\vdash\; Q}\ \text{MP}}{P[a] \Rightarrow Q, \underset{x}{\forall} P[x] \;\vdash\; Q}\ \forall\vdash}{\underset{x}{\exists}\left(P[x] \Rightarrow Q\right), \underset{x}{\forall} P[x] \;\vdash\; Q}\ \exists\vdash}{\underset{x}{\exists}\left(P[x] \Rightarrow Q\right) \;\vdash\; \left(\underset{x}{\forall} P[x]\right) \Rightarrow Q}\ \Rightarrow\vdash
$$

Exercise 2: $\underset{x}{\forall}\left(P[x] \Rightarrow Q\right) \;\vdash\; \left(\underset{x}{\exists} P[x]\right) \Rightarrow Q$

Sequent Calculus: Example 2

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\dfrac{}{P[a] \;\vdash\; P[a], Q, \underset{x}{\exists}(P[x] \Rightarrow Q)}\;\text{a}
}{\vdash\; P[a], P[a] \Rightarrow Q, \underset{x}{\exists}(P[x] \Rightarrow Q)}\;\vdash\Rightarrow
}{\vdash\; P[a], \underset{x}{\exists}(P[x] \Rightarrow Q)}\;\vdash\exists
}{\vdash\; \underset{x}{\forall} P[x], \underset{x}{\exists}(P[x] \Rightarrow Q)}\;\vdash\forall
\qquad
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\dfrac{}{Q, P[b] \;\vdash\; Q, \underset{x}{\exists}(P[x] \Rightarrow Q)}\;\text{a}
}{Q \;\vdash\; P[b] \Rightarrow Q, \underset{x}{\exists}(P[x] \Rightarrow Q)}\;\vdash\Rightarrow
}{Q \;\vdash\; \underset{x}{\exists}(P[x] \Rightarrow Q)}\;\vdash\exists
}{\left(\underset{x}{\forall} P[x]\right) \Rightarrow Q \;\vdash\; \underset{x}{\exists}(P[x] \Rightarrow Q)}\;\Rightarrow\vdash
$$

Exercise 3: $\left(\underset{x}{\exists} P[x]\right) \Rightarrow Q \;\vdash\; \underset{x}{\forall}(P[x] \Rightarrow Q)$

Sequent Calculus: Unit Propagation

Ground atom $\mathcal{A}$ (similar/dual for ground negated atom):

- match with $\forall$ assumptions, $\exists$ goals, add instances
- replace $\mathcal{A} \to \mathbb{T}$ if $\mathcal{A}$ in assumptions, $\mathcal{A} \to \mathbb{F}$ if $\mathcal{A}$ in goals
- simplify truth constants

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      \dfrac{}{\mathbb{F}, \underset{x}{\forall} P[x] \ \vdash} \ \text{a}
     }{\overline{P}[a], P[a], \underset{x}{\forall} P[x] \ \vdash} \ \text{UP (replace } P[a] \to \mathbb{F})
    }{\overline{P}[a], \underset{x}{\forall} P[x] \ \vdash} \ \text{UP (} P[a] \text{ matches } P[x]\text{: instantiate)}
   }{\underset{x}{\exists} \overline{P}[x], \underset{x}{\forall} P[x] \ \vdash} \ \exists\vdash
  }{\underset{x}{\exists} (P[x] \Rightarrow \mathbb{F}), \underset{x}{\forall} P[x] \ \vdash} \ \equiv \text{ (simplify truth constants)}
 }{\underset{x}{\exists} (P[x] \Rightarrow Q), \underset{x}{\forall} P[x] \ \vdash \ Q} \ \text{UP (replace } Q \to \mathbb{F})
}{\underset{x}{\exists} (P[x] \Rightarrow Q) \ \vdash \ \left(\underset{x}{\forall} P[x]\right) \Rightarrow Q} \ \Rightarrow\vdash
$$

Exercise 4: use unit propagation: $\left(\underset{x}{\forall} P[x]\right) \Rightarrow Q \ \vdash \ \underset{x}{\exists} (P[x] \Rightarrow Q)$

Sequent Calculus: Exercise 1

Witness

$$\boxed{\dfrac{\Phi \;\vdash\; \gamma_{x \to t}, \underset{x}{\exists}\gamma, \Psi}{\Phi \;\vdash\; \underset{\text{x}}{\exists}\gamma, \Psi} \vdash \forall}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Phi \;\vdash\; \gamma_{x \to t}, \underset{x}{\exists}\gamma, \Psi}{\Phi \;\vdash\; \gamma_{x \to t}, \neg\underset{x}{\forall}\neg\gamma, \Psi} \equiv}{\Phi, \neg\gamma_{x \to t}, \underset{x}{\forall}\neg\gamma \;\vdash\; \Psi} \vdash \neg}{\Phi, \underset{x}{\forall}\neg\gamma \;\vdash\; \Psi} \vdash \forall}{\Phi \;\vdash\; \neg\underset{x}{\forall}\neg\gamma, \Psi} \neg\vdash}{\Phi \;\vdash\; \underset{x}{\exists}\gamma, \Psi} \equiv$$

Sequent Calculus: Exercise 2

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Q, \underset{x}{\forall}(P[x] \Rightarrow Q), P[a] \;\vdash\; Q}\;\mathsf{a}}{P[a] \Rightarrow Q, \underset{x}{\forall}(P[x] \Rightarrow Q), P[a] \;\vdash\; Q}\;\mathsf{MP}}{\underset{x}{\forall}(P[x] \Rightarrow Q), P[a] \;\vdash\; Q}\;\forall\vdash}{\underset{x}{\forall}(P[x] \Rightarrow Q), \underset{x}{\exists}P[x] \;\vdash\; Q}\;\exists\vdash}{\underset{x}{\forall}(P[x] \Rightarrow Q) \;\vdash\; \left(\underset{x}{\exists}P[x]\right) \Rightarrow Q}\;\Rightarrow\vdash
$$

Sequent Calculus: Exercise 3

$$
\dfrac{\dfrac{\dfrac{}{P[a] \ \vdash \ P[a], \underset{x}{\exists} P[x], Q}\,\mathrm{a}}{P[a] \ \vdash \ \underset{x}{\exists} P[x], Q}\vdash\exists \qquad \dfrac{}{Q, P[a] \ \vdash \ Q}\,\mathrm{a}}{\dfrac{\left(\underset{x}{\exists} P[x]\right) \Rightarrow Q, P[a] \ \vdash \ Q}{\dfrac{\left(\underset{x}{\exists} P[x]\right) \Rightarrow Q \ \vdash \ P[a] \Rightarrow Q}{\left(\underset{x}{\exists} P[x]\right) \Rightarrow Q \ \vdash \ \underset{x}{\forall} (P[x] \Rightarrow Q)}\vdash\forall}\vdash\Rightarrow}\Rightarrow\vdash
$$

Sequent Calculus: Exercise 4

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\cfrac{}{\vdash\ \mathbb{T}, \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \text{a}
}{\vdash\ \mathbb{F} \Rightarrow Q, \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \equiv\ \text{(simplify truth constants)}
}{\vdash\ P[a], P[a] \Rightarrow Q, \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \text{UP (replace } P[a] \to \mathbb{F})
}{\vdash\ P[a], \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \text{UP}\ \ \begin{matrix}(P[a] \text{ matches } P[x]\text{:}\\ \text{instantiate)}\end{matrix}
}{\vdash\ \underset{x}{\forall}P[x], \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \vdash\forall
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\cfrac{}{\vdash\ \mathbb{T}}\ \text{a}
}{\vdash\ \underset{x}{\exists}(P[x] \Rightarrow \mathbb{T})}\ \equiv
}{Q\ \vdash\ \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \text{UP}
}{\left(\underset{x}{\forall}P[x]\right) \Rightarrow Q\ \vdash\ \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \vdash\Rightarrow
$$

Sequent Calculus: Exercise 4 (continued)

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\dfrac{}{\vdash\ \mathbb{T}, \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \text{a}
}{\vdash\ \mathbb{F} \Rightarrow Q, \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \equiv
}{\vdash\ P[a], P[a] \Rightarrow Q, \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \text{UP}
}{\vdash\ P[a], \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \text{UP}
}{\vdash\ \underset{x}{\forall} P[x], \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \vdash\forall
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\dfrac{}{\vdash\ \mathbb{T}}\ \text{a}
}{\vdash\ \underset{x}{\exists}(P[x] \Rightarrow \mathbb{T})}\ \equiv\ \text{(simplify truth constants)}
}{Q\ \vdash\ \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \text{UP (replace } Q \to \mathbb{T}\text{)}
}{\left(\underset{x}{\forall} P[x]\right) \Rightarrow Q\ \vdash\ \underset{x}{\exists}(P[x] \Rightarrow Q)}\ \Rightarrow\vdash
$$

Unit Propagation: Example 3

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ \mathbb{T}, \exists_x P[x]}\ \text{a}}{P[a]\ \vdash\ P[a], \exists_x P[x]}\ \text{UP (replace } P[a] \to \mathbb{T}\text{)}}{P[a]\ \vdash\ \exists_x P[x]}\ \text{UP}\quad \text{(}P[a]\text{ matches }P[x]\text{: instantiate)}}{\neg\left(\exists_x P[x]\right), P[a]\ \vdash}\ \neg\vdash}{\left(\exists_x P[x]\right) \Rightarrow \mathbb{F}, P[a]\ \vdash}\ \equiv\ \text{(simplify truth constants)}}{\left(\exists_x P[x]\right) \Rightarrow Q, P[a]\ \vdash\ Q}\ \text{UP (replace } Q \to \mathbb{F}\text{)}}{\left(\exists_x P[x]\right) \Rightarrow Q\ \vdash\ P[a] \Rightarrow Q}\ \vdash\Rightarrow}{\left(\exists_x P[x]\right) \Rightarrow Q\ \vdash\ \forall_x (P[x] \Rightarrow Q)}\ \vdash\forall
$$

Unit Propagation: Example 4

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              \dfrac{}{\mathbb{F}, \underset{x}{\forall}\overline{P}[x] \;\vdash}\;\text{a}
            }{\overline{\mathbb{T}}, \underset{x}{\forall}\overline{P}[x] \;\vdash}\;\equiv\ \text{(simplify truth constants)}
          }{\overline{P}[a], \underset{x}{\forall}\overline{P}[x], P[a] \;\vdash}\;\text{UP (replace } P[a] \to \mathbb{T}\text{)}
        }{\underset{x}{\forall}\overline{P}[x], P[a] \;\vdash}\;\text{UP} \quad \text{(} P[a] \text{ matches } P[x]\text{: instantiate)}
      }{\underset{x}{\forall}\overline{P}[x], \underset{x}{\exists}P[x] \;\vdash}\;\exists\vdash
    }{\underset{x}{\forall}\left(P[x] \Rightarrow \mathbb{F}\right), \underset{x}{\exists}P[x] \;\vdash\; Q}\;\equiv\ \text{(simplify truth constants)}
  }{\underset{x}{\forall}\left(P[x] \Rightarrow Q\right), \underset{x}{\exists}P[x] \;\vdash\; Q}\;\text{UP (replace } Q \to \mathbb{F}\text{)}
}{\underset{x}{\forall}\left(P[x] \Rightarrow Q\right) \;\vdash\; \left(\underset{x}{\exists}P[x]\right) \Rightarrow Q}\;\vdash\Rightarrow
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